HIROSHIMA MATH. J.
17 (1987), 471–487

An elementary proof of the Trombi theorem for the Fourier transform of $\mathscr{C}^p(G: F)$

Masaaki EGUCHI and Masato WAKAYAMA

(Received January 19, 1987)

§1. Introduction

Let G and $\mathfrak{g}$ be a real connected noncompact semisimple Lie group with finite center and its Lie algebra respectively. Let $G=KAN$ be an Iwasawa decomposition of G and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ the corresponding decomposition of $\mathfrak{g}$. Denote by $\hat{K}$ the set of all equivalence classes of irreducible unitary representations of K. Let $\boldsymbol{F}\subset\hat{K}$, $|\boldsymbol{F}|<\infty$ and $0<p\leq 2$. Let $\mathscr{C}^p(G: \boldsymbol{F})$ be the L^p Schwartz space on G of type $\boldsymbol{F}$. It follows from the definition that if $0<p'<p\leq 2$ then

$$C_c^\infty(G: \boldsymbol{F}) \subset \mathscr{C}^{p'}(G: \boldsymbol{F}) \subset \mathscr{C}^p(G: \boldsymbol{F}) \subset \mathscr{C}^2(G: \boldsymbol{F}) = \mathscr{C}(G: \boldsymbol{F}).$$

The images of $\mathscr{C}^p(G: \boldsymbol{F})$ by the Fourier transform are characterized by Harish-Chandra [9(c, d, e)] for $p=2$ and general rank cases, and by Trombi [12(c)] for $0<p<2$ and $\mathrm{rk}\,(G/K)=1$ case, respectively. One of the most difficult parts of the theory in [12(c)] is to show the continuity of the inverse Fourier transform. To prove the main theorem in [12(c)], Trombi [12(b)] investigated the asymptotic behavior of the Eisenstein integral at infinity. He gave, taking some terms of the Harish-Chandra expansion of the spherical function as an approximation for it, a uniform estimate for the difference between them for $\nu\in F$ apart from a compact set including the origin, where F denotes $(-1)^{1/2}\mathfrak{a}^*$ ($\mathfrak{a}^*$ the real dual space of $\mathfrak{a}$). But the use of the approximation, instead of the whole series expansion of the spherical function, and the exclusion of a compact set in the approximation theorem, made the proof of the continuity of the Fourier inverse map rather complicated.

On the other hand, Eguchi-Hashizume-Koizumi [4] obtained the Gangolli estimates for the coefficients of the Harish-Chandra expansions of Eisenstein integrals. Our purpose of this paper is to show that we can give an elementary proof of the continuity of the wave packets, the Fourier inverse map, by using the whole expansion and the Gangolli estimates. But unfortunately, our proof cannot remove the K finite condition on $\mathscr{C}^p$ functions (see Remark in Section 6).

In Section 3, we review the Harish-Chandra expansion of the Eisenstein integral and the Gangolli estimates for its coefficients. To explain the instruments which we use in Section 6, we recall in Sections 4 and 5, the notion of the Fourier transform of $\mathscr{C}^p(G: \boldsymbol{F})$ from [12(c)]. We give in Section 6 an elementary proof of the continuity of the wave packets.

§ 2. Notation

Let G and K be as in Section 1. In what follows Lie groups and their subgroups will be denoted by upper case Latin letters and their Lie algebras by the corresponding lower case Germann letters; the upper case Germann letters are reserved for the elements of the enveloping algebra.

If V is a vector space over $\boldsymbol{R}$, we shall denote by $V_{\boldsymbol{C}}$ its complexification. Let V^* (resp. $V_{\boldsymbol{C}}^*$) denote the real (resp. complex) dual of V (resp. $V_{\boldsymbol{C}}$); $S(V)$ (resp. $S(V_{\boldsymbol{C}})$) the symmetric algebra over V (resp. $V_{\boldsymbol{C}}$).

For any Lie group L we denote by $\hat{L}$ the set of equivalence classes of irreducible unitary representations of L.

Let θ be a Cartan involution of G which fixes K elementwise. We use also the same symbol θ for its differential. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ be the Cartan decomposition defined by θ. Let $\mathfrak{h}$ be a θ stable Cartan subalgebra of $\mathfrak{g}$ with maximal vector part and put $\mathfrak{a}=\mathfrak{h}\cap\mathfrak{s}$; $A=\exp\mathfrak{a}$. Throughout this paper we assume that $\dim\mathfrak{a}=1$. We denote by $P(A)$ the set of parabolic subgroups whose split component is A. Let M and M' be the centralizer and the normalizer of A in K, respectively. The finite group $W(A)=M'/M$ is called the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, and it acts on $\mathfrak{a}_{\boldsymbol{C}}^*$ and $\hat{M}$ in the usual manner: if $\chi\in\hat{M}$, $(V, \sigma)\in\chi$, $\nu\in\mathfrak{a}_{\boldsymbol{C}}^*$ and $w\in M'$ then w acts on $\mathfrak{a}_{\boldsymbol{C}}^*$ and $\hat{M}$ by $(w\nu)(H)=\nu(\mathrm{Ad}\,w^{-1}(H))$ $(H\in\mathfrak{a}_{\boldsymbol{C}})$ and $(w\sigma)(m)=\sigma(w^{-1}mw)$ $(m\in M)$, respectively.

Let $Q\in P(A)$ and $Q=MAN_Q$ be its Langlands decomposition. Let $\chi\in\hat{M}$, $\sigma\in\chi$, $\nu\in\mathfrak{a}_{\boldsymbol{C}}^*$ and put $\pi_{Q,\chi,\nu}=\mathrm{Ind}_Q^G(\sigma\otimes\xi_\nu)$, where $\xi_\nu(a)=e^{\nu(\log a)}$ and $\sigma\otimes\xi_\nu$ is extended to Q by making it trivial on N_Q. Let $\mathscr{H}_{Q,\chi,\nu}$ be the representation space of $\pi_{Q,\chi,\nu}$. Put $F=(-1)^{1/2}\mathfrak{a}^*$, $F_{\boldsymbol{C}}=\mathfrak{a}_{\boldsymbol{C}}^*$ and $F_{\boldsymbol{R}}=\mathfrak{a}^*$. We also put $F'=F-\{0\}$, $F'_{\boldsymbol{C}}=F_{\boldsymbol{C}}-\{0\}$ and $F'_{\boldsymbol{R}}=F_{\boldsymbol{R}}-\{0\}$. It is known that $\pi_{Q,\chi,\nu}$ is irreducible for all $\nu\in F'$ and that $\pi_{Q_1,\chi,\nu}$ is unitarily equivalent to $\pi_{Q_2,s\chi,s\nu}$ for all $\nu\in F'$, $\chi\in\hat{M}$, $s\in W(A)$ and Q_1, $Q_2\in P(A)$. The intertwining operator between them is denoted by $\mathscr{A}_{Q_1|Q_2}$, that is an isometry $\mathscr{H}_{Q_1,\chi,\nu}\to\mathscr{H}_{Q_2,s\chi,s\nu}$ such that

$$\mathscr{A}_{Q_1|Q_2}(s\colon\chi\colon\nu)\pi_{Q_1,\chi,\nu}(x)=\pi_{Q_2,s\chi,s\nu}(x)\mathscr{A}_{Q_1|Q_2}(s\colon\chi\colon\nu)\quad(x\in G).$$

Moreover, it is also known that, for fixed Q_1, Q_2, s and χ, the function $\nu\to\mathscr{A}_{Q_1|Q_2}(s\colon\chi\colon\nu)$ has a meromorphic extension to $F_{\boldsymbol{C}}$.

Suppose that $\mathrm{rk}(G)=\mathrm{rk}(K)$ and B is a Cartan subgroup contained in K. Then there exists a lattice $L_B\subset\mathfrak{b}_{\boldsymbol{C}}^*$ such that L_B is isomorphic to $\hat{B}$. Let $W(G/B)$ denote the finite group $N_G(B)/B$, where $N_G(B)$ denotes the normalizer of L_B in G. Then $W(G/B)$ acts on L'_B, the set of the regular elements of L_B. Let L_B^+ be a fundamental domain for this action. To each element $\Lambda\in L'_B$ a representation $\omega(\Lambda)$ corresponds, whose matrix elements are L^2 functions on G. It is known that if $\Lambda_1, \Lambda_2 \in L'_B$ then $\omega(\Lambda_1)$ is equivalent to $\omega(\Lambda_2)$ if and only if $\Lambda_1 = s\Lambda_2$ for some $s \in W(G/B)$. In particular, L_B^+ parametrizes the class of representations corresponding to B. We shall denote by $\mathscr{H}_\Lambda$ the representation space of $\omega(\Lambda)$.

Fix now a finite set $\boldsymbol{F} \subset \hat{K}$. We put $\hat{M}(\boldsymbol{F}) = \{\chi \in \hat{M}: [\delta_{|M}: \chi] > 0$ for some $\delta \in \boldsymbol{F}\}$. Then $|\hat{M}(\boldsymbol{F})| < \infty$ and we have by the Frobenius reciprocity theorem that $[\pi_{Q,\chi,\nu|K}: \delta] \neq 0$ for some $\delta \in \boldsymbol{F}$ if and only if $\chi \in \hat{M}(\boldsymbol{F})$. For each $\chi \in \hat{M}(\boldsymbol{F})$ we fix a representation (σ, V_σ) in χ. By the restriction map $\varphi \to \varphi|K$ of $\mathscr{H}_{Q,\chi,\nu}$ onto a Hilbert space $\mathscr{H}_{Q,\chi}$, which is independent of ν, we sometimes identify $\mathscr{H}_{Q,\chi}$ with $\mathscr{H}_{Q,\chi,\nu}$ if it is not neccesary to appeal to the parameter ν. For each $\gamma \in \hat{K}$, we denote by $\mathscr{H}_{Q,\chi,\gamma}$ the isotypic component of $\mathscr{H}_{Q,\chi}$ corresponding to γ and put $\mathscr{H}_{Q,\chi,\boldsymbol{F}} = \sum_{\gamma \in \boldsymbol{F}} \mathscr{H}_{Q,\chi,\gamma}$, $n(\chi, \gamma) = \dim \mathscr{H}_{Q,\chi,\gamma}$. Fix an orthonormal basis $\{\phi_{\gamma,l}(Q: \chi): 1 \le l \le n(\chi: \gamma)\}$ for $\mathscr{H}_{Q,\chi,\gamma}$.

In a manner similar to the above, we put $\mathscr{H}_{\Lambda,\boldsymbol{F}} = \sum_{\gamma \in \boldsymbol{F}} \mathscr{H}_{\Lambda,\gamma}$, $\mathscr{H}_{\Lambda,\gamma}$ denoting the isotypic component of $\mathscr{H}_\Lambda$ corresponding to γ. We put also $n(\Lambda, \gamma) = \dim \mathscr{H}_{\Lambda,\gamma}$ and fix an orthonormal basis $\{\phi_{\gamma,l}(\Lambda): 1 \le l \le n(\Lambda, \gamma)\}$ for $\mathscr{H}_{\Lambda,\gamma}$.

For $\gamma \in \boldsymbol{F}$ put $\xi_\gamma = \dim(\gamma)\text{conj.}\, \chi_\gamma$, where χ_γ denotes the character of γ, and put $\xi_{\boldsymbol{F}} = \sum_{\gamma \in \boldsymbol{F}} \xi_\gamma$. Let

$$\pi^{\boldsymbol{F}}_{Q,\chi,\nu}(x) = \pi(\xi_{\boldsymbol{F}})\pi(x)\pi(\xi_{\boldsymbol{F}}) \quad (x \in G),$$

where $\pi = \pi_{Q,\chi,\nu}$. Then $\pi^{\boldsymbol{F}}_{Q,\chi,\nu}(x) \in \operatorname{End}(\mathscr{H}_{Q,\chi,\boldsymbol{F}})$.

For $Q \in P(A)$ put $d_Q(m) = (\det \operatorname{Ad} m_{|\mathfrak{n}_Q})^{1/2}$ $(m \in MA)$ and $\rho_Q(H) = (1/2)\operatorname{tr}(\operatorname{ad} H_{|\mathfrak{n}_Q})$ $(H \in \mathfrak{a})$. We also put $A^+(Q) = \{a \in A: e^{\alpha(\log a)} > 1\}$, where $\alpha = \alpha_Q$ is the unique simple root in $\Delta(\mathfrak{g}, \mathfrak{a})$, the set of all roots of $(\mathfrak{g}, \mathfrak{a})$.

§ 3. The Harish-Chandra expansion of Eisenstein integrals

We shall review the Harish-Chandra expansion of the Eisenstein integral and the Gangolli estimate of the coefficients in the expansion.

On the Fréchet space $V = C^\infty(K \times K)$, equipped with the C^∞-topology, a double unitary representation $\tau = (\tau_1, \tau_2)$ of K is defined as follows. If $k_j, u_j \in K$ $(j = 1, 2)$ and $v \in V$ then let

$$\tau_1(k_1)v\tau_2(k_2)(u_1: u_2) = v(u_1k_1: k_2u_2).$$

It can be seen that τ is unitary with respect to the norm

$$|v|^2 = \int_{K \times K} |v(k_1: k_2)|^2 dk_1 dk_2.$$

We simply write τ for τ_1 and τ_2 when there is no ambiguity. Let

$$V_{\boldsymbol{F}} = \left\{v \in V: v = \int_K \xi_{\boldsymbol{F}}(k)\tau(k)v dk = \int_K v\tau(k)\xi_{\boldsymbol{F}}(k)dk\right\}.$$

Let (τ, V_F) be the double K-representation given by restricting τ to V_F. Let (σ, V_σ) be in the class χ, $\chi \in \hat{M}(F)$. A function $\mathscr{K}$ from $K \times K$ into End (V_σ) is called smooth if it is continuous and

$$\mathscr{K}(m_2 k_2 \colon k_1 m_1) = \sigma(m_2)\mathscr{K}(k_2 \colon k_1)\sigma(m_1) \quad (m_1, m_2 \in M, k_1, k_2 \in K).$$

It is known from Lemma 6.1 of [9(d)] that there exists a linear bijection $T \to \mathscr{K}_T$ of End $(\mathscr{H}_{Q,\chi,F})$ into the space of smooth functions such that

$$(Th)(k_2) = \int_K \mathscr{K}_T(k_2 \colon k_1) h(k_1^{-1}) dk_1 \quad (h \in \mathscr{H}_{Q,\chi,F}, k_2 \in K).$$

For $\chi \in \hat{M}(F)$, $L(\chi)$ denotes the subspace of all $f \in C^\infty(M \colon V_F \colon \tau_M)$ $(\tau_M = \tau_{|M})$ such that for all $m_1, m_2 \in M$, the function: $m \to f(m_1 \colon m \colon m_2)$ belongs to the span of the matrix elements of (σ, V_σ). For $T \in \operatorname{End}(\mathscr{H}_{Q,\chi,F})$, $\psi_T \in L(\chi)$ is defined as follows. If $m \in M$, $\psi_T(m)$ is the element $v \in V_F$ given by

$$v(k_1 \colon k_2) = \psi_T(k_1 \colon m \colon k_2) = \operatorname{tr}\{\mathscr{K}_T(k_2 \colon k_1)\sigma(m)\}.$$

By Lemma 7.1 of [9(e)] the map $T \to \psi_T$ is a bijection.

LEMMA 3.1. *We have*

$$\|\psi_T(1)\| \leq \dim(\mathscr{H}_{Q,\chi,F}) \|T\|.$$

PROOF. Let h_i $(1 \leq i \leq r)$ be an orthonormal basis for $\mathscr{H}_{Q,\chi,F}$ and u_j $(1 \leq j \leq d(\sigma))$ an orthonormal basis for V_σ. Then, from the argument in the proof of Lemma 6.1 of [9(e)], we have

$$\mathscr{K}_T(k_2 \colon k_1)u = \textstyle\sum_{1 \leq i \leq r} h_i(k_2)((T^*h_i)(k_1^{-1}), u) \quad (u \in V_\sigma),$$

where T^* denotes the adjoint operator of the linear operator T. Thus, from the definition of ψ_T we have for $k_1, k_2 \in K$

$$\begin{aligned} \psi_T(k_1 \colon 1 \colon k_2) &= \operatorname{tr}\{\mathscr{K}_T(k_2 \colon k_1)\} \\ &= \textstyle\sum_j (\sum_i h_i(k_2)((T^*h_i)(k_1^{-1}), u_j), u_j) \\ &= \textstyle\sum_i \sum_j (h_i(k_2), (u_j, (T^*h_i)(k_1^{-1}))u_j) \\ &= \textstyle\sum_i (h_i(k_2), (T^*h_i)(k_1^{-1})). \end{aligned}$$

Therefore we have

$$|\psi_T(k_1 \colon 1 \colon k_2)|^2 \leq \textstyle\sum_{1 \leq i, j \leq r} |(h_i(k_2), (T^*h_i)(k_1^{-1}))(h_j(k_2), (T^*h_j)(k_1^{-1}))|.$$

By using the Minkovsky-Schwarz inequality on the right hand side and integrating the both side, we obtain the desired inequality

$$\|\psi_T(1)\|^2 = \int_{K\times K} |\psi_T(k_1: 1: k_2)|^2 dk_1 dk_2 \leq \dim(\mathscr{H}_{Q,\chi,F})^2 \|T\|^2.$$

Let $Q \in P(A)$. According to the Iwasawa decomposition $G = KAN_Q$, each $x \in G$ can be written uniquely as $x = \kappa(x) \exp H(x) n(x)$ ($\kappa(x) \in K$, $H(x) \in \mathfrak{a}$, $n(x) \in N_Q$).

Given $\psi \in L(\chi)$ ($\chi \in \hat{M}(F)$), $\nu \in F_C$, $Q \in P(A)$, ψ is extended to a function on G by

$$\psi(kan) = \tau(k)\psi(1) \quad (k \in K,\ a \in A,\ n \in N_Q).$$

Then the integral

$$E(Q: \psi: \nu: x) = \int_K \psi(xk)\tau(k^{-1}) e^{(\nu - \rho_Q)(H(xk))} dk$$

is called the Eisenstein integral.

If $T \in \mathrm{End}(\mathscr{H}_{Q,\chi,F})$ then it is known ([9(e)]) that

$$E(Q: \psi_T: \nu: k_1: x: k_2) = \mathrm{tr}\{T\pi^F_{Q,\chi,\nu}(k_1 x k_2)\} \tag{3.1}$$

for k_1, $k_2 \in K$ and $x \in G$.

Fix $Q \in P(A)$ and let $\alpha = \alpha_Q$ be the unique simple root of $\Delta(\mathfrak{q}, \mathfrak{a})$. For the convenience we then identify $\boldsymbol{C}$ with F_C via the map $z \to z\alpha$. Under this identification ρ_Q corresponds to $(p+2q)/2$, where $p = \dim \mathfrak{g}_\alpha$ and $q = \dim \mathfrak{g}_{2\alpha}$. We put $V_F^M = \{v \in V_F: \tau(m)v = v\tau(m),\ m \in M\}$. Let $\omega_{\mathfrak{m}}$ denote the Casimir element of $\mathfrak{M}$ and let γ be the endomorphism of $\mathrm{Hom}_C(V_F^M, V_F^M)$ defined by

$$\gamma(T) = [\tau_2(\omega_{\mathfrak{m}}), T] \qquad (T \in \mathrm{Hom}_C(V_F^M, V_F^M)),$$

Let $\gamma_1, \ldots, \gamma_t$ be the set of all distinct eigenvalues of γ with multiplicities $m_1, \cdots, m_t$, respectively. Since the representations τ_1 and τ_2 of K are unitary, every eigenvalue of the transformation $v \to v\tau_2(\omega_{\mathfrak{m}})$ is real, whence the γ_i are real. Moreover, if $\{\theta_1, \ldots, \theta_l\}$ denotes an enumeration of the eigenvalues of $\tau_2(\omega_{\mathfrak{m}})$, it is then known that each γ_i is of the form $\theta_j - \theta_k$ ($1 \leq j, k \leq l$). We now put

$$\tau_{n,i} = n/2 - \rho_Q + \gamma_i/(2n\|\alpha\|^2) \quad (1 \leq i \leq t).$$

Put $\Gamma' = \boldsymbol{C} \setminus \{\tau_{n,i}: 1 \leq n < \infty,\ 1 \leq i \leq t\}$. Then Γ' is an open connected set. For $\nu \in \Gamma'$ and $n \geq 1$, we recursively define $\Gamma_n(\nu) \in \mathrm{End}(V_F^M)$ as follows: put $\Gamma_0(\nu) \equiv 1$, and for $n \geq 1$

$$\begin{aligned}
&\|\alpha\|^2\{2n\nu - n[n - 2\rho_Q]\}\Gamma_n(\nu) - [\tau_2(\omega_{\mathfrak{m}}), \Gamma_n(\nu)] \\
&= 2\sum_{l \geq 1} \{p\|\alpha\|^2(\nu - n + 2l)\Gamma_{n-2l}(\nu) + 2q\|\alpha\|^2(\nu - n + 4l)\Gamma_{n-4l}(\nu)\} \\
&\quad + 8\sum_{\lambda \in P^+, \bar{\lambda} = \alpha} \sum_{l \geq 1} \{(2l-1)\tau_1(Y_\lambda)\tau_2(Y_{-\lambda})\Gamma_{n-(2l-1)}(\nu)\}
\end{aligned}$$

$$+ 8 \sum_{\lambda \in P^+, \tilde{\lambda}=2\alpha} \sum_{l \geq 1} \{(2l-1)\tau_1(Y_\lambda)\tau_2(Y_{-\lambda})\Gamma_{n-4l+2}(\nu)\}$$

$$- 8 \sum_{\lambda \in P^+, \tilde{\lambda}=\alpha} \sum_{l \geq 1} l\{\tau_1(Y_\lambda Y_{-\lambda}) - \tau_2(Y_\lambda Y_{-\lambda})\}\Gamma_{n-2l}(\nu)$$

$$- 8 \sum_{\lambda \in P^+, \tilde{\lambda}=2\alpha} \sum_{l \geq 1} l\{\tau_1(Y_\lambda Y_{-\lambda}) - \tau_2(Y_\lambda Y_{-\lambda})\}\Gamma_{n-4l}(\nu).$$

Here $P^+ = P \setminus P^-$ and $\tilde{\lambda} = \lambda | \mathfrak{a}$, P being a positive system of roots for $\Delta(\mathfrak{g}, \mathfrak{h})$ and $P^- = \{\alpha \in P: \alpha|\mathfrak{a} = 0\}$. Moreover, we put $\Gamma_k = 0$ if $k < 0$. It is known that the functions $\nu \to \Gamma_n(\nu)$ are well defined and are rational functions in ν and holomorphic on Γ'. Let $\Gamma'' = \{\nu \in \boldsymbol{C}: \nu - \rho_Q \in \Gamma'\}$ and $\Gamma = \{\nu \in \boldsymbol{C}: \nu, -\nu \in \Gamma''\}$. Put

$$\Phi(\nu: a) = \sum_{n=0}^{\infty} \Gamma_n(\nu - \rho_Q) e^{(\nu - \rho_Q - n\alpha)(\log a)} \quad (\nu \in \Gamma,\ a \in A^+(Q)).$$

THEOREM 3.1 (Harish-Chandra (cf. [14])). *For any* $\nu \in \Gamma$, $t \in W(A)$, $Q \in P(A)$ *there exist uniquely determined elements* $C_{Q|Q}(t: \nu)$ *in* $\mathrm{End}(V_F^M)$ *such that if* $\psi \in C^\infty(M: V_F^M: \tau_M)$ *then*

$$E(Q: \psi: \nu: a) = \sum_{t \in W(A)} \Phi(t\nu: a) C_{Q|Q}(t: \nu)\psi(1)$$

for all $a \in A^+(Q)$. *Moreover,* $C_{Q|Q}(t: \nu)$ $(t \in W(A))$ *are meromorphic functions in* ν *and holomorphic on* Γ.

We list some properties of the Harish-Chandra C-functions and the Plancherel measure, which we shall use in the last section. For the details see [9(e)] and also [12(c)].

(1) There exists $\varepsilon_1 > 0$ such that if $\pi(\nu) = \langle \nu, \alpha_Q \rangle^{p+q}$ then $\pi(\nu) C_{Q_1|Q_2}(s: \nu)$ $(Q_1, Q_2 \in P(A))$ extends to a holomorphic function of ν on $F_{\boldsymbol{C}}^{\varepsilon_1} = \{\nu \in F_{\boldsymbol{C}}: |\mathrm{Re}\, \nu| < \varepsilon_1\}$.

(2) Let $s \in W(A)$. Then s acts on $L(\boldsymbol{F}) = \sum_{\chi \in \hat{M}(\boldsymbol{F})} L(\chi)$ in the usual manner. We then have

$$s C_{Q_2|Q_1}(t: \nu) = C_{Q_2^s|Q_1}(st: \nu); \quad C_{Q_2|Q_1}(t: \nu) s^{-1} = C_{Q_2|Q_1^s}(ts^{-1}: s\nu).$$

(3) $C_{Q_2|Q_1}$ extends to a meromorphic function on $F_{\boldsymbol{C}}$.

(4) $C_{\bar{Q}|Q}(1: \nu)$ and $C_{Q|Q}(1: -\nu)$ are holomorphic on the set $\langle \mathrm{Re}\, \nu, \alpha_Q \rangle < 0$. If $s \in W(A)$, $s \neq 1$, then $C_{Q|Q}(s: \nu)$ and $C_{\bar{Q}|\bar{Q}}(1: \nu)$ are also holomorphic there.

(5) For fixed $Q_1, Q_2 \in P(A)$, $s \in W(A)$, $\nu \in F'$ and $\chi \in \hat{M}$, $C_{Q_2|Q_1}(s: \nu)$ defines a bijection of $L(\chi)$ onto $L(s\chi)$. Let ${}^\circ C_{Q_1|Q_2}(s: \nu) = C_{Q_1|Q_2}(1: s\nu)^{-1} C_{Q_1|Q_2}(s: \nu)$. Then $\nu \to {}^\circ C_{Q_1|Q_2}(s: \nu)$ defines a rational mapping of $F_{\boldsymbol{C}}$ into $\mathrm{End}(L(\boldsymbol{F}))$.

(6) There exists a function $\mu: \hat{M} \times F_{\boldsymbol{C}} \to \boldsymbol{C}$ satisfying the following:

(A) For each $\chi \in \hat{M}$, $\nu \to \mu(\chi: \nu)$ is meromorphic on $F_{\boldsymbol{C}}$, holomorphic on $F_{\boldsymbol{C}}^{\varepsilon_2}$ for some $\varepsilon_2 > 0$, $\mu(\chi, \nu) > 0$ on F', and $\mu(\chi, \nu) \geq 0$ on F.

(B) There exists a constant depending only on A, say $C(A)$, such that for all $Q_1, Q_2 \in P(A)$, $t \in W(A)$, $\nu \in F'$, $\chi \in \hat{M}(F)$,

$$\mu(\chi: \nu) C_{Q_2|Q_1}(t: \nu)^* C_{Q_2|Q_1}(t: \nu) |_{L(\chi)} = C(A)^2 1_\chi ;$$

here for $T \in \mathrm{End}(L(F))$, T^* denotes the adjoint of T, and 1_χ denotes the identity operator on $L(\chi)$.

(C) $\mu(t\chi: t\nu) = \mu(\chi: \nu)$ $(\chi \in \hat{M}, \nu \in F, t \in W(A))$.

(D) The poles and zeroes of $\mu(\chi: \nu)$ are simple, with the exception of $\nu = 0$, where $\mu(\chi: \nu)$ may have a zero of multiplicity two. The poles and zeroes are all in $F_{\boldsymbol{R}}$; the poles are independent of χ and occur at the points $n\alpha/2$, $n \in \boldsymbol{Z}$ (cf. [11]).

(7) Fix $Q \in P(A)$, $\nu, \nu' \in F'$. Then the four linear transformations

$$C_{Q|Q}(1: \nu), \quad C_{\bar{Q}|\bar{Q}}(1: \nu), \quad C_{\bar{Q}|Q}(1: \nu'), \quad C_{Q|\bar{Q}}(1: \nu')$$

commute with each other. Moreover, we have

$$C_{Q|Q}(1: \nu)^* = C_{\bar{Q}|\bar{Q}}(1: \nu), \quad C_{\bar{Q}|Q}(1: \nu')^* = C_{Q|\bar{Q}}(1: \nu') ;$$

$$C_{Q|Q}(s: s^{-1}\nu)^* = s^* C_{\bar{Q}|Q}(1: \nu), \quad (s \in W(A), s \neq 1),$$

and as meromorphic functions on $F_{\boldsymbol{C}}$ we have

$$\mu(\chi: \nu) C_{\bar{Q}|\bar{Q}}(1: \nu) C_{Q|Q}(1: \nu) |_{L(\chi)} = C(A)^2 1_\chi ;$$

$$\mu(\chi: -\nu) s^{-1} \circ C_{\bar{Q}|Q}(1: \nu) C_{Q|\bar{Q}}(1: \nu) \circ s |_{L(\chi)} = C(A)^2 1_\chi ,$$

where $s \in W(A)$, $s \neq 1$.

(8) $\det C_{\bar{Q}|Q}(1: \nu) = 0$ and $\det C_{\bar{Q}|\bar{Q}}(1: \nu) = 0$ for at most finitely many points (all of which belong to $F_{\boldsymbol{R}}$) in $\mathrm{Re}\,\nu \leq 0$.

(9) Fix ε, $0 < \varepsilon < 1/4$, and $Q \in P(A)$, Then there exists a polynomial function $S \in S(\mathfrak{a}_{\boldsymbol{C}})$ such that if

$$A_1(\nu) = S(\nu) C_{\bar{Q}|\bar{Q}}(1: \nu)^{-1}, \quad A_2(\nu) = S(\nu) C_{\bar{Q}|Q}(1: \nu)^{-1}$$

$$B_1(\nu) = \pi(\nu) C_{Q|Q}(1: \nu), \quad B_2(\nu) = \pi(\nu) C_{Q|Q}(s: \nu) ,$$

where $s \in W(A)$, $s \neq 1$, then A_1, A_2 are holomorphic on $\langle \nu_{\boldsymbol{R}}, \alpha_Q \rangle < \varepsilon$. Further, given any $u \in S(F_{\boldsymbol{C}})$, $M > 0$ there exists $C = C_{u,M,\varepsilon} > 0$, $l = l_{u,M,\varepsilon} \geq 0$ such that

$$\|A_j(\nu; u)\| \leq C(1 + |\nu|)^l \quad (j = 1, 2, \nu \in F_{\boldsymbol{C}}, -M < \langle \mathrm{Re}\,\nu, \alpha_Q \rangle < \varepsilon) ;$$

$$\|B_j(\nu; u)\| \leq C(1 + |\nu|)^l \quad (\nu \in F^{\varepsilon}_{\boldsymbol{C}}, j = 1, 2) .$$

(10) There exist constants $C > 0$ and $r \geq 0$ such that

$$|\mu(\chi: \nu)| \leq C(1 + |\nu|)^r \quad (\nu \in F^{\varepsilon}_{\boldsymbol{C}}) .$$

We shall next review the Gangolli estimates for the coefficients Γ_n due to [4]. Without loss of generality, renumbering the eigenvalues of γ we can assume that

$$\gamma_1 < \cdots < \gamma_s < 0 < \gamma_{s+1} < \cdots < \gamma_t .$$

Let L_1' denote the finite set of all $n \in \boldsymbol{Z}$, $n>0$, such that $-n^2\|\alpha\|^2 \geq \gamma_1$. For each $n \in \boldsymbol{Z}_+$, we define polynomials p_n by

$$p_n(v) = 1 \quad \text{if} \quad n \in L \smallsetminus L_1' ;$$
$$p_n(v) = \prod (2n\|\alpha\|^2 v - n^2\|\alpha\|^2 - \gamma_i)^{m_i} \quad \text{if} \quad n \in L_1'$$

and set $d'(n) = \sum m_i$; where the products and the sums are taken for i such that $1 \leq i \leq s$ and $n^2\|\alpha\|^2 + \gamma_i \leq 0$. We also put

$$P(v) = \prod_{n \in L_1'} p_n(v), \quad d = \sum_{n \in L_1'} d'(n) ;$$
$$P_n(v) = \prod_{n' \in L', n' < n} p_{n'}(v), \quad d(n) = \sum_{n' \in L', n' < n} d'(n')$$

for $n \in L'$. Then remark that P is of finite degree and thus $d < \infty$. We put

$$\mathscr{R} = \{\xi + \eta \in F_{\boldsymbol{C}} \colon \xi \in F,\ \eta \in F_{\boldsymbol{R}},\ \eta \leq 0\} .$$

THEOREM 3.2 (Eguchi-Hashizume-Koizumi [4]). *There exist absolute constants D, $d_1 > 0$ such that*

$$\|P_n(v)\Gamma_n(v - \rho_Q)\| \leq D(1 + \|v\| + n)^{2d} \cdot n^{d_1} \quad (v \in \mathscr{R})$$

for all $n \in L$.

Let U denote the union of the following sets:

(1) $\{\tau_{n,i} \colon \tau_{n,i} - \rho_Q \leq 0\}$

(2) $\{\tau \in F_{\boldsymbol{C}} \colon \tau \leq 0$ and either $\det C_{\bar{Q}|\bar{Q}}(1 \colon \tau) = 0$ or $\det C_{\bar{Q}|Q}(1 \colon \tau) = 0\}$

(3) $\{0\}$ if either $C_{Q|Q}(1 \colon v)$ or $C_{Q|Q}(s \colon v)$ has a pole at $v = 0$.

For $\zeta \in U$ let $O_1(\zeta)$ denote the maximum order of the pole of the functions $v \rightarrow \Gamma_n(v - \rho_Q) C_{\bar{Q}|\bar{Q}}(1 \colon v)^{-1}$ at $v = \zeta$ if $\operatorname{Re} \zeta < 0$ and put $O_1(\zeta) = 0$ if $\zeta = 0$. Further, let $O_s(\zeta)$ ($s \in W(A)$, $s \neq 1$) denote the maximum order of the pole of the functions $v \rightarrow \Gamma_n(v - \rho_Q) C_{\bar{Q}|Q}(1 \colon v)^{-1} \cdot s$ at $v = \zeta$ if $\operatorname{Re} \zeta \leq 0$. Note that $O_t(\zeta) < \infty$ ($\forall t \in W(A)$). Fix $Q \in P(A)$ and $\chi \in \hat{M}$. For $\Delta \in \boldsymbol{F}^2 = \boldsymbol{F} \times \boldsymbol{F}$, say $\Delta = (\gamma, \delta)$, let

$$\boldsymbol{Z}(\Delta \colon \chi) = \{J \in \boldsymbol{Z}^2 \colon \text{if } J = (l, m) \text{ then } 1 \leq l \leq n(\chi \colon \gamma),\ 1 \leq m \leq n(\chi \colon \delta)\} .$$

Put

$$\pi(Q \colon \chi \colon v \colon \Delta \colon J \colon x) = \langle \pi_{Q,\chi,v}(x) \phi_{\gamma,l}(Q \colon \chi),\ \phi_{\delta,m}(Q \colon \chi) \rangle .$$

We now fix Q and set

$$\mathscr{E} = \{\pi(Q: \chi: t\zeta;\, \partial^k(\nu): \Delta: J): \chi \in \hat{M}(F),\, t \in W(A),\, \zeta \in U,\\ 0 \le k \le O_t(\zeta)-1,\, \Delta \in F^2,\, J \in Z(\Delta: \chi)\}.$$

We recall the basis for $\mathscr{E}$ which is given in [12(c)]. We first enumerate $U = \{\zeta_1, \dots, \zeta_m\}$ so that $0 \ge \zeta_1 \ge \cdots \ge \zeta_m$. We choose the least integer i_1 $(m \ge i_1 \ge 1)$ so that there exist some $\chi \in \hat{M}(F)$, $t \in W(A)$, $0 \le k \le O_t(\zeta_{i_1})-1$, $\Delta \in F^2$, and $J \in Z(\Delta: \chi)$ such that $\pi(Q: \chi: t\zeta_{i_1};\, \partial^k(\nu): \Delta: J) \not\equiv 0$. We choose a basis for the set

$$\mathscr{E}_1 = \{\pi(Q: \chi: t\zeta_{i_1};\, \partial^j(\nu): \Delta: J): \chi \in \hat{M}(F),\, t \in W(A),\\ 0 \le j \le O_t(\zeta_{i_1})-1,\, \Delta \in F^2,\, J \in Z(\Delta: \chi)\}.$$

Next let i_2 be such that $m \ge i_2 > i_1 \ge 1$ and there exist some $\chi \in \hat{M}(F)$, $t \in W(A)$, $0 \le j \le O_t(\zeta_{i_2})-1$, $\Delta \in F^2$, $J \in Z(\Delta: \chi)$ such that $\pi(Q: \chi: t\zeta_{i_2};\, \partial^j(\nu): \Delta: J)$ is independent from the already chosen basis elements. We extend the previously chosen basis elements by adding elements of the set

$$\mathscr{E}_2 = \{\pi(Q: \chi: t\zeta_{i_2};\, \partial^j(\nu): \Delta: J): \chi \in \hat{M}(F),\, t \in W(A),\\ 0 \le j \le O_t(\zeta_{i_2})-1,\, \Delta \in F^2,\, J \in Z(\Delta: \chi)\}.$$

By continuing this process we obtain a basis of $\mathscr{E}$. Let I be the subset of

$$\{Q\} \times \hat{M}(F) \times \Big(\bigcup_{t \in W(A)} tU\Big) \times Z \times F^2 \times Z^2$$

which indexes the elements of $\mathscr{E}$; and I' the subset of I which indexes the above chosen basis. For $i \in I$, say $i = (Q, \chi, t\zeta_{i_k}, j, \Delta, J)$ let us write $\pi(i)$ for $\pi(Q: \chi: t\zeta_{i_k};\, \partial^j(\nu): \Delta: J)$. For $i \in I$, $i' \in I'$ define constants $C(i: i') \in C$ by the equation

$$\pi(i) = \sum_{i' \in I'} C(i: i')\pi(i').$$

§4. The space $\mathscr{C}^p(G: F)$ and its Fourier transform

Let $Q \in P(A)$, $x = k \exp X$ $(k \in K,\ X \in \mathfrak{s})$ and put

$$\Xi(x) = \int_K e^{-\rho_Q(H_Q(xk))}dk; \quad \sigma(x) = \|X\|.$$

where $\|\cdot\|$ denotes the norm given by the Killing form. Let $f \in C^\infty(G)$, $a \in \mathfrak{G}$, $r \in R$, $0 < p \le 2$ and put

$$\nu^p_{a,r}(f) = \sup_{x \in G} |f(x;\, a)| \Xi^{-2/p}(x)(1+\sigma(x))^r.$$

Let

$$\mathscr{C}^p(G: F) = \{f \in C^\infty(G: F): \nu^p_{a,r}(f) < \infty \text{ for any } a \in \mathfrak{G},\, r \in R\}.$$

Then $\mathscr{C}^p(G: \boldsymbol{F})$ is a Fréchet algebra with convolution product. We denote by $S^p(G)$ the set of all continuous seminorms on $\mathscr{C}^p(G: \boldsymbol{F})$. It is also known that $\mathscr{C}^p(G: \boldsymbol{F}) \subset L^p(G)$, $C_c^\infty(G: \boldsymbol{F}) \subset \mathscr{C}^p(G: \boldsymbol{F})$, and that if $0<p<p' \leq 2$ then $\mathscr{C}^p(G: \boldsymbol{F}) \subset \mathscr{C}^{p'}(G: \boldsymbol{F}) \subset \mathscr{C}^2(G: \boldsymbol{F}) = \mathscr{C}(G: \boldsymbol{F})$. Moreover, the correspondence $f \to \check{f}$ ($\check{f}(x) = f(x^{-1})$) is a continuous involutive automorphism of $\mathscr{C}^p(G: \boldsymbol{F})$.

For $Q \in P(A)$, $\chi \in \hat{M}$, $\nu \in F_{\boldsymbol{C}}$ and $\alpha \in C_c^\infty(G: \boldsymbol{F})$, let $\mathscr{F}_H(\alpha)(Q: \chi: \nu) \in \mathrm{End}\,(\mathscr{H}_{Q,\chi,\boldsymbol{F}})$ be defined by

$$\mathscr{F}_H(\alpha)(Q: \chi: \nu)(f) = \pi_{Q,\chi,\nu}(\check{\alpha})(f) \quad (f \in \mathscr{H}_{Q,\chi,\nu}).$$

If $\mathrm{rk}\,(G) = \mathrm{rk}\,(K)$, $\Lambda \in L_B^+$, α as above, let $\mathscr{F}_B(\alpha)(\Lambda) \in \mathrm{End}\,(\mathscr{H}_{\Lambda,\boldsymbol{F}})$ be defined by

$$\mathscr{F}_B(\alpha)(\Lambda)v = \pi_\Lambda(\check{\alpha})v \quad (v \in \mathscr{H}_{\Lambda,\boldsymbol{F}}).$$

Let $\mathscr{F} = \mathscr{F}_H$ if $\mathrm{rk}\,(G) > \mathrm{rk}\,(K)$ and $\mathscr{F} = (\mathscr{F}_B, \mathscr{F}_H)$ if $\mathrm{rk}\,(G) = \mathrm{rk}\,(K)$.

§5. The space $\mathscr{C}_H^p(\hat{G}: \boldsymbol{F})$

As usual the symmetric algebra $S(F_{\boldsymbol{C}})$ can be considered as the algebra of differential operators on $F_{\boldsymbol{C}}$. If $F(Q: \chi)$ is a function defined on $F_{\boldsymbol{C}}(Q: 2/p-1)$ with values in $\mathrm{End}\,(\mathscr{H}_{Q,\chi,\boldsymbol{F}})$ and C^∞ in $\mathrm{Int}\,(F_{\boldsymbol{C}}(Q: 2/p-1))$, we put, for $u \in S(F_{\boldsymbol{C}})$ and $r \in \boldsymbol{R}$,

$$\nu_{u,r}^p(F) = \sup \|F(Q: \chi: \nu; u)\| (1+|\nu|)^r,$$

where $\|\cdot\|$ denotes the operator norm and the sup is taken over $\nu \in \mathrm{Int}\,(F_{\boldsymbol{C}}(Q: 2/p-1))$ and $\chi \in \hat{M}(\boldsymbol{F})$. Let I_p (resp. I_p') denote the set of $i \in I$ (resp. I'), $i = (Q, \chi, t\zeta, j, \Delta, J)$ such that $\zeta \in U \cap F_{\boldsymbol{C}}(Q: 2/p-1) = U_p$. We also let

$$A_{Q_1|Q_2}(s: \chi: \nu) = \pi_2(\xi_{\boldsymbol{F}}) \mathscr{A}_{Q_1|Q_2}(s: \chi: \nu) \pi_1(\xi_{\boldsymbol{F}}),$$

where $s \in W(A)$, $Q_1, Q_2 \in P(A)$, $\chi \in \hat{M}(\boldsymbol{F})$, $\nu \in F'$, and $\pi_1 = \pi_{Q_1,\chi,\nu}$, $\pi_2 = \pi_{Q_2,s\chi,s\nu}$.

DEFINITION 1. Let $\mathscr{C}_H^p(\hat{G}: \boldsymbol{F})$ denote the linear space of functions $G(Q: \chi)$: $F_{\boldsymbol{C}}(Q: 2/p-1) \to \mathrm{End}\,(\mathscr{H}_{Q,\chi,\boldsymbol{F}})$ ($Q \in P(A)$, $\chi \in \hat{M}$) such that $G(Q: \chi) \equiv 0$ if $\chi \notin \hat{M}(\boldsymbol{F})$ and

(1) $G(Q: \chi)$ is holomorphic on $\mathrm{Int}\,(F_{\boldsymbol{C}}(Q: 2/p-1))$;

(2) if $Q_1, Q_2 \in P(A)$, $s \in W(A)$, $\chi \in \hat{M}(\boldsymbol{F})$, $\nu \in F'$ then

$$A_{Q_1|Q_2}(s: \chi: \nu) G(Q_1: \chi: \nu) = G(Q_2: s\chi: s\nu) A_{Q_1|Q_2}(s: \chi: \nu);$$

(3) for all $r \in \boldsymbol{R}$, $u \in S(F_{\boldsymbol{C}})$, $\nu_{u,r}^p(G) < \infty$,

(4) in the notation of the previous section,

$$G(i) = \sum_{i' \in I_p'} C(i: i') G(i') \quad (i \in I_p).$$

Here, for $\Delta=(\gamma, \delta)$ and $J=(l, m)$,

$$G(Q: \chi: \nu: \Delta: J) = \langle G(Q: \chi: \nu)\phi_{\gamma,l}(Q: \chi), \phi_{\delta,m}(Q: \chi)\rangle .$$

Then $\mathscr{C}^p_H(\hat{G}: \boldsymbol{F})$ is a Fréchet space equipped with the topology defined by the seminorms $\nu^p_{u,r}$ ($u \in S(F_{\boldsymbol{C}})$, $r \in \boldsymbol{R}$). Let $S^p_H(\hat{G})$ be the set of all continuous seminorms on $\mathscr{C}^p_H(\hat{G}: \boldsymbol{F})$.

Definition 2. Assume rk$(G)=$rk(K) and let $B\subset K$ be the Cartan subgroup of G given in Section 2. Let $\mathscr{C}^p_B(\hat{G}: \boldsymbol{F})$ be the linear space of all functions L: $L^+_B \to \mathrm{End}(\mathscr{H}_{\Lambda,\boldsymbol{F}})$ such that $L(\Lambda)=0$ unless $\Lambda \in L^+_B(\boldsymbol{F})$ and $\mu^p_\alpha(L)=\sup(1+\|\Lambda\|)^\alpha \cdot \|L(\Lambda)\|<\infty$ (for any $\alpha \in \boldsymbol{R}$). Here the sup is taken over $\Lambda \in L^+_B$ and $\|L(\Lambda)\|$ denotes the norm introduced before relative to the basis $\{\phi_{\gamma,l}(\Lambda): \gamma \in \boldsymbol{F}, 1\leq l \leq n(\Lambda: \gamma)\}$.

We topologize $\mathscr{C}^p_B(\hat{G}: \boldsymbol{F})$ using the seminorms μ^p_α ($\alpha \in \boldsymbol{R}$). Then $\mathscr{C}^p_B(\hat{G}: \boldsymbol{F})$ is a Fréchet space with this topology. Denote by $S^p_\alpha(\hat{G})$ the set of the continuous seminorms on $\mathscr{C}^p_B(\hat{G}: \boldsymbol{F})$.

For each $i' \in I'$ choose $\alpha_{i'} \in C^\infty_c(G: \boldsymbol{F})$ such that Supp $\alpha_{i'}$ is contained in ${}^cG(1)$; here $G(s)=\{x \in G: \sigma(x)>s\}$ for $s>0$ and the suffix c denotes its complement, and

$$\int_G \alpha_{i'}(x^{-1})\pi(i'': x)dx = \delta_{i',i''} \quad (i'' \in I').$$

Definition 3. Let $\mathscr{C}^p(\hat{G}: \boldsymbol{F})$ be the space of functions defined as follows:

(1) If rk$(G)>$rk(K), let $\mathscr{C}^p(\hat{G}: \boldsymbol{F})=\mathscr{C}^p_H(\hat{G}: \boldsymbol{F})$ as topological spaces.

(2) If rk$(G)=$rk(K), let $\mathscr{C}^p(\hat{G}: \boldsymbol{F})$ be the linear subspace of $\mathscr{C}^p_B(G: \boldsymbol{F})\times \mathscr{C}^p_H(G: \boldsymbol{F})$ consisting of functions $G=(G_B, G_H)$ which satisfy the linear relation

$$G_B(\Lambda: \Delta: J) = \sum_{i' \in I'_p} \mathscr{F}_B(\alpha_{i'})(\Lambda: \Delta: J)G_H(i')$$

for all $\Lambda \in L^+_B$ such that $|\Lambda(H_\beta)|\leq k(\beta)$ for some positive (relative to some fixed ordering) non compact root β of the pair $(\mathfrak{g}, \mathfrak{b})$; here $k(\beta)=(1/2)\sum_{\alpha\in P}|\alpha(\overline{H}_\beta)|$, P a positive system for $\Delta(\mathfrak{g}, \mathfrak{b})$, $\beta(\overline{H}_\beta)=2$. $G_B(\Lambda: \Delta: J)$ denotes the matrix of $G_B(\Lambda)$ relative to the basis $\{\phi_{\gamma,l}(\Lambda): \gamma \in \boldsymbol{F}, 1\leq l\leq n(\gamma, l)\}$.

The space $\mathscr{C}^p(\hat{G}: \boldsymbol{F})$ is a closed subspace of the product space $\mathscr{C}^p_B(\hat{G}: \boldsymbol{F})\times \mathscr{C}^p_H(\hat{G}: \boldsymbol{F})$. Hence $\mathscr{C}^p(\hat{G}: \boldsymbol{F})$ is also a Fréchet space.

The following result is due to Trombi [12(c)].

Theorem 5.1. *$\mathscr{F}$ is an injective and continuous map of $\mathscr{C}^p(G: \boldsymbol{F})$ into $\mathscr{C}^p(\hat{G}: \boldsymbol{F})$.*

§ 6. Wave packets

Let $\alpha_{i'}$ be as in Section 5. Let us fix $F \in \mathscr{C}^p_H(\hat{G}: F)$ and put

$$\beta_F(x) = \sum_{i' \in I'_p} F(i')\alpha_{i'}(x) \quad (x \in G);$$

$$F_0 = F - \mathscr{F}_H(\beta_F).$$

Then F_0 has the properties stated in the following lemma.

LEMMA 6.1 ([12(c), Lemma 9.1]). *For all $i \in I_p$, $F_0(i)=0$. In particular, the function $F_0(Q: \chi)$ has a zero at every $\zeta \in U_p$ of order equal to the maximum of the order of the pole of the functions $\nu \to \Gamma_n(\nu - \rho_Q)C_{\bar{Q}|\bar{Q}^s}(1: \nu)^{-1} \circ s$ $(s \in W(A))$ at $\nu = \zeta$. If $\pm(2/p-1)\rho_Q \in U_p$ the above statement should be understood that the appropriate derivatives of F_0 when extended to $F_c(Q: 2/p-1)$ vanish at $\pm(2/p-1)\rho_Q$.*

We now want to compute $\phi_{F_0}(Q: \chi: x)$ $(\chi \in \hat{M}(F),\ x \in G)$ given by

$$\phi_{F_0}(Q: \chi: x) = \int_F \mathrm{tr}\,\{F_0(Q: \chi: \nu)\pi^F_{Q,\chi,\nu}(x)\}\mu(\chi: \nu)d\nu.$$

By (3.1), the last expression is equal to

$$\int_F E(Q: \psi_{F_0(Q:\chi:\nu)}: \nu: 1: x: 1)\mu(\chi: \nu)d\nu. \tag{6.1}$$

Let ε_1 and ε_2 be as in (1) and (6A) of Section 3 respectively. Let U be the set given in Section 3 and choose $\delta_0 > 0$ so that

$$0 < 2\delta_0 < \mathrm{Min}\,\{|\zeta|,\ \varepsilon_1,\ \varepsilon_2\}_{\zeta \in U \setminus \{0\}}.$$

Next choose $\varepsilon_0 \in F_R$ so that with the ordering induced on $\mathfrak{a}^*$ by $A^+(Q)$ we have

$$\rho_p = (1-2/p)\rho_Q < -\varepsilon_0 < \mathrm{Min}\,\{\zeta\},$$

where the minimum is taken over $\zeta \in U \cap \mathrm{Int}\,(F_c(Q: 2/p-1))$. We define contours as in Fig. 1.

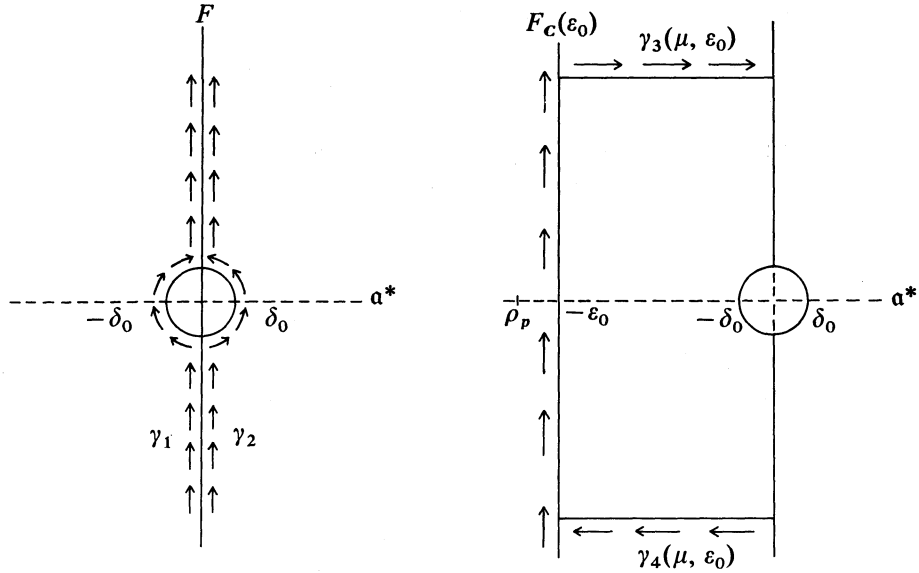

Figure. 1

Since the function

$$\nu \longrightarrow E(Q: \psi_{F_0(Q:\chi:\nu)}: \nu: 1: x: 1)\mu(\chi: \nu) = \operatorname{tr}\{F_0(Q: \chi: \nu)\pi^F_{Q,\chi,\nu}(x)\mu(\chi: \nu)\}$$

is analytic in the strip $F_C^{2\delta_0}$ for all $Q \in P(A)$, $\chi \in \hat{M}(F)$ and $x \in G$, by (6.1) we have

$$\phi_{F_0}(Q: \chi: x) = \int_{\gamma_1} E(Q: \psi_{F_0(Q:\chi:\nu)}: \nu: 1: x: 1)\mu(\chi: \nu)d\nu.$$

Hence, applying Theorem 3.1 and the above observation to the last formula we have for all $a \in A^+(Q)$,

$$\begin{aligned}\phi_{F_0}(Q: \chi: a) &= \int_{\gamma_1} \sum_{w \in W(A)} \Phi(w\nu: a)C_{Q|Q}(w: \nu)\psi_{F_0(Q:\chi:\nu)}(1)\mu(\chi: \nu)d\nu \\ &= \int_{\gamma_1} \Phi(\nu: a)C_{Q|Q}(1: \nu)\psi_{F_0(Q:\chi:\nu)}(1)\mu(\chi: \nu)d\nu \\ &\quad + \int_{\gamma_2} \Phi(\nu: a)C_{Q|Q}(w: w^{-1}\nu)\psi_{F_0(Q:\chi:w^{-1}\nu)}(1)\mu(\chi: w^{-1}\nu)d\nu.\end{aligned}$$

Recalling the equalities in Section 3, we have for $a \in A^+(Q)$,

$$\begin{aligned}&C(A)^{-2}\phi_{F_0}(Q: \chi: a) \\ &= \int_{\gamma_1} \Phi(\nu: a)C_{\bar{Q}|\bar{Q}}(1: \nu)^{-1}\psi_{F_0(Q:\chi:\nu)}(1)d\nu \\ &\quad + \int_{\gamma_2} \Phi(\nu: a)C_{\bar{Q}|Q}(1: \nu)^{-1}\circ w\psi_{F_0(Q:\chi:w^{-1}\nu)}(1)d\nu.\end{aligned}$$

LEMMA 6.2. *For all $u \in \mathfrak{A}$ and $a \in A^+(Q)$,*

$$\lim_{\mu \to \infty} \sum_{w \in W(A)} \sum_{j=3}^{4} \int_{\gamma_j(\mu, \varepsilon_0)} \Phi(v: a; u) C_{\bar{Q}|\bar{Q}^w}(1: v)^{-1} \circ w \psi_{F_0(Q: \chi: w^{-1}v)}(1) dv = 0.$$

PROOF. Fix $a \in A^+(Q)$ and let $\mu_0 > 0$. Since the polynomial $P_n(v)$ in Theorem 3.2 has zeroes only on the real axis, we can find constants $D > 0$ and $d > 0$ such that

$$\|\Gamma_n(v - \rho_Q)\| \leq D(1+n)^d$$

for all v ($|\mathrm{Im}\, v| > \mu_0$, $\mathrm{Re}\, v < \varepsilon$). In a similar manner, we have from (9) in Section 3 that for given $v \in S(F_C)$ and $M > 0$, there exists a constant $C_{v,M,\varepsilon} > 0$ and an integer $l > 0$ such that for all v ($|\mathrm{Im}\, v| > \mu_0$, $-M < \mathrm{Re}\, v < \varepsilon$) and $w \in W(A)$,

$$\|C_{\bar{Q}|\bar{Q}^w}(1: v; v)^{-1}\| \leq C_{v,M,\varepsilon}(1 + |v|)^l.$$

On the other hand, by Lemma 3.1 we have

$$\|\psi_{F_0(Q: \chi: v)}(1)\| \leq \dim(\mathscr{H}_{Q,\chi,F}) \|F_0(Q: \chi: v)\|.$$

Combining these estimates and the fact that $F_0 \in \mathscr{C}_H^p(\hat{G}: \boldsymbol{F})$, we have that for $u \in \mathfrak{A}$ there exists $l' \in \boldsymbol{Z}_+$ such that for μ ($\mu > \mu_0$)

$$\begin{aligned}
&\left\| \sum_{j=3}^{4} \int_{\gamma_j(\mu, \varepsilon_0)} \Phi(v: a; u) C_{\bar{Q}|\bar{Q}^w}(1: v)^{-1} \circ w \psi_{F_0(Q: \chi: w^{-1}v)}(1) dv \right\| \\
&\leq \text{const.} \sum_{j=3}^{4} D\left(\sum_{n=0}^{\infty} (1+n)^{d+l'} e^{-n\alpha(\log a)} \int_{\gamma_j(\mu, \varepsilon_0)} (1+|v|)^{-2} |dv|\right) \\
&\leq \text{const.} \int_0^1 (1 + \varepsilon_0^2 t^2 + \mu^2)^{-2} dt \leq \text{const.}\, (1+\mu^2)^{-2}.
\end{aligned}$$

This proves the assertion.

By Lemma 6.1 and Lemma 6.2 we have the following results.

COROLLARY. *For $a \in A^+(Q)$, we have*

$$\begin{aligned}
&\phi_{F_0}(Q: \chi: a) \\
&= C(A)^2 \sum_{w \in W(A)} \int_{F_C(\varepsilon_0)} \Phi(v: a) C_{\bar{Q}|\bar{Q}^w}(1: v)^{-1} \circ w \psi_{F_0(Q: \chi: w^{-1}v)}(1) dv,
\end{aligned} \tag{6.2}$$

where $F_C(\varepsilon_0) = \{v \in F_C: \mathrm{Re}\, v = -\varepsilon_0\}$. Further, all derivatives of $\phi_{F_0}(Q: \chi)$ by elements of $\mathfrak{A}$ can be computed differentiation under the integrals.

THEOREM 6.1. *Let notation be as above. If*

$$\phi_{F_0}(x) = D(G/A) \sum_{\chi \in \hat{M}(F)} d(\chi) \phi_{F_0}(Q: \chi: 1: x: 1),$$

then $\phi_{F_0} \in \mathscr{C}^p(G: \boldsymbol{F})$. *Moreover, the map* $F \to \phi_{F_0}$ *is continuous.*

PROOF. Since $F \to F_0$ is continuous map of $\mathscr{C}^p_H(\hat{G}: \boldsymbol{F})$ into itself, it suffices to show that for every $b \in \mathfrak{G}$, $r \in \boldsymbol{R}$ there exists $\mu \in S^p_H(\hat{G})$ such that

$$\sup_{x \in G} (1+\sigma(x))^r \Xi(x)^{-2/p} \|\phi_{F_0}(x;\, b)\| < \mu(F_0).$$

We first consider the above sup in the complement ${}^cG(1)$ of $G(1)$ in G. By [9(d), Lemma 17.1] we see that given $Q \in P(A)$, $a \in \mathfrak{G}$ there exist constants $C = C_a$ and $r = r_a$ such that

$$\|E(Q: \psi: \nu: x;\, a)\|_V \le C \|\psi\|_V (1+|\nu|)^r \Xi(x)(1+\sigma(x))^r$$

for $x \in G$, $\psi \in L(\chi)$ ($\chi \in \hat{M}(\boldsymbol{F})$), $\nu \in F$. And also by [9(c), Lemma 9.1] we have

$$\|\psi_{F_0(Q:\chi:\nu)}\|_V \le \dim(\chi)^{-1/2} \|F_0(Q: \chi: \nu)\| \quad (\nu \in F).$$

Since $\Xi(x)$ does not vanish on G and ${}^cG(1)$ is compact, we may choose $C > 0$ so that

$$C^{-1} \le \Xi(x)^{-1} \le C \quad (x \in {}^cG(1)),$$

combining these facts with (10) in Section 3. We see from this fact and the defining formula of ϕ_{F_0} that for any $r \in \boldsymbol{R}$ and $a \in \mathfrak{G}$ we can find $\mu' \in S^p_H(\hat{G})$ such that

$$\sup_{x \in {}^cG(1)} (1+\sigma(x))^r \Xi(x)^{-2/p} \|\phi_{F_0}(Q: \chi: \nu: x;\, a)\| \le C^{2/p+1} \mu'(F_0).$$

We next consider the sup on $G(1)$. Using the fact that $G = KCl(A^+(Q))K$ and the radial component formula for any $b \in \mathfrak{G}$ (cf. [14]), we see that it is sufficient to show that for every $u \in \mathfrak{A}$ and $r \in \boldsymbol{R}$ there exists $\mu \in S^p_H(\hat{G})$ such that

$$\sup (1+\sigma(a))^r e^{(2/p)\rho_Q(\log a)} \|\phi_{F_0}(a;\, u)\| < \mu(F_0),$$

the sup being taken over $A^+(Q) \cap A(1)$, where $A(s) = A \cap G(s)$ for s ($s > 0$). By the results in Section 3 we may write

$$\Gamma_n(\nu - \rho_Q) C_{\bar{Q}|\bar{Q}^w}(1: \nu)^{-1} \circ w = B_{n,w}(\nu)/(\nu - \rho_p)^{k_{n,w}}.$$

Here $0 \le k_{n,w} \le O_w(\rho_p)$ and $B_{n,w}(\nu)$ is holomorphic on $\{\nu \in F_{\boldsymbol{C}}: -\varepsilon_0' < \mathrm{Re}\, \nu \le -\varepsilon_0\}$, where $-\varepsilon_0' < \rho_p$; moreover for $u \in S(F_{\boldsymbol{C}})$ there exist constants $C = C_u > 0$, $d' > 0$ such that

$$\|B_{n,w}(\nu;\, u)\| \le C(1+n)^d (1+|\nu|)^{d'}$$

holds on the above domain. Take an element $\xi \in F_{\boldsymbol{R}}$ satisfying $\rho_p < \xi < -\varepsilon_0$.

Then from the above arguments it follows that the interchange of the summation and the integration in (6.2) is legitimate, and we have, for $a \in A^+(Q)$ and $v \in \mathfrak{A}$,

$$e^{(2/p)\rho_Q(\log a)}\phi_{F_0}(a;\, v) = D(G/A) \sum_{\chi \in \hat{M}(F)} d(\chi) \sum_{n=0}^{\infty} \sum_{w \in w(A)} e^{(\xi-\rho_P-n\alpha)(\log a)}$$

$$\times \int_F v(\xi + \nu - n\alpha - \rho_p) B_{n,w}(\nu) \psi_{F_{o,n,w}(Q:\chi:w^{-1}(\xi+\nu))}(1) e^{\nu(\log a)}\, d\nu;$$

here $F_{0,n,w}(Q\colon \chi\colon \nu) = (\nu - \rho_p)^{-k_{n,w}} F_0(Q\colon \chi\colon \nu)$ being rapidly decreasing in ν (cf. [12(c), Lemma 9.6]). On the other hand, we see (cf. [12(c), Lemma 9.7]) that if we put for ξ ($\xi \in F_{\boldsymbol{R}}$, $\rho_p < \xi < -\varepsilon_0$), $Q \in P(A)$ and $\chi \in \hat{M}(\boldsymbol{F})$,

$$G_\xi(Q\colon \chi\colon \nu\colon k_1\colon m\colon k_2) = \psi_{F_{o,n,w}(Q:\chi:\xi+\nu))}(k_1\colon m\colon k_2) \quad (k_1,\, k_2 \in K,\ m \in M),$$

then $G_\xi(Q\colon \chi) \in \mathscr{C}(F) \otimes C^\infty(M\colon V_{\boldsymbol{F}}\colon \tau)$ and if P is an arbitrary polynomial function on $F_{\boldsymbol{C}}$ and $u \in S(F_{\boldsymbol{C}})$ then there exists $\mu \in S_H^p(\hat{G})$ (possibly depending on P, u, n and w) such that

$$\sup \|P(\nu) G_\xi(Q\colon \chi\colon \nu;\, u)\|_V \le \mu(F_0)$$

for all above ξ, where the sup is taken over $Q \in P(A)$, $\chi \in \hat{M}$, and $\nu \in F$. Combining the usual arguments and these facts, we see that there exists $\mu'' \in S_H^p(\hat{G})$ such that

$$\|(1+\sigma(a))^r e^{(2/p)\rho_Q(\log a)} \phi_{F_0}(a;\, v)\| \le e^{(\xi-(1-2/p)\rho_Q)(\log a)} \mu''(F_0)$$

for $a \in A^+(Q) \cap A(1)$. Since this holds for all ξ ($\xi \in F_{\boldsymbol{R}}$, $\rho_p < \xi < -\varepsilon_0$) ξ can be replaced by ρ_p. This proves the theorem.

REMARK. (1) If we restrict ourselves to the special case that $\tau = (1, 1)$, then our proof gives a simple proof of [12(a)].

(2) In [6] Eguchi-Kowata studied the Fourier transform of the L^p Schwartz space $\mathscr{C}^p(G/K)$ on the symmetric space G/K when $\mathrm{rk}\,(G/K) = 1$, in the same way with ours (cf. also [10(a)]). But since the degree of the dependency of the constant in the Gangolli estimate for the Eisenstein integrals with respect to the K-types is higher than any polynomial order of the norm of K-types, we need to put the K-finite condition in the statement of the main theorem. For the proof of the theorem (general case), an argument like in [3(b)] is necessary.

Acknowledgement. The first author would like to state his gratitude to Professor T. Oshima for pointing out the mistake in [6] and both authors also would like to thank him for stimulating conversations.

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The Faculty of Integrated Arts and Sciences,
Hiroshima University

and

Department of Mathematics,
Fukuyama University