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On strongly exact sequences of cocommutative Hopf algebras

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An exact sequence

$$\text{(S)} \qquad k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of cocommutative Hopf algebras over a field k is said to be strongly exact if for any cocommutative coalgebra C the induced sequence

$$\text{(CS)} \quad e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J) \longrightarrow e$$

of groups is exact. In [6] we gave several equivalent conditions for (S) to be strongly exact in case H is irreducible (i.e., hyperalgebra).

Recently, Yanagihara has shown that when H is pointed, (S) is strongly exact if and only if the sequence

$$(\mathrm{S}^1) \qquad k \longrightarrow G^1 \longrightarrow H^1 \longrightarrow J^1 \longrightarrow k$$

of irreducible Hopf algebras extracted from (S) is strongly exact ([9], Theorem 2).

The main purpose of this paper is to generalize these results.

When H is irreducible, we showed in [6] that one of necessary and sufficient conditions for (S) to be strongly exact is that the Hopf subalgebra G has a coalgebra retraction in H, that is, there exists a coalgebra homomorphism η of H into G such that $\eta|_G = \mathrm{id}_G$. This is valid for cocommutative pointed Hopf algebras ([9], Theorem 2). But, generally, this is not sufficient. In fact, we show in Section 2 that we must demand G to have not only a coalgebra retraction but also a G-linear coalgebra retraction (Theorem 2.7 (3)).

When H is a cocommutative pointed Hopf algebra over k, the structure of H is completely determined by those of its irreducible component H^1 containing 1 and coradical H_0 ([7], §8.1). They are considered in Sections 3 and 4. In Section 3 we show that if a sequence (S) is strongly exact then so is the sequence (S^1) (Theorem 3.5). In Section 4 we prove that the coradical H_0 of a cocommutative Hopf algebra H over k is a Hopf subalgebra if and only if the dual algebra of H_0 is a direct product of separable extension fields of k (Theorem 4.7). In this case we show that if (S) is strongly exact then so is the sequence

$$(\mathrm{S}_0) \qquad k \longrightarrow G_0 \longrightarrow H_0 \longrightarrow J_0 \longrightarrow k$$

of the respective coradicals which are Hopf subalgebras.

In Section 5 we review Cartier-Gabriel's Decomposition Theorem ([3], Chapter II, §1, No. 4) in the context of the Hopf algebra action. We prove that a cocommutative Hopf algebra H such that H_0 is a Hopf subalgebra is isomorphic to the smash product $H^1\# H_0$ equipped with the tensor product coalgebra structure (Theorem 5.3). This is a generalization of a well-known theorem for cocommutative pointed Hopf algebras as stated above. With the aid of these theorems we obtain a generalization of Yanagihara's Theorem. We show that (S) is strongly exact if and only if both (S^1) and (S_0) are strongly exact (Theorem 5.5). The final section is devoted to construct examples.

§1. Preliminaries

Let k be a field. Let C, D be coalgebras over k and $f\colon C\to D$ a coalgebra homomorphism. For a group-like element g of C we put

$$\mathrm{K}_g(f) = \{c\in C\,|\,(f\otimes \mathrm{id}_C)\Delta_C(c)=f(g)\otimes c\}\,.$$

LEMMA 1.1 ([8], Proposition 1). *If C is cocommutative, then:*

(1) *$\mathrm{K}_g(f)$ is a subcoalgebra of C which contains g.*

(2) *$\mathrm{K}_g(f)$ is the largest subcoalgebra in those which are contained in $kg+\mathrm{Ker}(f)$.*

(3) *A subcoalgebra E of C is contained in $\mathrm{K}_g(f)$ if and only if $f(E)=kf(g)$, or equivalently, $f(x)=\varepsilon(x)f(g)$ for all x in E.*

Let H, J be cocommutative Hopf algebras over k. If $\rho\colon H\to J$ is a Hopf algebra homomorphism, then:

LEMMA 1.2 ([7], Lemma 16.1.1; [5], Corollary 3.4).

(1) *$\mathrm{K}_1(\rho)$ is a Hopf subalgebra of H.*

(2) *$\mathrm{Ker}(\rho)=K_1(\rho)^+H$, where $\mathrm{K}_1(\rho)^+=\mathrm{Ker}(\varepsilon)\cap \mathrm{K}_1(\rho)$.*

A sequence of cocommutative Hopf algebras and Hopf algebra homomorphisms

$$\cdots H_{i-1} \xrightarrow{\rho_i} H_i \xrightarrow{\rho_{i+1}} H_{i+1}\cdots$$

is said to be exact if $\mathrm{Im}(\rho_i)=\mathrm{K}_1(\rho_{i+1})$ for all i.

REMARK 1.3. k is itself a one dimensional Hopf algebra. For any Hopf algebra H the unit map $u_H\colon a\mapsto a1_H$ is the unique homomorphism of k into H, and the counit ε is the unique homomorphism of H into k. So we shall omit to state these maps explicitly in any sequence or diagram of Hopf algebras.

It is easy to see the following:

LEMMA 1.4. *A sequence*

$$k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

of cocommutative Hopf algebras is exact if and only if j is injective, $\mathrm{Im}(j) = \mathrm{K}_1(\rho)$, *and* ρ *is surjective.*

Let H be a Hopf algebra over k. Then for any cocommutative coalgebra C over k the set $\mathrm{Hom}_{coal}(C, H)$ of all coalgebra homomorphisms of C into H is a group under the convolution product, i.e.,

$$f * g = \mu_H(f \otimes g)\Delta_C$$

for $f,\ g \in \mathrm{Hom}_{coal}(C, H)$, where μ_H (resp. Δ_C) denotes the multiplication of H (resp. the comultiplication of C).

If $\rho: H \to J$ is a Hopf algebra homomorphism, then ρ induces a group homomorphism of $\mathrm{Hom}_{coal}(C, H)$ into $\mathrm{Hom}_{coal}(C, J)$. The following result is proved in [8].

LEMMA 1.5. *A sequence*

$$k \longrightarrow G \longrightarrow H \longrightarrow J$$

of cocommutative Hopf algebras over k is exact if and only if the induced sequence

$$e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J)$$

of groups is exact for any cocommutative coalgebra C over k.

For short exact sequences we have:

PROPOSITION 1.6. *Let a sequence*

$$\text{(S)} \qquad k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of cocommutative Hopf algebras be given. If for any cocommutative coalgebra C the induced sequence

$$\text{(CS)} \quad e \longrightarrow \mathrm{Hom}_{coal} \longrightarrow (C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \xrightarrow{\rho *} \mathrm{Hom}_{coal}(C, J) \longrightarrow e$$

of groups is exact, then (S) *is exact.*

PROOF. By Lemma 1.5 it suffices to show the surjectivity of ρ. Take J as C. Then the surjectivity of ρ_* implies that there is a coalgebra homomorphism $\lambda: J \to H$ such that $\rho \circ \lambda = \mathrm{id}_J$. This shows that ρ is surjective.

Conversely, if there is a coalgebra homomorphism λ of J into H such that $\rho\circ\lambda=\mathrm{id}_J$, then it is easy to see that ρ_* is surjective for every C. Therefore we have the following corollary which plays an important role in this paper.

COROLLARY 1.7. *The induced sequence* (CS) *is exact for every C if and only if the sequence* (S) *is exact and there is a coalgebra homomorphism λ of J into H such that $\rho\circ\lambda=\mathrm{id}_J$.*

DEFINITION 1.8 ([6]; [9]). An exact sequence

$$\text{(S)} \qquad k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of cocommutative Hopf algebras over k is said to be *strongly exact* if the induced sequence

$$\text{(CS)} \qquad e \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J) \longrightarrow e$$

of groups is exact for any cocommutative coalgebra C over k.

If C is a cocommutative coalgebra over k, then C is decomposed into a direct sum of irreducible subcoalgebras ([7], Theorem 8.0.5). It follows that:

PROPOSITION 1.9. *An exact sequence* (S) *is strongly exact if and only if the sequence* (CS) *is exact for every irreducible cocommutative coalgebra.*

Let ρ: $H\to J$ be a Hopf algebra homomorphism. We say that ρ has a *coalgebra splitting* if there exists a coalgebra homomorphism λ of J into H such that $\rho\circ\lambda=\mathrm{id}_J$. Let G be a Hopf subalgebra of H. We say that G has a *coalgebra retraction* if there exists a coalgebra homomorphism η of H into G such that $\eta\mid_G=\mathrm{id}_G$.

In [2] Blattner, Cohen, and Montgomery have shown that when a surjective homomorphism ρ: $H\to J$ has a coalgebra splitting, then H is isomorphic to a crossed product of $\mathrm{K}_1(\rho)$ and J as an algebra, and further, if H is cocommutative, then H is isomorphic as a coalgebra to the tensor product $\mathrm{K}_1(\rho)\otimes J$.

In [6] we gave several conditions for an exact sequence (S) of irreducible cocommutative Hopf algebras (i.e., hyperalgebras) to be strongly exact, namely,

LEMMA 1.10 ([6], Theorems 1.3, 1.8)). *Let*

$$k \longrightarrow G \longrightarrow H \xrightarrow{\ \rho\ } J \longrightarrow k$$

be an exact sequence of irreducible cocommutative Hopf algebras over k. Then the following coditions are equivalent:

(1) *The sequence is strongly exact.*

(2) *ρ has a coalgebra splitting.*

(3) *G has a coalgebra retraction.*

(4) *There exists a coalgebra isomorphism* θ: $H \to G \otimes J$ *such that* $(\varepsilon \otimes \mathrm{id}_J)\theta = \rho$.

Let H be a cocommutative Hopf algebra over k. We denote by H^1 the irreducible component of H that contains 1. H^1 is a Hopf subalgebra of H. The following result which is proved by Yanagihara states that the strong exactness is a local property in some sense.

LEMMA 1.11 ([9], Theorem 2). *When H is a pointed cocommutative Hopf algebra, then an exact sequence*

$$k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

is strongly exact if and only if the sequence

$$k \longrightarrow G^1 \longrightarrow H^1 \longrightarrow J^1 \longrightarrow k$$

is strongly exact.

§ 2. Strongly exact sequences

In this section we generalize Lemma 1.10 for not necessarily irreducible Hopf algebras.

Let

(S) $$k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

be an exact sequence of cocommutative Hopf algebras over a field k. By Lemma 1.4 we may assume that G is a Hopf subalgebra of H, so that H can be regarded as a left G-module by multiplication.

THEOREM 2.1. *The following conditions are equivalent*:

(1) *The sequence* (S) *is strongly exact.*

(2) ρ *has a coalgebra splitting.*

(3) G *has a coalgebra retraction which is left G-linear.*

(4) *There exists a coalgebra isomorphism* θ: $H \to G \otimes J$ *such that* $(\varepsilon \otimes \mathrm{id}_J)\theta = \rho$.

Proof of (1)$\Leftrightarrow$(2). This is just Corollary 1.7.

To prove (2)$\Rightarrow$(3) we need a lemma.

LEMMA 2.2. *If ρ has a coalgebra splitting, then ρ has a coalgebra splitting λ such that* $\lambda(1_J) = 1_H$.

PROOF. Let λ' be a coalgebra splitting of ρ. Then $g = \lambda'(1)$ is a group-like element of H and $\rho(g)=1_J$. Since H is a Hopf algebra, g^{-1} exists, and $\rho(g^{-1})=1_J$. Let $R_g: H\to H$ be defined by $R_g(x)=xg^{-1}$. Then R_g is a coalgebra automorphism of H. Put $\lambda=R_g\circ\lambda'$. It is easy to see that λ is a coalgebra splitting of ρ that satisfies $\lambda(1_J)=1_H$.

Proof of (2)⇒(3). By the above lemma we may assume that ρ has a coalgebra splitting λ such that $\lambda(1)=1$. Let $\eta=\mathrm{id}_H*S\lambda\rho$, where S is the antipode of H. Since H is cocommutative η is a coalgebra homomorphism of H into itself. Similar calculations as in the proof of Theorem 1.3 in [6] or of Theorem 4.14 in [2] (where η is denoted by Q) show that Im $(\eta)\subseteq G$ and η is a coalgebra retraction of G.

In order to prove that η is left G-linear, take $x\in G$ and $h\in H$. Then

$$\begin{aligned}
\eta(xh) &= \Sigma x_{(1)}h_{(1)}S\lambda\rho(x_{(2)}h_{(2)}) \\
&= \Sigma x_{(1)}h_{(1)}S\lambda(\rho(x_{(2)})\rho(h_{(2)})) \\
&\quad (\rho \text{ is an algebra homomorphism}) \\
&= \Sigma x_{(1)}h_{(1)}S\lambda(\varepsilon(x_{(2)})\rho(h_{(2)})) \\
&\quad (\text{by Lemma 1.1 (3)}) \\
&= \Sigma x h_{(1)}S\lambda\rho(h_{(2)}) \\
&= x\eta(h).
\end{aligned}$$

This shows that η is left G-linear.

Proof of (3)⇒(4). Let η be a coalgebra retraction of G which is left G-linear. Then $\bar{\eta}=(\eta\otimes\mathrm{id}_H)\Delta_H: H\to G\otimes H$ defines a left G-comodule structure on H. The left G-linearity of η asserts that H is a left G-Hopf module, i.e., the following diagram is commutative:

$$\begin{array}{ccccc}
G\otimes H & \xrightarrow{\;\cdot\;} & H & \xrightarrow{\;\bar{\eta}\;} & G\otimes H \\
{\scriptstyle \Delta\otimes\bar{\eta}}\big\downarrow & & & & \big\uparrow{\scriptstyle \cdot\otimes\cdot} \\
G\otimes G\otimes G\otimes H & & \xrightarrow{\;\mathrm{id}_G\otimes T\otimes\mathrm{id}_H\;} & & G\otimes G\otimes G\otimes H.
\end{array}$$

Therefore by Theorem 4.1.1 and its proof in [7] there is a Hopf module isomorphism ϕ between $G\otimes H'$ and H sending $x\otimes h$ into xh, where

$$H' = \{h\in H \mid \bar{\eta}(h)=1\otimes h\}.$$

By the definition of η we have $H'=K_1(\eta)$, which is a subcoalgebra of H by Lemma 1.1. Since the multiplication of a Hopf algebra is a coalgebra homomorphism, ϕ is a coalgebra isomorphism. As a coalgebra, H' is isomorphic to J by ρ. In fact, we have

$$J = \rho(H) = \rho(GH') = \rho(G)\rho(H') = \rho(H'),$$

which means that the restriction $\rho|_{H'}$ is surjective. The injectivity of $\rho|_{H'}$ is shown as follows. Since $G=k1\oplus G^+$ and since ϕ is an isomorphism, $H=H'\oplus G^+H$. On the other hand, by Lemma 1.2 (2), $\mathrm{Ker}(\rho)=G^+H=G^+GH'=G^+H'$. Hence we have $\mathrm{Ker}(\rho)\cap H'=0$, which means that $\rho|_{H'}$ is injective.

Put $\theta=(\mathrm{id}_G\otimes\rho|_{H'})\phi^{-1}$. Then θ is a coalgebra isomorphism of H onto $G\otimes J$ and satisfies the equation

$$(\varepsilon_G \otimes \mathrm{id}_J)\theta = \rho.$$

This proves (4).

Proof of (4)⇒(2). Define $\lambda: J\to H$ by $\lambda(y)=\theta^{-1}(1\otimes y)$. Clearly λ is a coalgebra splitting of ρ. Therefore Theorem is proved.

REMARK 2.3. Note that condition (3) of the theorem is different from condition (3) of Lemma 1.10. We know that when H is an irreducible cocommutative Hopf algebra and G is a Hopf subalgebra of H, then if G has a coalgebra retraction, G has also a coalgebra retraction which is left G-linear ([6], Corollary 1.4). In Section 5 we shall show by giving an example that it does not hold for non-irreducible Hopf algebras.

§ 3. Irreducible components containing 1

Let C be a cocommutative coalgebra over a field k. For a group-like element g of C we denote by C^g the irreducible component of C containing g. If H is a cocommutative Hopf algebra over k, then H^1 is a Hopf subalgebra of H.

Let

$$\text{(S)} \qquad k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

be a sequence of cocommutative Hopf algebras over k. By Theorem 8.0.8 in [7] we have $j(G^1)\subseteq H^1$ and $\rho(H^1)\subseteq J^1$. So we have a sequence

$$(\mathrm{S}^1) \qquad k \longrightarrow G^1 \xrightarrow{j^1} H^1 \xrightarrow{\rho^1} J^1 \longrightarrow k$$

of irreducible Hopf algebras, where j^1, ρ^1 denote the restrictions of j, ρ to G^1, H respectively.

We prove that if (S) is exact (resp. strongly exact), then so is (S^1). It is eas to see that if (S) is exact then j^1 is injective and $\mathrm{Im}(j^1)=\mathrm{K}_1(\rho^1)$. Thus th problem is the surjevtivity of ρ^1. In the following lemmas K denotes an extensio field of k.

LEMMA 3.1. *Let A be a finite-dimensional local algebra over k with the maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m}\cong k$. Then $A\otimes K$ is a local K-algebra with the maximal ideal $\mathfrak{m}\otimes K$.*

PROOF. $\mathfrak{m}\otimes K$ is a maximal ideal of $A\otimes K$ since

$$(A\otimes K)/(\mathfrak{m}\otimes K) \cong (A/\mathfrak{m}) \otimes K \cong k \otimes K \cong K.$$

In order to prove that $\mathfrak{m}\otimes K$ is the unique maximal ideal, it is enough to prove that any element of $A\otimes K$ not belonging to $\mathfrak{m}\otimes K$ is a unit.

By the assumption $\mathfrak{m}$ is a nilpotent ideal, so that $\mathfrak{m}\otimes K$ is also nilpotent. Since $A=k1\oplus\mathfrak{m}$ as a k-space, we have $A\otimes K=K1\oplus(\mathfrak{m}\otimes K)$ as a K-space. Let $x\in A\otimes K$ but $x\notin \mathfrak{m}\otimes K$. Then $x=\alpha 1+z$, where $\alpha\neq 0$ in K and $z\in\mathfrak{m}\otimes K$. Since z is nilpotent, it follows that x is a unit.

LEMMA 3.2. *If C is a pointed irreducible cocommutative coalgebra over k, then $C\otimes K$ is also pointed irreducible as a K-coalgebra.*

PROOF. C is a direct union of finite-dimensional subcoalgebras:

$$C = \bigcup_i C_i, \quad C_1 \subseteq C_2 \subseteq \cdots$$

So we have $C\otimes K=\cup_i (C_i\otimes K)$ and $C_1\otimes K\subseteq C_2\otimes K\subseteq\cdots$. Since every C_i is pointed, irreducible, and finite-dimensional over k, by Lemma 3.1 each $C_i\otimes K$ is pointed irreducible. Therefore $C\otimes K$ is pointed irreducible.

LEMMA 3.3. *Let C be a cocommutative coalgebra over k. If g is a group-like element of C, then $(C\otimes K)^{1\otimes 1}=C^g\otimes K$.*

PROOF. Note that $g\otimes 1$ is a group-like element of $C\otimes K$. By Theorem 8.0.5 in [7] we have $C=C^g\oplus D$, where D is the sum of all irreducible components of C different from C^g. Hence as a K-coalgebra we have

$$C \otimes K = (C^g\otimes K) \oplus (D\otimes K).$$

By Lemma 3.2 $C^g\otimes K$ is irreducible. This shows that $C^g\otimes K$ is an irreducible component of $C\otimes K$ that contains $g\otimes 1$, so that $C^g\otimes K=(C\otimes K)^{g\otimes 1}$.

PROPOSITION 3.4. *If ρ in* (S) *is surjective, then $\rho(H^1)=J^1$, i.e., ρ^1 is surjective.*

PROOF. Let K be an algebracially closed extension field of k. Then $H\otimes K$ is a pointed Hopf algebra over K, and $\rho\otimes K$ is surjective. Thus by Theorem 1 in [10] we have $(\rho\otimes K)((H\otimes K)^{1\otimes 1})=(J\otimes K)^{1\otimes 1}$. By Lemma 3.2 we have $(H\otimes K)^{1\otimes 1}=H^1\otimes K$ and $(J\otimes K)^{1\otimes 1}=J^1\otimes K$. Clearly $\rho^1\otimes K=(\rho\otimes K)|_{H^1\otimes K}$. It follows that ρ^1 is surjective.

If ρ in (S) has a coalgebra splitting λ, then $\lambda(J^1)\subseteq H^1$ and the restriction $\lambda|_{J^1}$ is a coalgebra splitting of ρ^1. Therefore we have shown the following:

THEOREM 3.5. *If a sequence*

$$k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of cocommutative Hopf algebras over k is exact (resp. strongly exact), then the extracted sequence

$$k \longrightarrow G^1 \longrightarrow H^1 \longrightarrow J^1 \longrightarrow k$$

of irreducible Hopf algebras is exact (resp. strongly exact).

REMARK 3.6. If H is pointed, then the converse of the theorem holds ([9], Theorem 2). But generally it is not valid as shown by an example later.

§ 4. Coradicals of cocommutative Hopf algebras

Let C be a cocommutative coalgebra over a field k. We denote by C_0 the coradical of C. If f: $C\to D$ is a homomorphism of coalgebras, then $f(C_0)\subseteq D_0$. If f is injective (resp. surjective) then we have $f(C_0)=f(C)\cap D_0$ (resp. $f(C_0)=D_0$) ([7], Lemma 9.0.1 and Theorem 8.0.8). We denote by f_0 the restriction of f to C_0.

PROPOSITION 4.1. (1) *If a sequence*

$$\text{(S)} \qquad k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

of cocommutative Hopf algebras is exact, then the sequence

$$k \longrightarrow G_0 \xrightarrow{j_0} H_0 \xrightarrow{\rho_0} J_0 \longrightarrow k$$

of coalgebras is exact, that is, j_0 is injective, ρ_0 is surjective, and $\mathrm{K}_1(\rho_0)=\mathrm{Im}\,(j_0)$.

(2) *If* (S) *is strongly exact, then ρ_0 has a splitting and H_0 is isomorphic to $(G_0\otimes J_0)_0$.*

PROOF (1) The injectivity of j_0 and the surjectivity of ρ_0 follow from the above remarks. Moreover, by definition, we have

$$\mathrm{K}_1(\rho_0) = H_0 \cap \mathrm{K}_1(\rho) = H_0 \cap G = G_0 .$$

(2) The first assertion is obvious. Since H is isomorphic to $G\otimes J$, the second assertion follows from Theorem 2.1 and the next lemma.

LEMMA 4.2. *If C, D are cocommutative coalgebras over k, then we have*

$$(C\otimes D)_0 = (C_0\otimes D_0)_0 .$$

PROOF. We may assume that both C and D are of finite dimension. Let A and B be the dual algebras of C and D respectively. Then $A\otimes B$ is the dual algebra of $C\otimes D$. Let $\mathfrak{R}(A)$, $\mathfrak{R}(B)$, $\mathfrak{R}(A\otimes B)$ be the Jacobson radicals of A, B, $A\otimes B$ respectively. Then it is easy to see that

$$\mathfrak{R}(A\otimes B) \supseteq \mathfrak{R}(A) \otimes B + A \otimes \mathfrak{R}(B).$$

It follows that $(C\otimes D)_0 \subseteq C_0\otimes D_0$. This implies that $(C\otimes D)_0 \subseteq (C_0\otimes D_0)_0$.

Conversely, it is clear that $(C_0\otimes D_0)_0 \subseteq (C\otimes D)_0$ because $C_0\otimes D_0$ is a subcoalgebra of $C\otimes D$.

REMARK 4.3. (1) G_0 has always a retraction in H_0 if j is injective. Indeed, then G_0 can be regarded as a subcoalgebra of a cosemisimple coalgebra H_0, so that there is a subcoalgebra H'_0 of H_0 such that $H_0 = G_0 \oplus H'_0$. If we put

$$\eta_0(x) = \begin{cases} x & \text{if} \quad x\in G_0 \\ \varepsilon(x)1 & \text{if} \quad x\in H'_0, \end{cases}$$

then it is clear that η_0 is a retraction of G_0 in H_0.

(2) $G_0\otimes J_0$ is not necessarily cosemisimple. But we see in the following that it is cosemisimple in case H_0 is a Hopf subalgebra of H.

Let C be a cocommutative cosemisimple coalgebra over k. Then the dual algerba C^* is a direct product of extension fields of k. C is called *separable* if C^* is a direct product of separable extension fields of k (cf. [1], §3.4). The following lemmas are consequences of the field theory (see, e.g., [4], Cap. IV, §10).

LEMMA 4.4. *Let C be a cocommutative cosemisimple coalgebra over k. Then the following conditions are equivalent*:

(1) *C is separable.*

(2) *For any extension field K of k, $C\otimes K$ is a cosemisimple K-coalgebra.*

(3) *If K is an algebraically closed extension field of k, then $C\otimes K$ is a direct sum of one dimensional subcoalgebras.*

PROOF. Since C is a direct sum of simple subcoalgebras, it is enough to prove the lemma in case C is simple.

(1)⇒(2). Assume that C is the dual coalgebra of a finite separable extension field L of k. Then $C\otimes K$ is the dual coalgebra of a K-algebra $L\otimes K$. Since L is separable over k, $L\otimes K$ has no nilpotent elements (*loc. cit.* Theorem 21, p. 197). It follows that $L\otimes K$ is a direct product of extension fileds of K, which means that $C\otimes K$ is a cosemisimple coalgebra.

(2)⇒(3) is trivial since $C\otimes K$ is a cocommutative pointed coalgebra.

(3)⇒(1). Assume that $L=C^*$ is not separable over k. Then there exists an algebraic extension field E of k such that $L\otimes E$ has a nonzero nilpotent element. Let K be an algebraically closed extension field of E. Then we have $L\otimes_k E\subseteq L\otimes_k K$. This contradicts the assumption.

LEMMA 4.5. *Let C be a cocommutative coalgebra over k. Let K be an extension field of k. Then we have, as K-coalgebras, $(C\otimes K)_0\subseteq C_0\otimes K$. The equality holds if C_0 is separable.*

PROOF. We may assume that C is of finite dimension. Since the Jacobson radical $\mathfrak{R}(C^*)$ of C^* is nilpotent, we have $\mathfrak{R}(C^*\otimes K)\supseteq\mathfrak{R}(C^*)\otimes K$. It follows that

$$(C\otimes K)_0 \subseteq C_0 \otimes K.$$

Next assume that C_0 is separable. Then $C_0\otimes K$ is a cosemisimple subcoalgebra of $C\otimes K$ by Lemma 4.4 (2). It follows that $C_0\otimes K\subseteq(C\otimes K)_0$. This proves the lemma.

LEMMA 4.6. *Let, C, D be cocommutative cosemisimple coalgebras over k. If either C or D is separable, then $C\otimes D$ is cosemisimple.*

PROOF. We may assume that both C and D are of finite dimension. Then $(C\otimes D)^*=C^*\otimes D^*$ has no nonzero nilpotent elements. It follows that $(C\otimes D)^*$ is a direct product of fields, which proves the lemma.

THEOREM 4.7. *Let H be a cocommutative Hopf algebra over k. Then H_0 is separable if and only if H_0 is a Hopf subalgebra of H.*

PROOF. Assume that H_0 is separable. In order to prove that H_0 is a Hopf subalgebra, it is sufficient to prove that H_0 is closed under the multiplication M of H.

Since H_0 is separable, $H_0\otimes H_0$ is cosemisimple and $H_0\otimes H_0=(H\otimes H)_0$ by Lemmas 4.6 and 4.2, respectively, so we have $M(H_0\otimes H_0)\subseteq H_0$, because M is a coalgebra homomorphism. Consequently, H_0 is a Hopf subalgebra of H.

Conversely, assume that H_0 is a Hopf subalgebra of H. Let K be an algebraically closed extension field of k. Then $H_0\otimes K$ is a cocommutative pointed Hopf algebra over K, so its irreducible components are isomorphic to each other as coalgebras over K. Since $H_0^1=k1$ we have $(H_0\otimes K)^{1\otimes 1}=H_0^1\otimes K=K1$ by Lemma 3.3. It follows that every irreducible component of $H_0\otimes K$ is one dimensional. Hence, by Lemma 4.4 (3), H_0 is separable. This proves the theorem.

Now consider an exact sequence

$$\text{(S)} \qquad k \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow k$$

of cocommutative Hopf algebras. If H_0 is separable, then both G_0 and J_0 are separable since the dual algebra G_0^* of G_0 (resp. J_0^* of J_0) is a homomorphic image (resp. isomorphic to a subalgebra) of H_0^*. Thus Proposition 4.1 becomes as follows:

COROLLARY 4.8. *Assume that* H_0 *is separable. If* (S) *is exact* (*resp. strongly exact*), *then the sequence*

$$(\mathrm{S}_0) \qquad k \longrightarrow G_0 \longrightarrow H_0 \longrightarrow J_0 \longrightarrow k$$

of coalgebras is an exact (*resp. a strongly exact*) *sequence of Hopf algebras.*

§ 5. Decomposition Theorem

In the previous section we showed that the coradical H_0 of a cocommutative Hopf algebra H is a Hopf subalgebra if and only if H_0 is separable. We prove that in this case H is a semidirect product of H^1 and H_0. This is known as Cartier-Gabriel's Decomposition Theorem in the theory of formal groups ([3], Chap. II, §1, No. 4). When H is pointed then H_0 is the group algebra of the group of all group-like elements of H, and it is well-known that H is isomorphic to the smash product $H^1 \# H_0$ ([7], §8.1). The notion of the smash product is refered to [7], Chapter VII.

Let H be a cocommutative Hopf algebra over a field k. Since the multiplication of H is a coalgebra homomorphism, the following map

$$\mu\colon H^1 \otimes H_0 \longrightarrow H, \quad h \otimes x \longrightarrow hx$$

is a coalgebra homomorphism. In case H is pointed, as stated above, μ is an isomorphism. In any case we have:

LEMMA 5.1. *μ is surjective.*

PROOF. Let K be an algebraically closed extension field of k. Then $H \otimes K$ is pointed Hopf algebra over K. Thus by Lemmas 3.3 and 4.5 we have

$$\begin{aligned}(\mu\otimes K)(H^1\otimes H_0\otimes K) &= (\mu\otimes K)((H^1\otimes K)\otimes_K (H_0\otimes K)) \\ &\supseteq (\mu\otimes K)((H\otimes K)^{1\otimes 1}\otimes_K (H\otimes K)_0) \\ &= H\otimes K.\end{aligned}$$

It follows that μ is surjective.

If H_0 is separable, then $(H\otimes K)_0 = H_0 \otimes K$ by Lemma 4.5. Therefore a similar argument shows the following:

PROPOSITION 5.2. *If H_0 is separable, then μ is a coalgebra isomorphism.*

If H_0 is separable, then H_0 is a Hopf subalgebra by Lemma 4.7. In this case we can consider the adjoint action of H_0 on H:

$$\alpha\colon H_0 \otimes H \longrightarrow H, \quad x \otimes h \longmapsto \Sigma x_{(1)} h S(x_{(2)}).$$

Under this action H is an H_0-module algebra. Note that if H is pointed α is the same action as that given in Theorem 8.1.5 in [7], so that in this case we have $\alpha(H_0 \otimes H^1) \subseteq H^1$. We show that in any case H^1 is stable under α.

Indeed, let K be an algebraically closed extension field of k. Then $(H\otimes K)^{1\otimes 1} = H^1 \otimes K$ and $(H\otimes K)_0 = H_0 \otimes K$ since H_0 is separable. So we have

$$\begin{aligned}(\alpha\otimes K)(H_0\otimes H^1 \otimes K) &= (\alpha\otimes K)((H_0\otimes K)\otimes_K (H^1\otimes K))\\ &= (\alpha\otimes K)((H\otimes K)_0 \otimes_K (H\otimes K)^{1\otimes 1})\\ &\subseteq (H\otimes K)^{1\otimes 1}\\ &= H^1 \otimes K,\end{aligned}$$

since, as easily seen, $\alpha\otimes K$ is the adjoint action of $(H\otimes K)_0 = H_0 \otimes K$. It follows that $\alpha(H_0 \otimes H^1) \subseteq H^1$.

Therefore H^1 is an H_0-module algebra under α. An easy calculation shows that μ is an algebra homomorphism of the smash product $H^1 \# H_0$ into H, where the multiplication of $H^1 \# H_0$ is given by

$$(h\#x)\cdot(l\#y) = \textstyle\sum_{(x)} h x_{(1)} l S(x_{(2)}) \# x_{(3)} y$$

for $h,\ l \in H^1,\ x,\ y \in H_0$.

$H^1 \# H_0$ becomes a Hopf algebra if we equip it with the tensor product coalgebra structure, and then μ is a Hopf algebra isomorphism. Thus we proved the Hopf algebraic version of Cartier-Gabriel's Decomposition Theorem:

THEOREM 5.3. *Let H be a cocommutative Hopf algebra over k and assume that the coradical H_0 is separable. Then*:

(1) *H_0 is a Hopf subalgebra of H.*

(2) *H^1 is stable under the adjoint action of H_0 on H:*
$\alpha(x\otimes h) = \sum_{(x)} x_{(1)} h S(x_{(2)}) \in H^1$ for $x \in H_0,\quad h \in H^1$.

(3) *The smash product $H^1 \# H_0$ has a Hopf algebra structure.*

(4) *$\mu\colon H^1 \# H_0 \to H$, $h\otimes x \mapsto hx$ is a Hopf algebra isomorphism.*

COROLLARY 5.4. *Let H be as in the theorem. Then we have a strongly exact sequence*

$$k \longrightarrow H^1 \longrightarrow H \xrightarrow{\ \rho\ } H_0 \longrightarrow k,$$

where ρ is given by $\rho(hx) = \varepsilon(h)x$ for $h \in H^1$, $x \in H_0$.

PROOF. Define j: $H^1 \to H^1 \# H_0$ and ρ': $H^1 \# H_0 \to H_0$ as follows:

$$j(h) = h\#1, \quad \rho'(h\#x) = \varepsilon(h)x,$$

for $h \in H^1$, $x \in H_0$. Then we have an exact sequence

$$k \longrightarrow H^1 \xrightarrow{j} H^1 \# H_0 \xrightarrow{\rho'} H_0 \longrightarrow k$$

of cocommutative Hopf algebras. This is strongly exact since ρ' has a coalgebra splitting λ: $H_0 \to H_0{}^1 \# H$, $x \mapsto 1\#x$.

We are in a position to give a generalization of Theroem 2 in [9]. Let H be a cocommutative Hopf algebra such that H_0 is separable, or equivalently, H_0 is a Hopf subalgebra. Let

$$\text{(S)} \qquad k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

be a sequence of Hopf algebras. Then we have the following two sequences:

$$(\mathrm{S}^1) \qquad k \longrightarrow G^1 \xrightarrow{j^1} H^1 \xrightarrow{\rho^1} J^1 \longrightarrow k$$

$$(\mathrm{S}_0) \qquad k \longrightarrow G_0 \xrightarrow{j_0} H_0 \xrightarrow{\rho_0} J_0 \longrightarrow k$$

of Hopf subalgebras extracted from (S).

If (S) is exact (resp. strongly exact), then both (S^1) and (S_0) are exact (resp. strongly exact) by Theorem 3.5 and Corollary 4.8 respectively.

Conversely, assume that both (S^1) and (S_0) are exact.

Clearly the following diagram is commutative:

$$\begin{array}{ccccccccc} k & \longrightarrow & G & \xrightarrow{j} & H & \xrightarrow{\rho} & J & \longrightarrow & k \\ \| & & \uparrow{\scriptstyle \mu_G} & & \uparrow{\scriptstyle \mu_H} & & \uparrow{\scriptstyle \mu_J} & & \| \\ k & \longrightarrow & G^1\#G_0 & \xrightarrow{j^1\#j_0} & H^1\#H_0 & \xrightarrow{\rho^1\#\rho_0} & J^1\#J_0 & \longrightarrow & k, \end{array}$$

where μ_G, μ_H, and μ_J are Hopf algebra isomorphisms as in Theorem 5.3 (4). It is easy to see that $\mathrm{K}_{1\otimes 1}(\rho^1 \otimes \rho_0) = G^1 \otimes G_0$. It follows that $\mathrm{K}_1(\rho) = G$, so that (S) is exact.

Further, assume that both (S^1) and (S_0) are strongly exact. Then both ρ^1 and ρ_0 have coalgebra splitting, so that ρ has a coalgebra splitting. This shows that (S) is strongly exact. Therefore we have:

THEOREM 5.5. *Assume that H_0 is separable. Then* (S) *is exact* (*resp. strongly exact*) *if and only if both* (S^1) *and* (S_0) *are exact* (*resp. strongly exact*).

REMARK 5.6. When H is pointed and when (S) is exact, then (S_0) is always strongly exact as stated in the proof of Theorem 2 in [9].

REMARK 5.7. In the next section, we shall construct an exact sequence of cosemisimple Hopf algebras which is not strongly exact.

§ 6. Examples

In this section we give examples which show that Corollary 1.4 in [6] is not true for non-irreducible Hopf algebras (Remark 2.3), the converse of Theorem 3.7 does not hold generally (Remark 3.8), and there exists an exact sequence of cosemisimple Hopf algebras which is not strongly exact (Remark 5.7).

First, we briefly construct a certain kind of Hopf algebra generated by a coalgebra.

Let C be a coalgebra over a filed k. The tensor algebra $\mathrm{T}C$ on C has a bialgebra structure ([7], Proposition 3.2.4). If C is cocommutative, then $\mathrm{T}C$ is a cocommutative bialgebra.

Let C be cocommutative. Let S' be a linear transformation of $\mathrm{T}C$ defined by

$$S'(c_1c_2\cdots c_n) = c_nc_{n-1}\cdots c_1,$$

for $c_1,\dots, c_n$ in C. Let I be an ideal of $\mathrm{T}C$ that is generated by the set

$$\{\textstyle\sum_{(c)} c_{(1)}c_{(2)} - \varepsilon(c)1 \mid c \in C\}.$$

Then it is easy to see that I is an S'-stable biideal of $\mathrm{T}C$, i.e., we have

$$\Delta_{\mathrm{T}C}(I) \subseteq \mathrm{T}C \otimes I + I \otimes \mathrm{T}C, \quad \varepsilon_{\mathrm{T}C}(I) = 0, \quad \text{and} \quad S'(I) \subseteq I$$

and that $I \cap C = 0$. Therefore $\mathrm{H}(C) = \mathrm{T}C/I$ is a cocommutative Hopf algebra with the antipode S induced by S'. By the natural injection $\iota: C \to \mathrm{H}(C)$, C is regarded as a subcoalgebra of $\mathrm{H}(C)$.

Let $f: C \to D$ be a homomorphism of cocommutative coalgebras. Then we have a unique Hopf algebra homomorphism $\mathrm{H}(f): \mathrm{H}(C) \to \mathrm{H}(D)$ which makes the following diagram commutative:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \iota_C}\downarrow & & \downarrow{\scriptstyle \iota_D} \\ \mathrm{H}(C) & \xrightarrow[\mathrm{H}(f)]{} & \mathrm{H}(D). \end{array}$$

If f is surjective, then $\mathrm{H}(f)$ is also surjective since $\mathrm{H}(D)$ is generated by D as an algebra over k.

Let $D = kg$, a one dimensional coalgebra generated by a single group-like element g. Then we see easily that $\mathrm{H}(D)$ is a two dimensional Hopf algebra $k1\oplus kg$, with $\Delta(g)=g\otimes g$, $\varepsilon(g)=1$, and $g^2=1$.

Let C be a cocommutative coalgebra and let D be as above. If we define $f: C\to D$ by $f(c)=\varepsilon(c)g$ for $c\in C$, then f is a surjective coalgebra homomorphism. Put $\bar{f}=\mathrm{H}(f)$. Let $L_1=\mathrm{K}_1(\bar{f})$ and $L_g=\mathrm{K}_g(\bar{f})$. Then we have an exact sequence

$$\text{(S)} \qquad k \longrightarrow L_1 \longrightarrow \mathrm{H}(C) \xrightarrow{\bar{f}} \mathrm{H}(D) \longrightarrow k.$$

It is easy to see that

$$L_1 \cap L_g = 0, \quad L_1L_g \subseteq L_g, \quad L_gL_1 \subseteq L_g, \quad L_g^2 \subseteq L_1.$$

By Lemma 1.1 (3), we have $C\subseteq L_g$. Thus we have

$$C^{2n} \subseteq L_1, \quad C^{2n+1} \subseteq L_g$$

for $n=0, 1, 2,\dots$. Since $1\in C^2$ (see the definition of the ideal I), we see that $C^n\subseteq C^{n+2}$ for $n=0, 1, 2,\dots$. Therefore

$$L_1 = \sum_{n=0}^{\infty} C^{2n}, \quad L_g = \sum_{n=0}^{\infty} C^{2n+1}, \quad \text{and} \quad \mathrm{H}(C) = L_1 \oplus L_g,$$

since $H=\sum_{n=0}^{\infty} C^n$.

EXAMPLE 1. Assume that C is the dual coalgebra of a purely inseparable extension field K of k with degree more than 1. Then C is a simple coalgebra and has no group-like elements. Since $K\otimes K\otimes\cdots\otimes K$ (n times) is a local k-algebra, the dual coalgebra $C\otimes C\otimes\cdots\otimes C=(K\otimes K\otimes\cdots\otimes K)^*$ is irreducible, so that its homomorphic image C^n in $\mathrm{H}(C)$ is irreducible for any $n\in\mathbf{N}$. Let D be as above. Then it follows that L_1 and L_g are irreducible subcoalgebras and that $(L_1)_0=k1$ and $(L_g)_0=C$. Since $\mathrm{H}(D)^1=k1$, we have an exact sequence

$$\text{(S}^1\text{)} \qquad k \longrightarrow L_1 \longrightarrow L_1 \longrightarrow k1 \longrightarrow k.$$

Trivially, this is strongly exact.

Since $\mathrm{H}(C)_0=k1+C$, it follows that $\mathrm{H}(C)$ has only one group-like element, namely, 1. This implies that f has no coalgebra splittings because $\mathrm{H}(D)$ has two group-like elements. Thus (S) is not strongly exact. This shows that the converse of Theorem 3.5 does not hold.

Next, define $\eta: L_1\oplus L_g\to L_1$ by

$$\eta(x) = \begin{cases} x & \text{for} \quad x\in L_1 \\ \varepsilon(x)1 & \text{for} \quad x\in L_g. \end{cases}$$

Then η is a coalgebra retraction of L_1. This and Theorem 2.1 show that Corollary 1.4 in [6] is not true for non-irreducible Hopf algebras.

EXAMPLE 2. Assume that $\mathrm{ch}(k)\neq 2$, and there is an $a\in k$ such that $\sqrt{a}\notin k$. Let H be a four dimensional vector space over k with a basis $\{e, x, y, z\}$. Dcfine a multiplication on H by the following table:

	e	x	y	z
e	e	x	y	z
x	x	$\frac{1}{2}(e+z)$	0	x
y	y	0	$\frac{1}{2a}(e-z)$	$-y$
z	z	x	$-y$	e

Then calculations show that H is an associative, commutative k-algebra with unit element e which we denote by 1 from now on.

Define linear mappings

$$\Delta: H \longrightarrow H\otimes H \quad \text{and} \quad \varepsilon: H\longrightarrow k$$

by

$$\Delta(1)=1\otimes 1, \quad \varepsilon(1)=1$$
$$\Delta(x)=x\otimes x+ay\otimes y, \quad \varepsilon(x)=1$$
$$\Delta(y)=x\otimes y+y\otimes x, \quad \varepsilon(y)=0$$
$$\Delta(z)=z\otimes z, \quad \varepsilon(z)=1.$$

Then we see easily that Δ and ε are k-algebra homomorphisms, so that H is a bialgebra over k. Further, the multiplication table of H shows that $\mathrm{id}_H*\mathrm{id}_H=\varepsilon$. It follows that H is, in fact, a Hopf algebra with id_H as its antipode.

Put $C=kx+ky$. Then C is a subcoalgebra of H, and the dual algebra C^* of C is isomorphic to the quadratic extension field $k(\sqrt{a})$ of k, so that C is a simple subcoalgebra. It follows that H is a cosemisimple Hopf algebra with simple subcoalgebras $k1$, kz, and C.

Let D be a coalgebra as in Example 1. Consider a linear mapping $\rho: H\rightarrow \mathrm{H}(D)$ defined by

$$\rho(1)=1, \quad \rho(x)=g, \quad \rho(y)=0, \quad \text{and} \quad \rho(z)=1.$$

Then ρ is a Hopf algebra homomorphism of H onto $\mathrm{H}(D)$. By Lemma 1.1 we have

$$K_1(\rho) = k1 + kz \quad \text{and} \quad K_g(\rho) = C.$$

Thus we have an exact sequence

$$\text{(S)} \qquad k \longrightarrow K_1(\rho) \longrightarrow H \xrightarrow{\rho} H(D) \longrightarrow k$$

of cosemisimple Hopf algebras. ρ has no coalgebra splittings because C has no group-like elements. Therefore (S) is not strongly exact.

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