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Two-timing methods with applications to heterogeneous reaction-diffusion systems

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1. Introduction

There are so many mathematical models characterized by quasi-linear parabolic systems of differential equations which describe actual problems arising in physics, chemistry, biology and many other fields. When such systems take of the form

$$\frac{\partial u}{\partial t} = \operatorname{div}(D(x, t, u)\nabla u + E(x, t, u)) + f(x, t, u), \tag{1.1}$$

$u=(u_1,\dots, u_n)$, in which f may be replaced by $f(x, t, u, \nabla u)$, they are often called *reaction-diffusion-advection systems.* Here, $D(x, t)$ is a nonnegative definite matrix. If D is a constant matrix, f is independent of t and x and $E\equiv 0$, (1.1) is reduced to *homogeneous* reaction-diffusion systems

$$\frac{\partial u}{\partial t} = D\Delta u + f(u). \tag{1.2}$$

In many of applications, D is diagonal. (1.2) are extensively investigated by numerous authors from both analytical and numerical points of view. In this paper, we shall not touch upon them but refer to excellent reviews by Fife [9] and Smoller [46].

On the other hand, we encounter *spatially inhomogeneous* or *heterogeneous* reaction-diffusion-advection systems such as

$$\frac{\partial u_i}{\partial t} = \operatorname{div}(d_i(x)\nabla u_i + u_i\nabla e_i(x)) + f_i(x, u) \quad (i=1,\dots, n), \tag{1.3}$$

in which $f_i(x, u)$ and $d_i(x)$, $e_i(x)$ explicitly depend on space variables x. These models occur widely as ones for dynamics of chemical substances or biological species in heterogeneous media or environments (Okubo [32], Fife [9], etc.). Let us show one simple but very suggestive model equation introduced by Fisher [11] in population genetics though there are many other models described by heterogeneous reaction-diffusion equations (Gurney and Nisbet [15], Kawasaki and Teramoto [20], Kurland [23], Nagylaki [30], Pacala and Roughgarden [33], Skellam [45] and their references therein). It is described by

(1.4) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda m(x)u(1-u), \quad x \in I = (0, 1), \quad t > 0$$

with Neumann baundary conditions. Since the unknown function u in (1.4) denotes the frequency of a gene, we pay attention to only solutions of (1.4), for which $0 \leq u \leq 1$. Assume $\int_I m(x)dx = 1$. When $m(x)$ is constant, that is, $m(x) \equiv 1$, we know that any solution $u(t, x)$ satisfying $0 \leq u \leq 1$, and $u \not\equiv 0$ tends to 1 as $t \to \infty$ for any $\lambda > 0$. On the other hand, when $m(x)$ is not constant and $m(x) < 0$ on a set of positive measure, Fleming [12] showed that there exists $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$, $u(t, x)$ satisfying $0 \leq u \leq 1$ and $u \not\equiv 0$ tends to 1 as $t \to \infty$, while for $\lambda > \lambda_0$, there appears a unique stable inhomogeneous equilibrium $\phi_\lambda(x)$ of (1.4), $0 < \phi_\lambda(x) < 1$ on $\bar{I}$ and $u(t, x)$ satisfying $0 \leq u \leq 1$, and $u \not\equiv 0$, $u \not\equiv 1$ tends to $\phi_\lambda(x)$ as $t \to \infty$. Thus, the appearance of heterogeneity of $m(x)$ drastically changes the situation. For the study of such heterogeneous systems, there have been at least three approaches. The first two, which are essentially powerful to scalar equations, are the super- and sub-solutions methods (Fife and Peletie [10], Howes [19], Leung and Bendjilali [25], Matano [27], Pauwelussen and Peletier [34]) and the variational ones (Fleming [12], Kawasaki and Teramoto [20], Yanagida [49]), which give the existence and sometimes the stability of inhomogeneous stationary solutions. The third is perturbed bifurcation techniques, which can widely be applied to homogeneous reaction-diffusion systems of equations perturbed by weak heterogeneity (e.g. Mimura and Nishiura [29]). Although we know these three approaches, it is still hard to study spatial and/or temporal pattern formation in heterogeneous reaction- diffusion systems of equations. For this purpose, Su Yu [48] considered a fairly general class of systems

(1.5) $$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + \varepsilon f(x, \varepsilon t, u), \quad t > 0, \quad x \in \Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \end{cases}$$

with Neumann boundary conditions. Here, Ω is a bounded domain in $\mathbf{R}^m$. Under certain assumptions, he showed that when ε is sufficiently small, there exist $K > 0$ and $T(\varepsilon) > 0$ such that

$$\|u(t, \cdot) - \bar{u}(\varepsilon t)\|_{L^2(\Omega)} \leq K\varepsilon$$

for $t \geq T(\varepsilon)$, where $u(t, x)$ is the solution of (1.5) and $\bar{u}(\varepsilon t)$ is the solution of

(1.6) $$\begin{cases} \dfrac{d\bar{u}}{dT} = \bar{f}(T, \bar{u}), \quad T > 0, \\ \bar{u}(0) = \bar{u}_0, \end{cases}$$

with $\bar{u}_0 = (meas.\ \Omega)^{-1} \int_\Omega u_0(x)dx$ and $\bar{f}(T, \bar{u}) = (meas.\ \Omega)^{-1} \int_\Omega f(x, T, \bar{u})dx$.

Thus, we can study the effect of heterogeneity of the reaction term f on the asymptotic behavior of solutions of (1.5) through the analysis of O.D.E. (1.6), which is much easier than that of P.D.E. Unfortunately, his result did not discuss the transient behavior of solutions for $0<t\leq T(\varepsilon)$. Furthermore if we are interested in dynamics of populations in heterogeneous environments for instance, we meet heterogeneous reaction-diffusion-advection systems of (1.3) rather than (1.5) (Okubo [32], Comins and Blatt [4], Fife and Peletier [10], Shigesada et. al [44], Yanagida [49], Howes [19]).

For such systems, Shigesada [43], Ei and Mimura [7] analyzed the transient behavior as well as asymptotic behavior of solutions for ecological model equations of the form

$$(1.7)\quad \begin{cases} \dfrac{\partial u_i}{\partial t} = \operatorname{div}(d_i \nabla u_i + u_i \nabla e_i(x)) + \varepsilon f_i(x, u), \quad x \in \Omega \subset \mathbf{R}^m, \quad t > 0, \\ u_i(0, x) = u_{0i}(x), \quad x \in \Omega, \quad u = (u_1, \dots, u_n) \quad (i=1, \dots, n) \end{cases}$$

with a positive small parameter ε. Here, Ω is a bounded domain in $\mathbf{R}^m$ and the boundary conditions are no-flux ones and $d_i>0$ $(i=1,\dots, n)$. To investigate the spatial and/or temporal pattern formation of solutions of (1.7), Shigesada [43] applied the two-timing method (see Section 3) to (1.7) and constructed the lowest-order approximate solution of the form

$$(1.8)\qquad u_i^0(t, \varepsilon t, x) = N_i(\varepsilon t)\varphi_i(x) + w_i(t, x) \quad (i=1,\dots, n),$$

where $\varphi_i(x)=\exp\{-e_i(x)/d_i\}/\int_\Omega \exp\{-e_i(x)/d_i\}dx$ and $N_i(T)$ is the solution of

$$(1.9)\quad \begin{cases} \dfrac{dN_i}{dT} = F_i(N_1, \dots, N_n) = \displaystyle\int_\Omega f_i(x, N_1\varphi_1(x), \dots, N_n\varphi_n(x))dx, \\ N_i(0) = \displaystyle\int_\Omega u_{i0}(x)dx \quad (i=1,\dots, n) \end{cases}$$

and $w_i(t, x)$ is the solution of

$$(1.10)\quad \begin{cases} \dfrac{\partial w_i}{\partial t} = \operatorname{div}(d_i \nabla w_i + w_i \nabla e_i(x)), \\ w_i(0, x) = u_{0i}(x) - \displaystyle\int_\Omega u_{0i}(x)dx \cdot \varphi_i(x) \quad (i=1,\dots, n), \end{cases}$$

with the no-flux boundary conditions. Ei and Mimura [7] proved that if the solution of (1.9) converges to an asymptotically stable equilibrium of (1.9), then the solution $u_i(t, x; \varepsilon)$ of (1.7), satisfies

$$\|u_i(t, \cdot\,; \varepsilon) - u_i^0(t, \varepsilon t, \cdot)\|_{L^\infty(\Omega)} \leq O(\varepsilon),$$

specially,

$$\left|\int_\Omega u_i(t, x; \varepsilon)dx - N_i(\varepsilon t)\right| \leq O(\varepsilon)$$

uniformly on $t \in [0, \infty)$ for $i=1,\dots, n$.

On the other hand, it is possible that solutions of (1.9) exhibit stable limit cycles or strange attractors such as chaos. Especially, in the case of limit cycles, it is observed by computer simulations that $u(t, x; \varepsilon)$ behaves like a periodic solution in t and the orbit described by $\left(\int_\Omega u_1(t, x; \varepsilon)dx, \dots, \int_\Omega u_n(t, x; \varepsilon)\, dx\right)$ agrees fairly well with the orbit $(N_1(\varepsilon t),\dots, N_n(\varepsilon t))$ in phase plane $\mathbf{R}^n$ though Ei and Mimura [7] have not discussed this case.

This observation motivates us to study case that (1.9) possesses stable limit cycles. Our result (mainly stated in Section 4) is as follows: Assume that there exists a stable limit cycle γ of (1.9) and that $(N_1(\varepsilon t),\dots, N_n(\varepsilon t)) \to \gamma$ as $t \to \infty$. Then there exist $C>0$ and $t_\varepsilon>0$ such that

$$\operatorname{dist}\left\{\gamma, \left(\int_\Omega u_1(t, x; \varepsilon)dx, \dots, \int_\Omega u_n(t, x; \varepsilon)dx\right)\right\} \leq C\varepsilon$$

for any $t \geq t_\varepsilon$ when ε is sufficiently small. In Section 5, we will analyze (1.7) in detail with applications to more concrete actual problems of two-species prey-predator model and illustrate, in a very explicit manner, the effect of heterogeneity on the behaviors of solutions.

Results in this paper contain another application. Let us consider the following systems:

$$(1.11) \qquad \frac{\partial u_i}{\partial t} = d_i \Delta u_i + \varepsilon\left\{\sum_{j=1}^m a_{ij}(x, u)\frac{\partial u_i}{\partial x_j} + f_i(u)\right\}, \quad x \in \Omega,\ t > 0 \qquad (i=1,\dots, n)$$

with Neumann boundary conditions and

$$(1.12) \qquad u_i(0, x) = u_{i0}(x), \quad x \in \Omega,$$

where $u=(u_1,\dots, u_n)$, Ω is a bounded domain in $\mathbf{R}^m$ and $d_i>0$. Conway, Hoff and Smoller [5] proved that when ε is sufficiently small and (1.11) admit a compact positively invariant set $\Sigma \subset \mathbf{R}^n$ independent of small ε, then any solution $(u_1(t, x),\dots, u_n(t, x))$ of (1.11), (1.12) with values in Σ converges uniformly and exponentially to their spatial averages $\bar{u}_i = (meas.\ \Omega)^{-1}\int_\Omega u_i(t, x)dx$ $(i=1,\dots, n)$, where $\bar{u}_i(t)$ satisfies

$$(1.13) \qquad \frac{d\bar{u}_i}{dt} = \varepsilon f_i(\bar{u}_1,\dots, \bar{u}_n) + O(\varepsilon e^{-\sigma t}) \quad \text{as} \quad t \longrightarrow \infty \quad (i=1,\dots, n)$$

for some $\sigma>0$. If we apply our results to (1.11), (1.12), the above statement is valid without assuming the existence of such an invariant set Σ. Details will be stated in Section 5.

The outline of this paper is the following: In Section 2, we formulate (1.7), (1.11) and (1.12) as abstract initial value problems of parabolic type in a Banach space. Results and proofs are stated in an abstract form in Sections 4 and 6, respectively. In Section 5, we apply results in Section 4 to (1.7), (1.11) and (1.12). The two-timing method will be stated in Section 3.

2. Setting the problem and assumptions

Let B be a Banach space with norm $\|\cdot\|$. We consider the following initial value problems in B;

$$\text{(2.1)}\qquad \begin{cases} \dfrac{du}{dt} + Au = \varepsilon F(u), \quad t > 0, \\ u(0) = u_0, \end{cases}$$

where ε is a positive small parameter. Suppose A, F and u_0 satisfy the following four assumptions.

A1) *A is a sectorial operator in B and $\sigma(A)$, the spectrum of A, consists of $\sigma_1=\{0\}$, isolated eigenvalue of A, and $\sigma_2\subset\{\lambda\in\mathbf{C}\,|\,\mathrm{Re}\,\lambda>a$ for some constant $a>0\}$.*

We note that $\sigma(A)$ decomposes the space $B=B_1\oplus B_2$ corresponding to the spectral sets σ_1 and σ_2, and let Q, P be the projections onto B_1, B_2, respectively (Dunford and Schwartz [6, v. 1, Ch. 7]).

A2) *$B_1=\mathrm{Ker}\,A$ and B_1 is a finite dimensional space.*

Let $\mathscr{L}(E, E')$ denote the Banach space of continuous linear operators from E to E' with the sup-norm and we set $\mathscr{L}(E)=\mathscr{L}(E, E)$ if E and E' are Banach spaces. B^α represents the space $D(A^\alpha)$, domain of A^α, with norm $\|u\|_\alpha=\|u\|+\|(A+Id)^\alpha u\|$ for $u\in D(A^\alpha)$, where Id is the identity mapping on B.

A3) *$F\in C^2(B^\alpha; B)$ for some $0\le\alpha<1$ and for each bounded set B_0 in B, there exists $M>0$ depending on B_0 such that $\|F(u)\|$, $\|F'(u)\|_{\mathscr{L}(B^\alpha,B)}$, $\|F''(u)\|_{\mathscr{L}(B^\alpha,\mathscr{L}(B^\alpha,B))}\le M$ for any $u\in B_0$, where $'$ represents the Fre'chet derivative.*

A4) $u_0\in B^\alpha$.

3. The two-timing method and formal expansion procedure

To explain basic conception of the two-timing method, we consider the simple but typical evolution equations in a Banach space B of the form

$$\frac{du}{dt} + Au = \varepsilon F(u, t, \varepsilon), \quad t > 0 \tag{3.1}$$

with a positive small parameter ε, where F is an analytic mapping in ε and A is a linear operator whose spectrum consists of simple eigenvalues on imaginary axis and the spectral set whose elements have positive real parts. In many problems in which (3.1) appear, transient and large time behaviors of solutions $u(t; \varepsilon)$ of (3.1) are often our main interest and to study them, we may first consider Taylor expansions of $u(t; \varepsilon)$ with respect to ε of the form $u(t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j u^j(t)$. But, it is a well known fact that finite Taylor expansions usually furnish asymptotic approximations to exact solutions only on finite time interval though we need perturbation procedures which give asymptotic expansions approximating exact solutions uniformly on the whole time interval $0 \leq t < \infty$, which is quite difficult. To overcome this difficulty, one of the most powerful perturbation methods was first developed by Cole and Kevorkian [3] and was simplified by Reiss [38] later. The method is known under the various names, the *method of multiple scales*, or simply the *two-timing method*. We briefly explain this procedure following Reiss [38]. This method is based on a conjecture that more than one time scale is involved in the evolutional behavior of the solutions of (3.1); a *slow* time scale and a *fast* time scale. Reiss used $T = \varepsilon t$ as the slow time, while retaining the original variable t as the fast time.

Let $u(t; \varepsilon)$ be solutions of (3.1). We seek solutions in the form

$$u(t; \varepsilon) = \tilde{u}(t, \varepsilon t; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j u^j(t, \varepsilon t) \tag{3.2}$$

assuming that every term $u^j(t, \varepsilon t)$ satisfies

$$|u^j(t, T)| \leq C_{j,T} \tag{3.3}$$

for any $t \in [0, \infty)$, arbitrarily fixed $T > 0$ and some $C_{j,T} > 0$. We formally insert *the two-timing approximation* (3.2) into (3.1), in which equations we regard t and T $(= \varepsilon t)$ as independent variables and equate coefficients like powers of ε. Consequently, we can obtain the equation for each u^j of the form

$$\frac{du^j}{dt} + Au^j = H_j, \tag{3.4}$$

where H_j depends on ε, T, t and the previous u^k's. (3.4) involves indeterminacy of T-dependence of $u^j(t, T)$, which becomes determinate by requiring (3.3).

Later on, this procedure will be demonstrated for the equations (2.1). An immediate consequence of this procedure is that there is a positive number T such that all functions $u^j(t, \varepsilon t)$ exist and are bounded on $[0, T/\varepsilon]$, that is, the expansion (3.2) is at least formally an approximation to $u(t; \varepsilon)$ on $[0, T/\varepsilon]$. Here, we say that the expansion (3.2) is *uniformly valid* on some time interval, say, I, if

$$\|u(t;\varepsilon)-\textstyle\sum_{j=0}^{n} u^j(t, \varepsilon t)\|_B \leq O(\varepsilon^{n+1}) \tag{3.5}$$

uniformly for $t \in I$. In fact, many authors proved that these two-timing approximations are uniformly valid on such *expanding intervals* $[0, T/\varepsilon]$ in the sense of (3.5) (Perko [35], Kollet [22], Hoppensteadt [18], Sanders [40], Persek and Hoppensteadt [36]). However, in almost all cases to which the procedure can be applied, two-timing approximations of the form (3.2) seem to be approximate to exact solutions on the whole time interval in a certain sense, and in chemistry, physics and many other fields, there have been already a lot of applications of the two-timing method to various problems under the *belief* in the validity on the whole time interval of two-timing approximations (Benney and Lange [1], Lange and Larson [24] and reviews in Nayfeh [31]), which is the original purpose of this method. In mathematics, we must clarify under what assumptions and in what senses the approximations can be regarded as valid on the whole time interval. To our knowledge, there have not been many papers on their mathematical validity on the whole time interval (Greenlee and Snow [14], keener [21], Sethna and Moran [42], Hoppensteadt [17], Ei and Mimura [7]). Greenlee and Snow [14], Keener [21], Sethna and Moran [42] dealt with the oscillation equations such as unforced Duffing equations, Van der Pol equations and equations arising in satellite problems. The typical equations are of the forms $y_{tt}+\varepsilon f(\varepsilon, t, y, y_t)+y=0$, which correspond to (3.1) in the case that A has a pair of conjugate pure imaginary eigenvalues. For these equations, the two-timing method is much available to study the existence of limit cycles. Hoppensteadt [17] considered systems of O.D.E. of the form $\varepsilon \frac{dx}{dt}=f(t, x, y, \varepsilon), \frac{dy}{dt}=g(t, x, y, \varepsilon)$ arising from certain singular perturbation problems. Ei and Mimura [7] and this paper study the validity of the lowest order approximation $u^0(t, \varepsilon t)$ of (3.2) for certain heterogeneous reaction-diffusion systems stated in Section 1, which correspond to (3.1) in the case that A has a simple 0 eigenvalue.

On the other hand, there is one of the most remarkable applications of the two-timing method, the application to bifurcation phenomena (Matkovsky [28], Reiss [39], Segel and Levin [41]). Consider the nonlinear bifurcation problem

$$\frac{dv}{dt} = F[v; \lambda] \tag{3.6}$$

with a parameter λ assumed to have a stationary solution $v_0(\lambda)$ for all values of λ and let λ_c be a bifurcation point. Then, defining $\varepsilon=(\lambda-\lambda_c)^\alpha$ for some $\alpha>0$, $u=v-v_0(\lambda)$ and $A=-F_v[v_0(\lambda_c), \lambda_c]$, (3.6) is often reduced to equations with nearly the same forms as (3.1). If the two-timing method can be applied to the equations, we will obtain the structure containing transient behaviors of solutions as well as asymptotic behaviors such as stationary solutions and limit cycles. Thus, the two-timing method gives quite a new point of view to studies of bifurcation phenomena. On the other hand, we see, for example, that both linear operators corresponding to A of (3.1) in the case of Hopf bifurcation and in the case of oscillation equations have the same property of spectrum, that is, both linear operators have pairs of conjugate pure imaginary simple eigenvalues. Then, it may be possible that the same structures exist in both cases and that results in oscillation equations are applied to Hopf bifurcation problems, and vice versa. Thus, we find it also important to investigate the relation between bifurcation phenomena and other problems.

Let us construct the two-timing approximations for the problem (2.1) with Assumptions A1), A2), assuming formally $F(u)=\sum_{n=0}^{\infty} F_n u^n$. Substituting (3.2) into (2.1), we have

$$\frac{\partial \tilde{u}}{\partial t} + \varepsilon \frac{\partial \tilde{u}}{\partial T} + A\tilde{u} = \varepsilon F(\tilde{u}), \tag{3.7}$$

where $T=\varepsilon t$. Since ε is sufficiently small, we may assume that t and T are independent variables. We first determine u^0. Equating coefficients of ε^0 and ε^1 in (3.6) respectively, we obtain

$$\frac{\partial u^0}{\partial t} + Au^0 = 0 \tag{3.8}$$

with $u^0(0, 0)=u_0$ and

$$\frac{\partial u^1}{\partial t} + \frac{\partial u^0}{\partial T} + Au^1 = F(u^0) \tag{3.9}$$

with $u^1(0, 0)=0$. The general solution of (3.8) is

$$u^0(t, T) = e^{-tA}y^0(T), \tag{3.10}$$

where e^{-tA} is a semigroup generated by A and $y^0(T)=u^0(0, T)$. Let us determine $y^0(T)$. Substituting (3.10) into (3.9), we have

$$\frac{\partial u^1}{\partial t} + Au^1 = G^0(t, T). \tag{3.11}$$

where $G^0(t, T)=-e^{-tA}\dfrac{dy^0}{dT}+F(e^{-tA}y^0(T))$.

LEMMA 3.1. *The following two conditions are equivalent*:

i) (3.2) *holds for* $i=1$;

ii) $\lim_{t\to\infty} QG^0(t, T)=0$.

PROOF. From A1), A2), we have $\|e^{-tA}\|\le M$ and $\|e^{-tA}P\|\le Me^{-at}$ for some $M>0$, so that $\|u^0(t, T)-Qy^0(T)\|\le Me^{-at}\|Qy^0(T)\|$ and

$$\left\|G^0(t, T)-\left\{-Q\frac{dy^0}{dT}+F(Qy^0(T))\right\}\right\|\le M_1e^{-at} \tag{3.12}$$

holds for some $M_1>0$. Let $G^0(\infty, T)1=\text{im}_{t\to\infty} G^0(t, T)=-Q\dfrac{dy^0}{dT}+F(Qy^0(T))$. We will show that i) implies ii). If $QG^0(\infty, T)\neq 0$, then there exists $t_0>0$ and $\sigma_T>0$ such that $\|QG^0(\infty, T)\|\ge\sigma_T$ and $\|QG^0(t, T)-QG^0(\infty, T)\|\le\sigma_T/2$ for any $t\ge t_0$. Since

$$\begin{aligned} u^1(t, T) &= e^{-tA}u^1(0, T)+\int_0^t e^{-(t-s)A}G^0(s, T)ds \\ &= Qu^1(0, T)+\int_0^t QG^0(s, T)ds+e^{-tA}Pu^1(0, T) \\ &\quad +\int_0^t e^{-(t-s)A}PG^0(s, T)ds, \end{aligned} \tag{3.13}$$

we have $Qu^1(t, T)=Qu^1(0, T)+\int_0^t QG^0(s, T)ds$. Hence, $\|Qu^1(t, T)\|\ge(\sigma_T/2)\times(t-t_0)-\left\|\int_0^{t_0} QG^0(s, T)ds\right\|-\|Qu^1(0, T)\|$, which contradicts i). Suppose that ii) holds. (3.11) leads to $\|QG^0(t, T)\|\le M_1e^{-at}$, so that it follows from (3.13) that

$$\begin{aligned} \|u^1(t, T)\| &\le M\|u^1(0, T)\|+\int_0^t \{M_1e^{-as}+Me^{-a(t-s)}\|PG^0(s, T)\|\}ds \\ &\le M_2 \end{aligned}$$

for some $M_2>0$, which implies i). ∎

Noting that $QG^0(\infty, T)=0$ and $y^0(0)=u_0$, we have

$$\begin{cases} \dfrac{d}{dT}Qy^0(T)=QF(Qy^0(T)) \\ Qy^0(0)=Qu_0. \end{cases} \tag{3.14}$$

Defining $Y(T)$ by $Qy^0(T)$ reduces (3.14) to

$$\begin{cases} \dfrac{dY}{dT}=QF(Y) \\ Y(0)=Qu_0, \end{cases} \tag{3.15}$$

which is the initial value problem of O.D.E. in B_1 $(=QB)$. By solving (3.15), $Y=Qy^0$ can be determined. Although $Py^0(T)$ remains undetermined, an arbitrary given $Py^0(T)$ implies (3.3) for $i=1$ because there is some $C_T>0$ such that $\|PG^0(t, T)\| \leq C_T$ uniformly for any $t \in [0, \infty)$. Therefore, we may define $Py^0(T) \equiv Pu_0$ as the simplest form. Consequently, we have

$$u^0(t, T) = Y(T) + e^{-tA}Pu_0. \tag{3.16}$$

In order to determine $u^1(t, T)$, we equate the coefficients of ε^2 and then we have

$$\frac{\partial u^2}{\partial t} + Au^2 = G^1(t, T) \tag{3.17}$$

with $u^2(0, 0)=0$, where $G^1(t, T) = -\frac{\partial u^1}{\partial T} + F'(u^0)u^1$. The general solution of (3.9) is

$$u^1(t, T) = e^{-tA}y^1(T) + \tilde{G}^0(t, T), \tag{3.18}$$

where $y^1(T)=u^1(0, T)$ and $\tilde{G}^0(t, T) = \int_0^t e^{-(t-s)A}G^0(s, T)ds$. In a similar way to Lemma 3.1, we know that the assumption (3.3) for $i=2$ is equivalent to the condition

$$\lim_{t\to\infty} QG^1(t, T) = 0. \tag{3.19}$$

Since $u^1(t, T) \to Qy^1(T) + \tilde{G}^0(\infty, T)$, (3.19) implies

$$-\frac{dY^1}{dT} + F'(Y(T))(Y^1 + \tilde{G}^0(\infty, T)) = 0, \tag{3.20}$$

where $Y^1(T)=Qy^1(T)$ and $\tilde{G}^0(\infty, T) = \lim_{t\to\infty} \tilde{G}^0(t, T)$. The equation (3.20) with $Y^1(0)=0$ is an initial value problem of O.D.E. in B_1 where $Y(T)$ and $\tilde{G}^0(\infty, T)$ have already been determined, so that $Y^1(T)$ is determined. Consequently, letting $Py^1(T) \equiv 0$, we have $u^1(t, T) = Y^1(T)$.

In the same way as above, $u^i(t, T)$ can be determined for $i>2$.

Remark. Though $Pu^i(t, T)$ remains undetermined for any i, we may set $Pu^i(0, T) \equiv 0$ as the simplest form such that $Pu^0(0, 0)=0$ $(i=1, 2, 3, \ldots)$. In fact, in the same way as that of Py^0, we see that an arbitrarily given $Pu^i(0, T)$ satisfies the assumption (3.2).

4. Main results

In this section, we consider abstract equations (2.1) under Assumptions A1)–A4).

We first consider the following O.D.E.:

$$\left\{\begin{array}{l} \dfrac{d\varphi}{dT} = QF(\varphi) \\ \varphi(0) = \varphi_0 \end{array}\right. \quad \text{in} \quad B_1, \tag{4.1}$$

and let $\pi(T; \varphi_0)$ be a solution of (4.1). Then $u^0(t, T)$ constructed in Section 3 is represented as $u^0(t, T) = \pi(T; Qu_0) + e^{-tA}Pu_0$. When $\pi(T; Qw)$ exists on $[0, T']$ for $w \in B^\alpha$, we define $b_{T'}(w)$ by $b_{T'}(w) = \|Pw\|_\alpha + \sup_{0 \le T \le T'} |\pi(T; Qw)|$ and especially when $\pi(T; Qw)$ exists on $[0, \infty)$ for $w \in B_0$, we define $b(w)$ by $b(w) = b_\infty(w)$. Here $|\cdot|$ is the Euclidean norm of E if it is a finite dimensional space.

THEOREM 4.1. *Suppose that $\pi(T; Qu_0)$ exists for $T \in [0, T_1]$ for some $T_1 < \infty$. Then, there exist $C > 0$, $\varepsilon_1 > 0$ such that a solution $u(t; \varepsilon)$ of (2.1) exists on $[0, T_1/\varepsilon]$ and satisfies*

$$\|u(t; \varepsilon) - u^0(t, \varepsilon t)\| \le C\varepsilon$$

for any $\varepsilon \in (0, \varepsilon_1]$ and any $t \in [0, T_1/\varepsilon]$, where C and ε_1 depend only on $b_{T_1}(u_0) + T_1$.

DEFINITION. We call a closed bounded set Γ in B_1 an *exponentially asymptotically stable attractor* if there exist an open bounded set $V \supset \Gamma$ and constants $M_0 > 0$, $\beta > 0$ such that

$$\operatorname{dist}(\pi(T; v), \Gamma) \le M_0 e^{-\beta T} \operatorname{dist}(v, \Gamma)$$

for any $v \in V$ and any $T > 0$, where $\operatorname{dist}(V_1, V_2) = \inf\{|v_1 - v_2|;\ v_1 \in V_1 \text{ and } v_2 \in V_2\}$ for closed bounded sets V_1 and V_2 in B_1.

THEOREM 4.2. *Suppose that $\pi(T; Qu_0)$ converges, as $T \to \infty$, to some exponentially asymptotically stable attractor Γ. Then there exist constants $\varepsilon_1 > 0$ and $C_1 > 0$ such that $u(t; \varepsilon)$ exists and satisfies*

$$\|Pu(t; \varepsilon) - e^{-tA}Pu_0\|_\alpha \le C_1\varepsilon$$

for any $\varepsilon \in (0, \varepsilon_1]$ and any $t \in [0, \infty)$. Moreover, there exists $t_\varepsilon > 0$ for any $\varepsilon \in (0, \varepsilon_1]$ such that

$$\operatorname{dist}(\Gamma, Qu(t; \varepsilon)) \le C_1\varepsilon$$

for any $t \ge t_\varepsilon$. Here ε_1 and C_1 depend only on $b(u_0)$ and T_1 such that $\pi(T_1; Qu_0) \in V$, where V is a neighborhood of Γ stated in Definition.

COROLLARY 4.1. *In addition to the assumptions of Theorem 4.2, assume that Γ is an equilibrium of (4.1). Then, there exist constants $\varepsilon_1 > 0$ and $C_1 > 0$ such that*

$$\|u(t; \varepsilon) - u^0(t, \varepsilon t)\|_\alpha \le C_1\varepsilon$$

for any $\varepsilon \in (0, \varepsilon_1]$ *and any* $t \in [0, \infty)$.

COROLLARY 4.2. *Suppose that* F *maps* B_1 *into itself and that there exist* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that* $|Qu(t; \varepsilon)| \leq C_0$ *for any* $t \geq 0$ *and any* $\varepsilon \in (0, \varepsilon_0]$. *Then, there exist* $\tilde{a} \in (0, a)$, $\varepsilon_1 \in (0, \varepsilon_0]$ *and* $C_1 > 0$ *such that*

$$\|Pu(t; \varepsilon)\|_\alpha \leq C_1 e^{-\tilde{a}t} \|Pu_0\|_\alpha$$

and that $Qu(t; \varepsilon)$ *satisfies*

$$\frac{1}{\varepsilon}\frac{d}{dt} Qu(t; \varepsilon) = QF(Qu(t; \varepsilon)) + g_\varepsilon(t)$$

for any $t \geq 0$, *any* $\varepsilon \in (0, \varepsilon_1]$ *and some* $g_\varepsilon(t)$ *with* $|g_\varepsilon(t)| \leq C_1 e^{-\tilde{a}t}$, *where* a *is the constant given in the assumption* $A1$).

5. Applications

EXAMPLE 1. A model of rapidly dispersing animals in heterogeneous environments.

We consider systems (1.7) with $u_{0i} \geq 0$ in Ω and no-flux boundary conditions

$$\langle J_i(x, u_i), \nu \rangle = 0 \quad \text{on} \quad x \in \partial\Omega \quad (i = 1, \dots, n). \tag{5.1}$$

Here, $J_i(x, u_i) = -d_i \nabla u_i - u_i \nabla e_i(x)$ ($d_i > 0$) and Ω is a bounded domain in $\mathbf{R}^m$ with reasonably smooth boundary. $\langle \cdot, \cdot \rangle$ and ν denote the Euclidean inner product of $\mathbf{R}^m$ and the outward normal vector on $\partial\Omega$, respectively. Moreover, we assume that $u_{0i}(x) \in C(\bar{\Omega}; \mathbf{R})$, $e_i(x) \in C^2(\bar{\Omega}; \mathbf{R})$ and $f_i(x, u) \in C^2(\bar{\Omega} \times \mathbf{R}^n; \mathbf{R})$ $(i = 1, \dots, n)$.

We briefly explain the biological meanings of this model. Every $u_i = u_i(t, x)$ represents the population density of i-th species in the habitat Ω. J_i is the flux of the i-th species due to the dispersal process associated with random movements of individuals and the flow due to directed movements of individuals toward favorable environments respectively. $e_i(x)$ is called the *evironmental potential* in the sense that $e_i(x)$ induces the advection velocity, $-\nabla e_i(x)$, toward the minimum points of $e_i(x)$ in Ω. The second term of (1.7), $\varepsilon f_i(x, u)$, represents the net growth rate due to ecological interactions among n-species. We express the net growth rate by the product of ε and f_i so that the dispersal term, $\operatorname{div} J_i(x, u)$, and f_i are of the same order of magnitude. In many ecological systems, we may sssume ε to be very small because it is frequently seen in nature that the change in population density as a result of the dispersal process occurs more rapidly than the change due to the growth process (see also Shigesada [43]).

We now give the correspondence between (1.7) and the abstract equations

(2.1) in order to construct the O.D.E. corresponding to (4.1). Let the Banach space B be $C(\bar{\Omega}; \mathbf{R}^n)$ with sup-norm $\|\cdot\|$ and the linear operator A be $\operatorname{diag}(A_1,\dots, A_n)$, where $A_j = \operatorname{div} J_i(x, v)$ for $v \in D(A_j) = \{v \in C(\bar{\Omega}; \mathbf{R}) \cap W^{2,p}(\Omega; \mathbf{R}); A_j v \in C(\bar{\Omega}; \mathbf{R})$ with $\langle J_j(x, v), v\rangle = 0$ on $\partial\Omega$ for some $p > m\}$ $(j=1,\dots, n)$. Then, we see that A with the domain $D(A) = D(A_1) \times \cdots \times D(A_n)$ is a sectorial operator in B (Stewart [47]) and has compact resolvents. Moreover, when we define $\varphi_i(x) = \exp\{-e_i(x)/d_i\}\Big/\int_\Omega \exp\{-e_i(x)/d_i\}dx$ $(i=1,\dots, n)$, it is easily shown that $\operatorname{Ker} A_i = \operatorname{span}\{\varphi_i\}$ and

$$\int_\Omega (A_i v_1)(x)\cdot v_2(x)\cdot \frac{1}{\varphi_i(x)}\, dx = \int_\Omega \frac{1}{d_i}\cdot J_i(x, v_1)\cdot J_i(x, v_2)\cdot \frac{1}{\varphi_i(x)}\, dx$$

for $v_1, v_2 \in D(A_i)$ $(i=1,\dots, n)$. Hence, we have $\operatorname{Ker} A = \operatorname{span}\{\psi_1,\dots, \psi_n\}$, $\sigma(A) = \{\lambda_0 = 0 < \lambda_1 < \cdots\}$ and $Qv = {}^t\left(\int_\Omega v_1(x)dx\cdot\varphi_1,\dots, \int_\Omega v_n(x)dx\cdot\varphi_n\right)$ for $v = {}^t(v_1,\dots, v_n) \in B$, where $\psi_i(x) = {}^t(0,\dots, 0, \underset{\hat{i}}{\varphi_i(x)}, 0,\dots, 0)$ $(i=1,\dots, n)$, which imply that Assumptions A1) and A2) hold for some $0 < a < \lambda_1$. Here, we note that $P = Id - Q$, so that $Pv = {}^t\left(v_1 - \int_\Omega v_1(x)dx\cdot\varphi_1,\dots, v_n - \int_\Omega v_n(x)dx\cdot\varphi_n\right)$ for $v \in B$. A3) and A4) obviously hold for $\alpha = 0$ by assumptions of $u_{0i}(x)$ and $f_i(x, u)$ $(i=1,\dots, n)$. Therefore, if we define $\pi(T; Qu_0)$, which is a solution of (4.1) with $\pi(0; Qu_0) = Qu_0$, by ${}^t(N_1(T)\varphi_1(x),\dots, N_n(T)\varphi_n(x))$ and $e^{-tA}Pu_0$ by ${}^t(w_1(t, x),\dots, w_n(t, x))$, then it is obvious that $u_0(t, \varepsilon t, x) = {}^t(u_1^0(t, \varepsilon t, x),\dots, u_n^0(t, \varepsilon t, x))$ in Section 4 is given by (1.8) and $N_i(T)$, $w_i(t, x)$ $(i=1,\dots, n)$ are the solutions of (1.9) and (1.10), respectively. Thus, we can apply results in Section 4 to this model when ε is sufficiently small and by solving (1.9), we know behaviors of solutions of (1.7), expecially, whether species can exist or not. Note that $\int_\Omega u_i^0(t, \varepsilon t, x)dx = N_i(\varepsilon t)$ because $\int_\Omega \varphi_i(x)dx = 1$ and $\int_\Omega w_i(t, x)dx = 0$ $(i=1,\dots, n)$.

We consider a two-species prey-predator model in one dimensional heterogeneous environments. It is described by

$$(5.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} J_1(x, u) = \varepsilon f(x, u, v) & \\ & t > 0, \quad x \in I \rightleftharpoons (0, 1) \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x} J_2(x, v) = \varepsilon g(x, u, v) & \\ u(0, x) = u_0(x) \geq 0, \quad \not\equiv 0, \quad v(0, x) = v_0(x) \geq 0, \quad \not\equiv 0, \quad x \in I, & \\ J_1(x, u) = J_2(x, v) = 0 \quad \text{on} \quad x \in \partial I, & \end{cases}$$

where $J_i(x, w) = -d_i \dfrac{\partial w}{\partial x} - \dfrac{d}{dx} e_i(x)\cdot w$ $(i=1, 2)$ and $f(x, u, v) = \{(1 - u/k(x))(1 + ru) - bv\}u$, $g(x, u, v) = \{-c(x) + hu\}v$. d_i $(i=1, 2)$, r and b, h are positive constants and $k(x)$, $c(x)$ are positive in $\bar{I}$. $u=u(t, x; \varepsilon)$ (or $v=v(t, x; \varepsilon)$) represents the density of prey (or predator) and $k(x)$, $c(x)$ denote the carrying capacity of u and the death rate of v at position x, respectively.

In many ecological systems, we often observe that species can coexist merely by having different favorable regions each other, otherwise there are no factors. From an ecological point of view, we are interested in the possibility of coexistence of u and v in such heterogeneous environments with different favorable regions. To study it, we assume that $e_i(x)$ $(i=1, 2)$, $k(x)$ and $c(x)$ can be extended to functions of $C^2(\mathbf{R}; \mathbf{R})$ with period 1 and for simplicity that each $e_i(x)$ has at most one minimal point on I. A neighborhood of the minimal point of every $e_i(x)$ ecologically corresponds to the *favorable region* of each species. Let θ be a phase difference between heterogeneity of environments of u and v, that is, we assume that $c(x)$ and $e_2(x)$ are represented as forms of $c(x)=c^*(x-\theta)$ and $e_2(x)=e^*(x-\theta)$, respectively, for some functions $c^*(x)$, $e^*(x)$ of $C^2(\mathbf{R}; \mathbf{R})$ with period 1. Then, we can regard θ as a parameter denoting the degree of overlaps between favorable regions of u and v. We are interested in the possibility of coexistence of u and v by changing only θ and fixing all coefficients of (5.2) except for θ. To analyze such problems of (5.2), we consider the O.D.E. corresponding to (1.9), which is of the following form

$$(5.3) \qquad \begin{cases} \dfrac{d}{dT} N_1 = F(N_1, N_2) \\ \qquad\qquad\qquad\qquad\qquad T > 0, \\ \dfrac{d}{dT} N_2 = G(N_1, N_2) \end{cases}$$

with $N_1(0) = \int_I u_0(x)dx > 0$, $N_2(0) = \int_I v_0(x)dx > 0$, where $F(N_1, N_2) = \{(1 - N_1/K)(1+RN_1) - b_\theta N_2\}N_1$ and $G(N_1, N_2) = \{-C + h_\theta N_1\}N_2$ for some positive constants K, R, C and b_θ, h_θ. Only b_θ and h_θ depend on the phase difference θ. Behaviors of solutions of (5.3) have been completely analyzed as follows (Cheng [2]):

Case 1. If $C/h_\theta > K$, then the equilibrium $(K, 0)$ of (5.3) is the only one stable attractor of (5.3);

Case 2. if $1/2(K-1/R) < C/h_\theta < K$, then the equilibrium (N_1^+, N_2^+) of (5.3) with positive components is the only one stable attractor of (5.3);

Case 3. if $0 < C/h_\theta < 1/2(K-1/R)$, then there uniquely exists a stable limit cycle γ, which is the only one stable attractor of (5.3).

It is also known that a unique stable attractor in each case is exponentially asymptotically stable and that almost all orbits of (5.3) converge to the stable attractor, so that assumptions of Theorem 4.2 or Corollary 4.1 are satisfied. Above Case 1 ecologically means the extinction of the predator v. Cases 2 and 3 imply stationary coexistence and periodic coexistence of u and v, respectively.

We are interested in the possibility of the occurrence of each case. If either $e_1(x)$ or $e_2(x)$ is constant, then it follows directly that both b_θ and h_θ are independent of θ, in other word, the fortune of the species is determined independently of θ when ε is sufficiently small.

We next consider the following case in which $e_i(x)$ $(i=1, 2)$, $k(x)$ and $c(x)$ are non-constant:

$$d_i = 1 \quad (i=1, 2); \quad k(x) = 8((8/9)\sin 2\pi x + 1); \quad e_1(x) = -2\log k(x); \quad r = 2;$$

$$b = 1; \quad c(x) = \frac{3}{(8/9)\sin 2\pi(x-\theta)+1} \quad e_2(x) = 2\log c(x); \quad h = 1.$$

$e_i(x)$ $(i=1, 2)$ have effects that u and v move toward the higher place of $k(x)$ and the lower place of $c(x)$, respectively. These are ecologically plausible assumptions. Figure 1 is the graph of C/h_θ. Note that it suffices to restrict θ within $0 \leq \theta \leq 0.5$. Figure 1 indicates that there exist θ_1, θ_2 $(0<\theta_1<\theta_2<0.5)$ such that the ranges of $\theta_2<\theta\leq 0.5$, $\theta_1<\theta<\theta_2$ and $0\leq\theta<\theta_1$ correspond to Cases 1～3, respectively. In Figures 2–1～2–3, evolutional behaviors of solutions of (5.2) and (5.3) in each case are described. In Figure 3, orbits of $\left(\int_I u(t, x; \varepsilon)dx, \int_I v(t, x; \varepsilon)dx\right)$ and $(N_1(\varepsilon t), N_2(\varepsilon t))$ are drawn in phase plane $\mathbf{R}^2$, which indicates that these two orbits are *orbitally* close. Figures 3 and 4 denote that solutions of (5.2) in Case 3) are nearly periodic.

Thus, the relation of positions of favorable regions can play a much important role for coexistence of species, which is the problem appearing not until we take

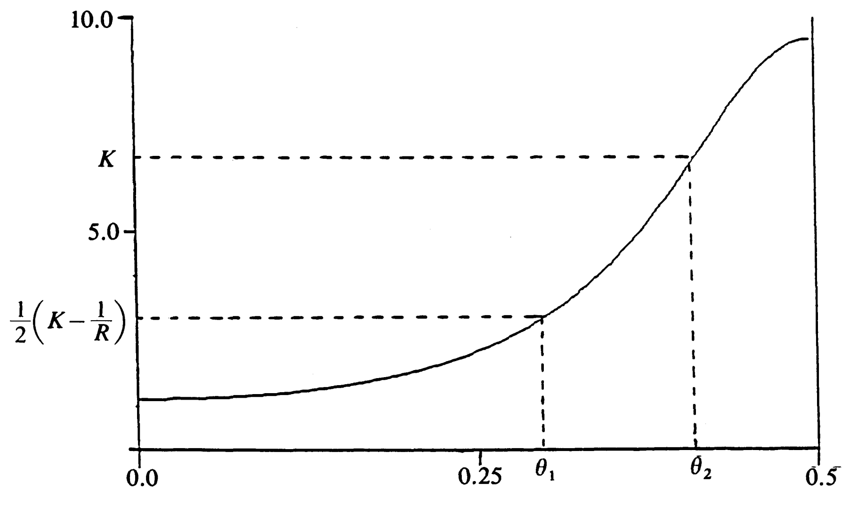

Figure 1; The graph of $\frac{C}{h_\theta}$ for $\theta \in [0, 0.5]$.

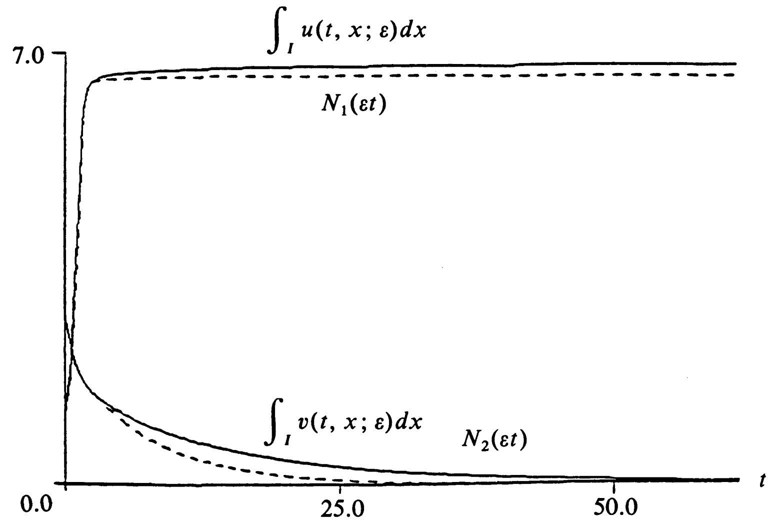

Figure 2-1; Evolutional behaviors of $\int_I u(t, x; \varepsilon)dx$, $\int_I v(t, x; \varepsilon)dx$ and $N_1(\varepsilon t)$, $N_2(\varepsilon t)$ with $\theta=0.5$ and $\varepsilon=0.1$ (Case 1). Solid lines denote $\int_I u(t, x; \varepsilon)dx$ and $\int_I v(t, x; \varepsilon)dx$ and broken ones do $N_1(\varepsilon t)$ and $N_2(\varepsilon t)$.

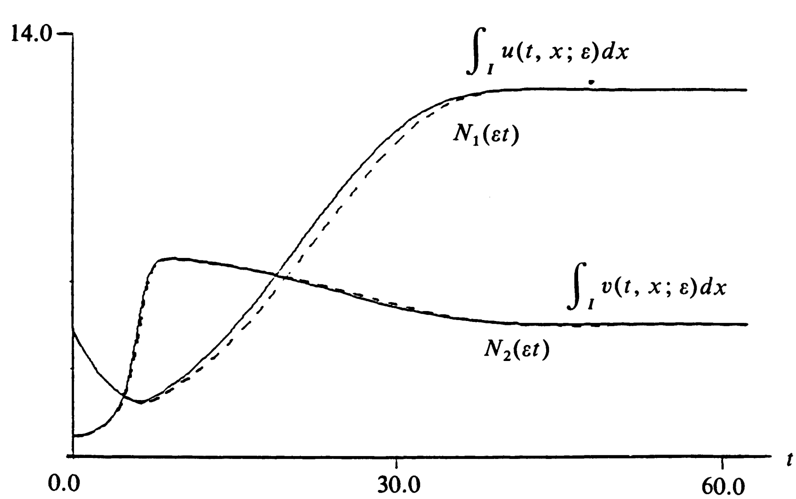

Figure 2-2; Evolutional behaviors of $\int_I u(t, x; \varepsilon)dx$, $\int_I v(t, x; \varepsilon)\,dx$ and $N_1(\varepsilon t)$, $N_2(\varepsilon t)$ with $\theta=0.35$ $(\theta_1<\theta<\theta_2)$ and $\varepsilon=0.1$ (Case 2). Solid lines denote $\int_I u(t, x; \varepsilon)dx$ and $\int_I v(t, x; \varepsilon)dx$ and broken ones do $N_1(\varepsilon t)$ and $N_2(\varepsilon t)$.

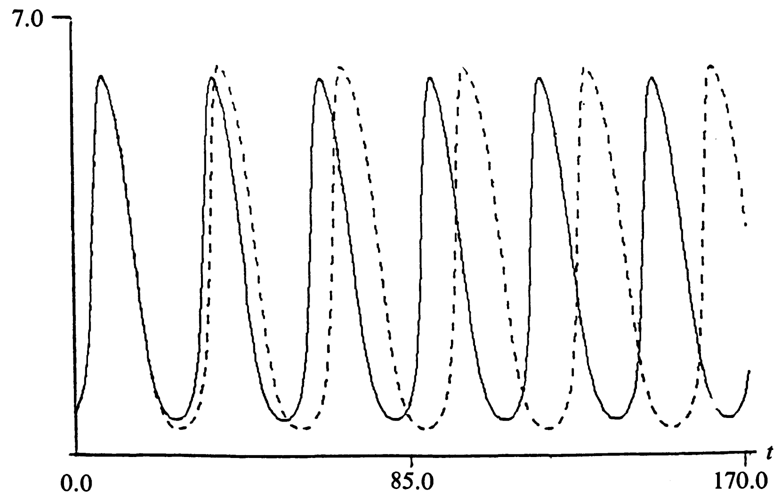

Figure 2–3; Evolutional behaviors of $\int_I u(t, x; \varepsilon)dx$ and $N_1(\varepsilon t)$ with $\theta=0.28$ $(0<\theta<\theta_1)$ and $\varepsilon=0.1$ (Case 3). Solid line denotes $\int_I u(t, x; \varepsilon)dx$ and broken one does $N_1(\varepsilon t)$.

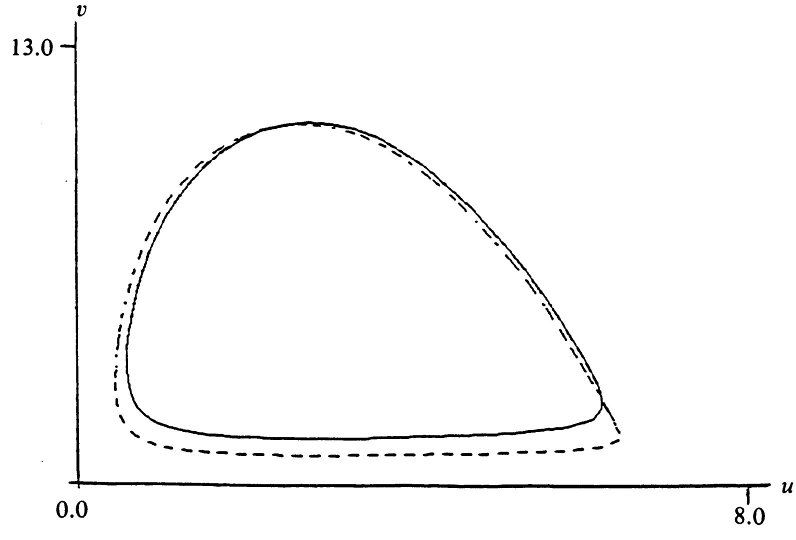

Figure 3; Orbits described by $\left(\int_I u(t, x; \varepsilon)dx, \int_I v(t, x; \varepsilon)dx\right)$ and $(N_1(\varepsilon t), N_2(\varepsilon t))$with $\theta=0.28$ and $\varepsilon=0.1$ (Case 3). A solid line denotes the orbit described by $\left(\int_I u(t, x; \varepsilon)dx, \int_I v(t, x; \varepsilon)dx\right)$and broken one does the orbit described by $(N_1(\varepsilon t), N_2(\varepsilon t))$.

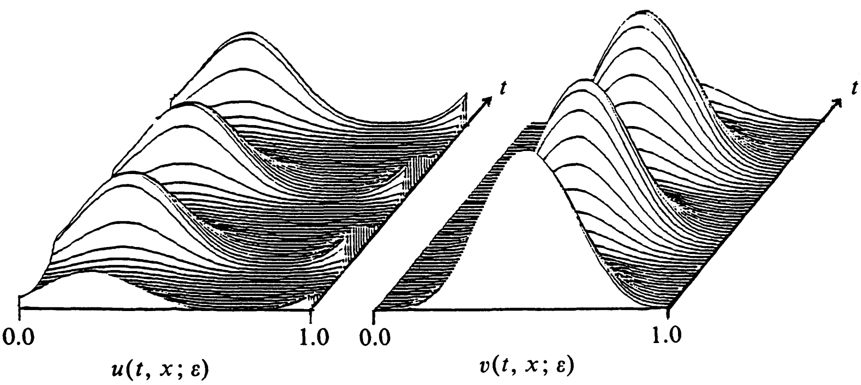

Figure 4; Evolutional behaviors of $u(t, x; \varepsilon)$ and $v(t, x; \varepsilon)$ with $\theta = 0.28$ and $\varepsilon = 0.1$ (Case 3).

account of the environmental heterogeneity. More detailed consideration in various ecological models will be stated in Ei, Mimura and Yamanoue [8].

EXAMPLE 2. On the validity of the *lumped parameter assumption*.

Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with reasonably smooth boundary $\partial\Omega$ and ν be the outward normal vector on $\partial\Omega$. We consider the following systems more general than (1.11);

$$\frac{\partial u}{\partial t} = D\Delta u + \varepsilon f(\nabla u, u), \quad x \in \Omega, \quad t > 0 \tag{5.4}$$

with Neumann boundary conditions on $\partial\Omega$ and the initial condition

$$u(0, x) = u_0(x) \in C^2(\bar{\Omega}; \mathbf{R}^n), \quad x \in \Omega, \tag{5.5}$$

where $u = {}^t(u_1, \ldots, u_n)$, D is a positive definite constant matrix and $f \in C^2(\mathbf{R}^{mn} \times \mathbf{R}^n; \mathbf{R}^n)$. Let the Banach space B be $C(\bar{\Omega}; \mathbf{R}^n)$ with sup-norm and A be $-D\Delta$ with domain $\{u \in B \cap W^{2,p}(\Omega; \mathbf{R}^n); D\Delta u \in B$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ for some $p > m\}$. Then, it is easily seen that Assumptions A1), A2) are satisfied with $B_1 = \ker A = \mathbf{R}^n$ and $Qu = (meas.\ \Omega)^{-1} \int_\Omega u(x)dx$ for $u \in B$ (e.g. Stewart [46]) and that B^α is imbedded into $C^1(\bar{\Omega}; \mathbf{R}^n)$ for some $1/(2 - m/p) < \alpha < 1$ (e.g. Friedman [13]), which implies that when we define $F(u) = f(\nabla u, u)$ for $u \in B^\alpha$, F satisfies the Assumption A3). A4) obviously holds for (5.5).

Now, we consider the proposition that the asymptotic behavior of solutions of (5.4) and (5.5) is determined by the dynamics of the O.D.E. of the form;

$$\frac{dc}{dt} = \varepsilon f(0, c), \quad t > 0,$$

or equivalently,

$$\frac{dc}{dT} = f(0, c), \quad T > 0, \tag{5.6}$$

where $c = {}^t(c_1, \ldots, c_n)$ and $T = \varepsilon t$, which is called the *lumped parameter assumption* by Conway, Hoff and Smoller [5]. (5.6) is just the O.D.E. corresponding to (4.1), that is, (5.6) is rewritten as

$$\frac{dc}{dT} = QF(c), \quad T > 0.$$

Instead of the assumption in [5] that (5.4) admits a compact invariant set in $\mathbf{R}^n$, we assume the existence of an open bounded set $V \subset B_1$ $(=\mathbf{R}^n)$ with following property:

(5.7) V is a positively invariant open bounded set of (5.6) and there exists $\delta_0 > 0$, $T_0 > 0$ such that $\mathrm{dist}(\pi(T_0; \bar{V}), \partial V) \geq \delta_0$, where $\pi(T; v)$ is a solution of (5.6) with $\pi(0; v) = v \in B_1$.

Define $V_{T_1} = \{v \in B_1;\ \pi(T_1; v) \in V\}$ for any fixed $T_1 > 0$. Then, it follows from the assertion of (6.10) in Section 6 that when $Qu_0 \in V_{T_1}$, there exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that $|Qu(t, \cdot\,; \varepsilon)| \leq C_0$ for any $t \geq 0$ and any $0 < \varepsilon \leq \varepsilon_0$, where $u(t, x; \varepsilon)$ is a solution of (5.4), (5.5). Moreover, F maps $\mathbf{R}^n$ into itself. Therefore, we see from Corollary 4.2 that there exist $\tilde{a} > 0$, $\varepsilon_1 > 0$ and $C_1 > 0$ such that

$$\|Pu(t, \cdot\,; \varepsilon)\|_\alpha \leq C_1 e^{-\tilde{a}t} \|Pu_0\|_\alpha \tag{5.8}$$

and that $Qu(t, \cdot\,; \varepsilon)$ satisfies

$$\frac{1}{\varepsilon}\frac{d}{dt} Qu(t, \cdot\,; \varepsilon) = QF(Qu(t, \cdot\,; \varepsilon)) + g_\varepsilon(t) \tag{5.9}$$

for any $t \geq 0$, any $\varepsilon \in (0, \varepsilon_1]$ and some $g_\varepsilon(t)$ with $|g_\varepsilon(t)| \leq C_1 e^{-\tilde{a}t}$. (5.8) indicates that any solution of (5.4), (5.5) whose spatial average $Qu(t, \cdot\,; \varepsilon)$ lies in V_{T_1} converges exponentially to the spatial average itself. (5.9) corresponds to (1.13) and show that the asymptotic behavior of solutions of (5.4) and (5.5) is subject to the dynamics of (5.6).

Thus, the lumped parameter assumption is interpreted as one of assertions for validity of two-timing approximations (see Section 3).

6. Proofs

Throughout this section, we assume that $0<\varepsilon\leq 1$.

6.1. Proof of Theorem 4.1

We simply write T_1 as T and several constants depending on T (or independent of T) as the same symbol C_T (or C).

Transforming (2.1) by $v(t;\varepsilon)=u(t;\varepsilon)-u^0(t,\varepsilon t)$, we have

$$\begin{cases} \dfrac{dv}{dt}+Av=\varepsilon\{F(v+u^0(t,\varepsilon t))-QF(\pi(\varepsilon t;Qu_0))\}, & t>0,\\ v(0)=0, \end{cases}$$

which is written as

$$\begin{cases} \dfrac{dv}{dt}+\{A-\varepsilon F'(u^0(t,\varepsilon t))\}v=\varepsilon\{F(u^0(t,\varepsilon t)) \\ \qquad\qquad -QF(\pi(\varepsilon t;Qu_0))+N(t,v;\varepsilon)\}, \quad t>0,\\ v(0)=0, \end{cases} \tag{6.1}$$

where

$$N(t,v;\varepsilon)=F(v+u^0(t,\varepsilon t))-F(u^0(t,\varepsilon t))-F'(u^0(t,\varepsilon t))v. \tag{6.2}$$

Denote by $X(t,\tau;\varepsilon)$ the solution of the operator equation

$$\begin{cases} \dfrac{dX}{dt}+\{A-F'(u^0(t,\varepsilon t))\}X=0, & t>\tau,\\ X(\tau,\tau)=Id, \end{cases} \tag{6.3}$$

where Id is the identity on B. Then (6.1) is reduced to

$$v(t;\varepsilon)=\varepsilon\int_0^t X(t,s;\varepsilon)\{F(u^0(s,\varepsilon s))-QF(\pi(\varepsilon s;Qu_0)) + N(s,v(s;\varepsilon);\varepsilon)\}ds. \tag{6.4}$$

Lemma 6.1. *Suppose M_1, $M_2\geq 0$, $0\leq\alpha$, $\beta<1$ and $w(t)$ is nonnegative and locally integrable with*

$$w(t)\leq\frac{M_1}{t^\alpha}+M_2\int_0^t\frac{1}{(t-s)^\beta}w(s)ds$$

on $0\leq t<T$ Then, there is a constant $C_T>0$ depending on T such that $w(t)\leq C_TM_1/t^\alpha$.

PROOF. One can find a proof of the lemma in Henry [16, Chapter 7, Lemma 7.1.1]. So we omit the proof. ▮

LEMMA 6.2. *There exists* $C_T>0$ *such that*

$$\|X(t, \tau; \varepsilon)\|_{\mathscr{L}(X,X^\alpha)} \leq C_T\left(1+\frac{1}{(t-\tau)^\alpha}\right)$$

for $0\leq\tau<t\leq T/\varepsilon$.

PROOF. Now, we have for $b\in B$,

$$\|e^{-tA}b\| = \|e^{-tA}Qb\|_\alpha + \|e^{-tA}Pb\|_\alpha = \|Qb\|_\alpha + \|e^{-tA}Pb\|_\alpha$$
$$\leq C\|b\| + \frac{Ce^{-at}}{t^\alpha}\|b\| \leq C\left(1+\frac{1}{t^\alpha}\right)\|b\|$$

for some $C>0$ and

$$\|u^0(t, \varepsilon t)\|_\alpha \leq |\pi(\varepsilon t; Qu_0)| + \|e^{-tA}Pu_0\|_\alpha \leq Cb_T(u_0)$$

for some $C>0$, so that $\|F'(u^0(t, \varepsilon t))\|_{\mathscr{L}(B^\alpha,B)}\leq C_T$ holds for some $C_T>0$ and for $0\leq t\leq T/\varepsilon$. We have also from (6.3),

$$X(t, \tau; \varepsilon)b = e^{-(t-\tau)A}b + \varepsilon\int_\tau^t e^{-(t-s)A}F'(u^0(s, \varepsilon s))X(s, \tau; \varepsilon)bds \tag{6.5}$$

for $b\in B$ and $0\leq\tau<t\leq T/\varepsilon$. Hence,

$$\|X(t, \tau; \varepsilon)b\|_\alpha \leq C\left(1+\frac{1}{(t-\tau)^\alpha}\right)\|b\| + C_T\varepsilon\int_\tau^t\left(1+\frac{1}{(t-s)^\alpha}\right)$$
$$\times\ \|X(s, \tau; \varepsilon)b\|_\alpha ds$$
$$= C\left(1+\frac{1}{(t-\tau)^\alpha}\right)\|b\| + C_T\int_{\varepsilon\tau}^{\varepsilon t}\left(1+\frac{\varepsilon^\alpha}{(\varepsilon t-s)^\alpha}\right)\|x(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds$$
$$\leq C\left(1+\frac{1}{(t-\tau)^\alpha}\right)\|b\| + C_T\int_{\varepsilon t}^{\varepsilon t}\frac{T^\alpha+\varepsilon^\alpha}{(\varepsilon t-s)^\alpha}\|X(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds$$

for some $C>0$ and $C_T>0$. If $0\leq t-\tau\leq 1$, then $(t-\tau)^{-\alpha}\geq 1$ holds and we have

$$\|X(t, \tau; \varepsilon)b\|_\alpha \leq \frac{2C\varepsilon^\alpha}{(\varepsilon t-\varepsilon\tau)^\alpha}\|b\| + C_T\int_{\varepsilon\tau}^{\varepsilon t}\frac{1}{(\varepsilon t-s)^\alpha}\|X(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds$$

for some $C_T>0$, which implies by Lemma 6.1,

$$\|X(t, \tau; \varepsilon)b\|_\alpha \leq C'_T\frac{2C\varepsilon^\alpha}{(\varepsilon t-\varepsilon\tau)^\alpha}\|b\| \leq C''_T\frac{1}{(t-\tau)^\alpha}\|b\|$$

for some C'_T and C''_T (depending on T). If $t-\tau\geq 1$, then $(t-\tau)^{-\alpha}\leq 1$ holds and we have

$$
\begin{aligned}
\|X(t, \tau; \varepsilon)b\|_\alpha &\leq 2C\|b\| + C_T\int_{\varepsilon\tau}^{\varepsilon t}\frac{1}{(\varepsilon t-s)^\alpha}\|X(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds \\
&\leq 2C\|b\| + C_T\int_{\varepsilon\tau}^{\varepsilon(\tau+1)}\frac{1}{(\varepsilon t-s)^\alpha}\times\frac{C''_T}{(s/\varepsilon-\tau)^\alpha}\|b\|ds \\
&\quad + C_T\int_{\varepsilon(\tau+1)}^{\varepsilon t}\frac{1}{(\varepsilon t-s)^\alpha}\|X(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds \\
&\leq C'''_T\|b\| + C_T\int_{\varepsilon(\tau+1)}^{\varepsilon t}\frac{1}{(\varepsilon t-s)^\alpha}\|X(s/\varepsilon, \tau; \varepsilon)b\|_\alpha ds
\end{aligned}
$$

for some C'''_T (depending on T). Hence by Lemma 6.1, we have

$$\|X(t, \tau; \varepsilon)b\|_\alpha \leq C_T\|b\|$$

for some C_T and $0<\tau+1\leq t\leq T/\varepsilon$. Consequently, we have

$$\|X(t, \tau; \varepsilon)b\|_\alpha \leq C_T\left(1+\frac{1}{(t-\tau)^\alpha}\right)\|b\|$$

for $0\leq\tau<t\leq T/\varepsilon$, $b\in B$ and some $C_T>0$. ∎

LEMMA 6.3. *There exist C and $C_T>0$ such that*

$$\|X(t, \tau; \varepsilon)P\|_{\mathscr{L}(B,B^\alpha)} \leq \frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha} + \varepsilon C_T$$

for $0\leq\tau<t\leq T/\varepsilon$.

PROOF. By (6.5), we have

$$
\begin{aligned}
\|PX(t, \tau; \varepsilon)Pb\|_\alpha &\leq \frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha}\|b\| \\
&\quad + \varepsilon\int_\tau^t\frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha}\|F'(u^0(s; \varepsilon s))\|_{\mathscr{L}(B^\alpha,B)}\|X(s, \tau; \varepsilon)\|_\alpha ds
\end{aligned}
$$

for $b\in B$ and $0\leq\tau<t\leq T/\varepsilon$. Since it follows from Lemma 6.2 that $\|X(s, \tau; \varepsilon)Pb\|_\alpha\leq C_T(1+(s-\tau)^{-\alpha})\|b\|$ and $\|F'(u^0(s, \varepsilon s))\|_{\mathscr{L}(B^\alpha,B)}\leq C_T$ for $0\leq s\leq T/\varepsilon$ and for some C_T and that $\int_\tau^t e^{-a(t-\tau)}(t-s)^{-\alpha}(1+(s-\tau)^{-\alpha})ds\leq C$ for $0\leq\tau<t<+\infty$ and some C, we have

$$\|PX(t, \tau; \varepsilon)Pb\|_\alpha\leq\frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha}\|b\| + \varepsilon C_T\|b\| \tag{6.6}$$

for $0\leq\tau<t\leq T/\varepsilon$ and some C, C_T. Since $Qe^{-tA}Q=Q$ and $PQ=QP=0$ hold, (6.5) gives

$$QX(t,\tau;\varepsilon)Pb = \varepsilon\int_\tau^t QF'(u^0(s,\varepsilon s))QX(s,\tau;\varepsilon)Pbds + \varepsilon\int_\tau^t QF'(u^0(s,\varepsilon s))PX(s,\tau;\varepsilon)Pbds.$$

Hence it follows from (6.6) that

$$\begin{aligned}\|QX(t,\tau;\varepsilon)Pb\|_\alpha &\le C_T\varepsilon\int_\tau^t \|QX(s,\tau;\varepsilon)Pb\|_\alpha ds + \varepsilon C_T\int_\tau^t \|PX(s,\tau;\varepsilon)Pb\|_\alpha ds\\ &\le C_T\varepsilon\int_\tau^t \|QX(s,\tau;\varepsilon)Pb\|_\alpha ds + \varepsilon C'_T\end{aligned}$$

for some C_T and C'_T, so that, by Gronwall's inequality, we have

$$\|QX(t,\tau;\varepsilon)Pb\|_\alpha \le C_T\varepsilon\|b\| \tag{6.7}$$

for some C_T. Consequently from (6.6), (6.7) we have

$$\begin{aligned}\|X(t,\tau;\varepsilon)Pb\|_\alpha &\le \|QX(t,\tau;\varepsilon)Pb\|_\alpha + \|PX(t,\tau;\varepsilon)Pb\|_\alpha\\ &\le C_T\varepsilon\|b\| + \left(\frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha} + \varepsilon C_T\right)\|b\|\\ &\le \left(\frac{Ce^{-a(t-\tau)}}{(t-\tau)^\alpha} + \varepsilon C'_T\right)\|b\|\end{aligned}$$

for some C'_T, which completes the proof. ∎

The equation (6.4) is rewritten as

$$v(t;\varepsilon) = H_\varepsilon(v)(t), \tag{6.8}$$

where

$$H_\varepsilon(v)(t) = \varepsilon U(t;\varepsilon) + \varepsilon\int_0^t X(t,s;\varepsilon)N(s,v(s;\varepsilon);\varepsilon)ds$$

and

$$U(t;\varepsilon) = \int_0^t X(t,s;\varepsilon)\{F(u^0(s;\varepsilon s)) - QF(\pi(\varepsilon s;Qu_0))\}ds.$$

Hence, it suffices to show that H_ε has a unique solution $v(t,\varepsilon)$ such that $\|v(t;\varepsilon)\| \le O(\varepsilon)$ uniformly for $t\in[0,T/\varepsilon]$.

LEMMA 6.4. *There exists $C_T>0$ such that $\|U(t;\varepsilon)\|_\alpha \le C_T$ for $0\le t\le T/\varepsilon$.*

PROOF. It follows from Lemmas 6.2 and 6.3 that

$$\|U(t;\varepsilon)\|_\alpha \le \int_0^t \|X(t,s;\varepsilon)P\|_{\mathscr{L}(B,B^\alpha)}\|PF(u^0(s,\varepsilon s))\|ds$$
$$+\int_0^t \|X(t,s;\varepsilon)Q\|_{\mathscr{L}(B,B^\alpha)}\|QF(u^0(s,\varepsilon s))-QF(\pi(\varepsilon s;Qu_0))\|ds$$
$$\le C_T\int_0^t\left(\frac{Ce^{-a(t-s)}}{(t-s)^\alpha}+\varepsilon\right)ds$$
$$+C_T\int_0^t\left(1+\frac{1}{(t-s)^\alpha}\right)\int_0^1\|QF'(\pi(\varepsilon s;Qu_0)+\theta e^{-sA}Pu_0)\|_{\mathscr{L}(B^\alpha,B)}d\theta$$
$$\times\ \|e^{-sA}Pu_0\|_\alpha ds$$
$$\le C'_T+C'_T\int_0^t\left(1+\frac{1}{(t-s)^\alpha}\right)e^{-as}ds$$
$$\le C''_T$$

for $0\le t\le T/\varepsilon$ and for some C_T, C'_T and C''_T depending on T. This shows the result. ∎

We now consider the equation (6.8). Let $C([0, T/\varepsilon]; B^\alpha)$ be the Banach space of all bounded continuous functions from $[0, T/\varepsilon]$ into B^α with the norm $|||v|||_T=\sup_{0\le t\le T/\varepsilon}\|v(t)\|_\alpha$ and define $V_r=\{v\in C([0, T/\varepsilon]; B^\alpha); |||v|||_T\le r\}$ for some $r\ge 0$. Then it follows from Lemmas 6.2 and 6.4 that

$$\|H_\varepsilon(v)(t)\|\le \varepsilon C_T+\varepsilon\int_0^t C_T\left(1+\frac{1}{(t-s)^\alpha}\right)\|v(s)\|_\alpha^2 ds$$
$$\le \varepsilon C_T+\varepsilon C_T\int_0^t\left(1+\frac{1}{(t-s)^\alpha}\right)ds|||v|||_T^2$$
$$\le C'_T(\varepsilon+|||v|||_T^2)$$

for some C_T, C'_T and $0\le t\le T/\varepsilon$. Thus, we have $|||H_\varepsilon(v)|||_T\le C_T(\varepsilon+|||v|||_T^2)$ for some C_T. If we define $\varepsilon_T=\min\{1/(4C_T)^2, r/(2C_T)\}$, then it turns out that for any $\varepsilon\in(0,\varepsilon_T]$, $V_{2C_T\varepsilon}\subset V_r$ and H_ε maps $V_{2C_T\varepsilon}$ into itself. Moreover, we find that

$$|||H_\varepsilon(v_1)-H_\varepsilon(v_2)|||_T\le C'_T\varepsilon v|||v_1-v_2|||_T$$

for some C'_T and v_1, $v_2\in V_{2C_T\varepsilon}$. Consequently there exist $\varepsilon_T>0$ and $C_T>0$ such that H_ε is a contraction on $V_{C_T\varepsilon}$ for any $\varepsilon\in(0,\varepsilon_T]$, which implies that H_ε has a unique solution $v(t;\varepsilon)$ satisfying $|||v|||_T\le C_T\varepsilon$. The proof of Theorem 4.1 is complete. ∎∎

6.2. Proof of Theorem 4.2

LEMMA 6.5. *Γ has an open neighborhood V with the following property*:
(6.9) *V is a positively invariant open bounded set in B_1, that is, $\pi(T; V)\subset V$*

for any $T>0$, *and there exist* $\delta_0>0$, $T_0>0$ *such that* $\operatorname{dist}(\pi(T_0; \bar{V}), \partial V) \geq \delta_0$.

PROOF. Since Γ is exponentially asymptotically stable, there exist constants $r_0>0$, $M_0>1$ and $\beta>0$ such that $\operatorname{dist}(\pi(t; V), \Gamma) \leq M_0 e^{-\beta T} \operatorname{dist}(v, \Gamma)$ for any $v \in U_{r_0}(\Gamma)$ and any $T>0$, where $U_r(E)=\{x \in B_1;\ |x-y|<r$ for any $y \in E\}$ if E is a set in B_1 and r is a non-negative constant. Fix δ_0 satisfying $0<2\delta_0<r_0/M_0$ and define $V=\cup_{T\geq 0}\pi(T; U_{2\delta_0}(\Gamma))$. Then, we find V is a positively invariant open bounded set satisfying $U_{2\delta_0}(\Gamma) \subset V \subset U_{r_0}(\Gamma)$. If we choose a constant T_0 such that $M_0 r_0 e^{-\beta T_0}<\delta_0$, then we see that $\operatorname{dist}(\pi(T_0; v), \Gamma) \leq M_0 e^{-\beta T_0} \operatorname{dist}(v, \Gamma) \leq M_0 r_0 e^{-\beta T_0} \leq \delta_0$ for any $v \in V$, which implies $\operatorname{dist}(\pi(T_0; \bar{V}), \partial V) \geq \delta_0$. ∎

Keeping the result of Lemma 6.5 in mind, we show the following assertion as the first step of this proof:

(6.10) *Suppose that* V *has the property* (6.9) *and* $\pi(T_1; Qu_0) \in V$ *for some* $T_1>0$. *Then, there exist* $\varepsilon_0>0$ *and* $C>0$ *such that for any* $0<\varepsilon\leq\varepsilon_0$ *we have* $Qu(t; \varepsilon) \in U_{C\varepsilon}(V)$ *for any* $t \geq T_1/\varepsilon$ *and* $\|Pu(t; \varepsilon)-e^{-tA}u_0\|_\alpha \leq C\varepsilon$ *for any* $t>0$. *Moreover* $Qu(t; \varepsilon)$ *satisfies*

$$\frac{1}{\varepsilon}\frac{d}{dt} Qu(t; \varepsilon) = QF(Qu(t; \varepsilon)) + g_\varepsilon(\varepsilon t)$$

for some g_ε *and any* $t \geq T_1/\varepsilon$ *with* $|g_\varepsilon(\varepsilon t)| \leq C\varepsilon$.

[PROOF of (6.10)]. Let $\varphi=Qu(t; \varepsilon)$ and $w=Pu(t; \varepsilon)$. Then φ and w satisfy

$$\begin{cases} \dfrac{d\varphi}{dT} = QF(\varphi+w), \\ \varphi(0) = Qu_0, \end{cases} \tag{6.11}$$

where $T=\varepsilon t$, and

$$\begin{cases} \dfrac{dw}{dt} + Aw = \varepsilon F(\varphi+w), \\ w(0) = Pu_0. \end{cases} \tag{6.12}$$

Hereafter, let the argument of φ be $T\,(=\varepsilon t)$ and that of w be t, namely, we write $\varphi=\varphi(T; \varepsilon)$ and $w=w(t; \varepsilon)$. From assumptions of (6.10), there exist $\delta_0>0$ and $T_0>0$ such that $\operatorname{dist}(\pi(T_0; \bar{V}), \partial V) \geq \delta_0$. Moreover, we can assume that $\operatorname{dist}(\pi(T_1; Qu_0), \partial V) \geq \delta_0$ and $\pi(T; Qu_0) \in V$ for any $T \geq T_1$.

Theorem 4.1 implies that there exist $\varepsilon_1=\varepsilon_T$ and $C_1=C_{T_1}>0$ such that

$$|\varphi(T; \varepsilon)-\pi(T; Qu_0)| \leq C_1\varepsilon, \tag{6.13}$$

$$\|w(t; \varepsilon)-e^{-tA}Pu_0\|_\alpha \leq C_1\varepsilon \tag{6.14}$$

for any $0<\varepsilon\leq\varepsilon_1$ and any $0\leq T\leq T_1$. From (6.13), we obtain

$$(6.15)\qquad |\varphi(T;\,\varepsilon)| \leq C_1\varepsilon + |\pi(T;\,Qu_0)| \leq C_2$$

for some $C_2>0$, any $0<\varepsilon\leq\varepsilon_1$ and any $0\leq T\leq T_1$. Let $K_1=\overline{U_1(V)}$. Here, we define several constants as follows:

M_0 is the constant satisfying $\|e^{-tA}P\|_{\mathscr{L}(B,B^\alpha)} \leq M_0e^{-at}t^{-\alpha}$ and

$\|e^{-tA}P\|_{\mathscr{L}(B^\alpha)} \leq M_0e^{-at}$;

M_1 is the constant satisfying $b(u_0) \leq M_1$;

$M_2 = \max\{C_2, \sup_{x\in K_1}|x|\}$;

$M_3 = \sup\{\|x+y\|_\alpha;\ |x|\leq M_2,\ \|y\|_\alpha\leq 1+M_0M_1 \text{ for } x\in B_1 \text{ and } y\in B_2^\alpha\}$;

$M_4 = \sup\{\|F(x+y)\|;\ |x|\leq M_2,\ \|y\|_\alpha\leq 1+M_0M_1 \text{ for } x\in B_1 \text{ and } y\in B_2^\alpha\}$;

$M_5 = M_0M_4\int_0^\infty \frac{e^{-as}}{s^\alpha}ds\left(=M_0M_4\times\lim_{t\to\infty}\int_0^t \frac{e^{-a(t-s)}}{(t-s)^\alpha}ds\right)$;

$M_6 = \sup\{\|F'(x+y)\|_{\mathscr{L}(B^\alpha,B)};\ |x|\leq M_2 \text{ and } \|y\|_\alpha\leq 1+M_0M_1\}$;

$M_7 = M_0M_5\int_0^\infty \frac{e^{-as}}{s^\alpha}ds.$

We also define $I_\varepsilon(T')=\{T'';\ |\varphi(T;\,\varepsilon)|\leq M_2 \text{ for any } T'\leq T\leq T''\}$. Note that $I_\varepsilon(T_1)$ is a nonempty closed interval containing T_1.

Lemma 6.6. *There exists ε_2 $(0<\varepsilon_2\leq\varepsilon_1)$ such that*

$$\|w(t;\,\varepsilon)-e^{-tA}Pu_0\|_\alpha \leq M_5\varepsilon$$

for any $0<\varepsilon\leq\varepsilon_2$ and any $\varepsilon t\in I_\varepsilon(0)$.

Proof. Let $J_\varepsilon(v)(t)=e^{-tA}Pu_0+\varepsilon\int_0^t e^{-(t-s)A}PF(\varphi(\varepsilon s;\,\varepsilon)+v(s))ds$ for $v\in C(I_2^\varepsilon;\,B^\alpha)$, where $I_2^\varepsilon=\{t;\ \varepsilon t\in I_\varepsilon(0)\}$. Suppose that $v\in C(I_2^\varepsilon;\,B^\alpha)$ satisfies

$$(6.16)\qquad \|v(t)-e^{-tA}Pu_0\|_\alpha \leq M_5\varepsilon$$

for $t\in I_2^\varepsilon$. Then,

$$(6.17)\qquad \|J_\varepsilon(v)(t)-e^{-tA}Pu_0\|_\alpha \leq \varepsilon\int_0^t \frac{M_0e^{-a(t-s)}}{(t-s)^\alpha}\|F(\varphi(\varepsilon s;\,\varepsilon)+v(s))\|ds.$$

Now, it follows that $|\varphi(\varepsilon s;\,\varepsilon)|\leq M_2$ and $\|v(s)\|_\alpha\leq M_5\varepsilon+\|e^{-sA}Pu_0\|_\alpha\leq M_5\varepsilon_2'+M_0e^{-sA}\|Pu_0\|_\alpha\leq 1+M_0M_1$ for $s\in I_2^\varepsilon$, where $\varepsilon_2'=\min(\varepsilon_1,\,1/M_5)$, which implies $\|F(\varphi(\varepsilon s;\,\varepsilon)+v(s))\|\leq M_4$. Therefore, from (6.17),

$$\|J_\varepsilon(v)(t) - e^{-tA}Pu_0\|_\alpha \le \varepsilon \int_0^t \frac{M_0 e^{-a(t-s)}}{(t-s)^\alpha} M_4 ds$$
$$\le \varepsilon M_0 M_4 \int_0^\infty \frac{e^{-as}}{s^\alpha} ds = \varepsilon M_5$$

for any $0<\varepsilon\le\varepsilon_2'$ and any $t\in I_2^\varepsilon$. We also have

$$\|J_\varepsilon(v_1)(t) - J_\varepsilon(v_2)(t)\|_\alpha \le \varepsilon \int_0^t \frac{M_0 e^{-a(t-s)}}{(t-s)^\alpha}$$
$$\times \|F'(\varphi(\varepsilon s;\varepsilon)+\theta v_1(s)+(1-\theta)v_2(s))\|_{\mathscr{L}(B^\alpha,B)} d\theta ds$$
$$\times \sup_{t\in I_2^\varepsilon} \|v_1(t)-v_2(t)\|_\alpha$$
$$\le \varepsilon \int_0^t M_0 M_6 \frac{e^{-as}}{s^\alpha} ds \cdot \sup_{t\in I_2^\varepsilon} \|v_1(t)-v_2(t)\|_\alpha$$
$$= \varepsilon M_7 \sup_{t\in I_2^\varepsilon} \|v_1(t)-v_2(t)\|_\alpha$$

for any $t\in I_2^\varepsilon$ and $v_i\in C(I_2^\varepsilon; B^\alpha)$ satisfying (6.16) ($i=1, 2$). Consequently, fixing $0<\varepsilon_2<\min\{\varepsilon_2', 1/M_7\}$, we see that J_ε is a contraction on $\{v\in C(I_2^\varepsilon; B^\alpha);\ \|v(t)-e^{-tA}Pu_0\|_\alpha\le M_5\varepsilon\}$ for any $0<\varepsilon\le\varepsilon_2$ and that $w(t;\varepsilon)$ is a unique fixed point of J_ε, as required. ∎

LEMMA 6.7. *There exist C_3 and $\varepsilon_3>0$ such that $\varphi(T;\varepsilon)$ satisfies:*

$$\frac{d\varphi}{dT} = QF(\varphi) + g_\varepsilon(T) \tag{6.18}$$

for any $0<\varepsilon\le\varepsilon_3$, any $T\in I_\varepsilon(T_1)$ and some $g_\varepsilon(T)$ with $|g_\varepsilon(T)|\le C_3\varepsilon$.

PROOF. From Lemma 6.6 we have $\|w(T/\varepsilon;\varepsilon)\|_\alpha\le 1+M_0M_1$ and $|\varphi(T;\varepsilon)|\le M_2$ for any $T\in I_\varepsilon(T_1)$, which implies $\|F'(\varphi(T;\varepsilon)+w(T/\varepsilon;\varepsilon))\|_{\mathscr{L}(B^\alpha,B)}\le M_6$. Hence, it follows that

$$|QF(\varphi(T;\varepsilon)+w(T/\varepsilon;\varepsilon)) - QF(\varphi(T;\varepsilon))| \tag{6.19}$$
$$\le C_3' \int_0^1 \|QF'(\varphi(T;\varepsilon)+\theta w(T/\varepsilon;\varepsilon))\|_{\mathscr{L}(B^\alpha,B)} d\theta \cdot \|w(T/\varepsilon;\varepsilon)\|_\alpha$$
$$\le C_3' M_6 \|w(T/\varepsilon;\varepsilon)\|_\alpha$$

for any $T\in I_\varepsilon(T_1)$ and some $C_3'>0$. It follows also from Lemma 6.6 that $\|w(T/\varepsilon;\varepsilon)\|_\alpha\le M_5\varepsilon+M_0M_1e^{-aT/\varepsilon}\le M_5\varepsilon+M_0M_1e^{-aT_1/\varepsilon}\le(M_5+M_0M_1)\varepsilon$ for any $0<\varepsilon\le\varepsilon_3$ and any $T\in I_\varepsilon(T_1)$, where $\varepsilon_3=\sup\{0<\varepsilon\le\varepsilon_2;\ aT_1\ge\varepsilon\log(1/\varepsilon)\}$. Consequently, we see from (6.11) that $\varphi(T;\varepsilon)$ satisfies $d\varphi/dT=QF(\varphi)+g_\varepsilon(T)$ with $|g_\varepsilon(T)|\le C_3\varepsilon$, where $g_\varepsilon(T)=QF(\varphi(T;\varepsilon)+w(T/\varepsilon;\varepsilon))-QF(\varphi(T;\varepsilon))$ and $C_3=C_3'\times M_6(M_5+M_0M_1)$. ∎

LEMMA 6.8. *There exist* C_4 *and* $\varepsilon_4>0$ *such that if* $\varphi(T';\varepsilon)\in V$ *for some* $T'\geq T_1$, *then it follows that* $[T', T'+T_0]\subset I_\varepsilon(T')$ *and* $|\varphi(T;\varepsilon)-\pi(T-T';\varphi(T';\varepsilon))|\leq C_4\varepsilon\ (\leq 1)$ *for any* $T\in[T', T'+T_0]$ *and any* $0<\varepsilon\leq\varepsilon_4$.

PROOF. Fix T' ($T'\geq T_1$) satisfying the assumption of Lemma 6.8. Define $Y(T;\varepsilon)=\varphi(T;\varepsilon)-\pi(T-T';\varphi(T';\varepsilon))$. Since $T'\geq T_1$ and $|\varphi(T;\varepsilon)|\leq M_2$ on $I_\varepsilon(T')$, similarly to the proof of Lemma 6.7 it follows that φ satisfies (6.18) for any $0<\varepsilon\leq\varepsilon_3$ and any $T\in I_\varepsilon(T')$. That is,

$$(6.20)\qquad \begin{cases} \dfrac{dY}{dT}=QF(Y+\pi)-QF(\pi)+g_\varepsilon(T), \\ Y(T';\varepsilon)=0 \end{cases}$$

with $|g_\varepsilon(T)|\leq C_3\varepsilon$, where $\pi=\pi(T-T';\varphi(T';\varepsilon))$. Fix $0<\varepsilon_4<\min\{\varepsilon_3, 1/(C_3T_0\exp(M_6T_0))\}$ and suppose that there exists $T_2\in(T', T'+T_0]$ such that $|Y(T;\varepsilon)|<1$ for $T\in[T', T_2)$ and $|Y(T_2;\varepsilon)|=1$. Note that $[T', T_2]\subset I_\varepsilon(T')$ because $\pi\in V$ and $|Y(T;\varepsilon)|<1$, which implies $\varphi(T;\varepsilon)\in K_1$. Then from (6.20), we have

$$\begin{aligned}|Y(T;\varepsilon)| &\leq \int_{T'}^{T}\int_0^1\|QF'(\pi+\theta Y(s;\varepsilon))\|_{\mathscr{L}(B^\alpha,B)}d\theta|Y(s;\varepsilon)|ds+\int_{T'}^{T}|g_\varepsilon(d)|ds \\ &\leq \int_{T'}^{T}M_6|Y(s;\varepsilon)|ds+C_3\varepsilon(T-T') \\ &\leq M_6\int_{T'}^{T}|Y(s;\varepsilon)|ds+C_3T_0\varepsilon,\end{aligned}$$

so that by Gronwall's inequality,

$$|Y(T;\varepsilon)|\leq\varepsilon C_3T_0e^{M_6(T-T')}\leq\varepsilon C_3T_0e^{M_6T_0}<1$$

for any $0<\varepsilon\leq\varepsilon_4$ and any $T'\leq T\leq T_2$, which is contradictory to $|Y(T_2;\varepsilon)|=1$. Hence, $|Y(T;\varepsilon)|<1$ for any $0<\varepsilon\leq\varepsilon_4$ and any $T'\leq T\leq T'+T_0$. Consequently, we obtain $|Y(T;\varepsilon)|\leq C_4\varepsilon$ (<1) for any $0<\varepsilon\leq\varepsilon_4$ and any $T'\leq T\leq T'+T_0$, where $C_4=C_3T_0\exp(M_6T_0)$. ∎

Let $\varepsilon_5=\min\{\varepsilon_4, \delta_0/C_1, \delta_0/C_4\}$. Then it follows from (6.13) that $\varphi(T_1;\varepsilon)\in V$ for any $0<\varepsilon\leq\varepsilon_5$, so that by Lemma 6.8 we have

$$(6.21)\qquad |\varphi(T;\varepsilon)-\pi(T-T_1;\varphi(T_1;\varepsilon))|\leq C_4\varepsilon\leq\min\{\delta_0, 1\}$$

for any $0<\varepsilon\leq\varepsilon_5$ and any $T_1\leq T\leq T_1+T_0$. Since $\pi(T_0;\varphi(T_1;\varepsilon))\in V$ with $\mathrm{dist}(\pi(T_0;\varphi(T_1;\varepsilon)), \partial V)\geq\delta_0$, we see from (6.21) that $\varphi(T_1+T_0;\varepsilon)\in V$ and $\varphi(T;\varepsilon)\in U_{C_4\varepsilon}(V)\subset K_1$ for any $T_1\leq T\leq T_1+T_0$. Consequently, it follows from Lemma 6.8 inductively that $\varphi(T_1+nT_0;\varepsilon)\in V$ and $\varphi(T;\varepsilon)\in U_{C_4\varepsilon}(V)\subset K_1$ for any $n\in\mathbf{N}=\{1, 2, 3,\dots\}$, any $0<\varepsilon\leq\varepsilon_5$ and any $T_1\leq T\leq nT_0$, which shows $I_\varepsilon(T_1)=[T_1, \infty)$ and (6.10) [End of the proof of (6.10)].

Using (6.10) and Lemma 6.5, we continue the proof of Theorem 4.2.

It follows from the assumption of Theorem 4.2, Lemma 6.5 and (6.10) that there exist $T_1>0$, $E_0>0$ and $C>0$ such that $\pi(T_1; Qu_0)\in V$ and $\varphi(T;\varepsilon)\in U_{C\varepsilon}(V)\subset K_1$ for any $0<\varepsilon\leq\varepsilon_0$ and any $T\geq T_1$. We have also by Lemma 6.8

$$|\varphi(T;\varepsilon)-\pi(T-T';\varphi(T';\varepsilon))|\leq C\varepsilon \tag{6.22}$$

for any $0<\varepsilon\leq\varepsilon_0$, any $T'\geq T_1$ and any $T'\leq T\leq T'+T_0$. Here we can assume that T_0 satisfies $M_0\exp(-\beta T_0)<1/3$, where M_0 and β are constants such that dist $(\pi(T; v), \Gamma)\leq M_0\exp(-\beta T)\,\mathrm{dist}\,(v,\Gamma)$ for any $v\in V$ and any $T>0$. Then for arbitrary $\eta>0$ it follows from (6.22) that

$$\begin{aligned}\mathrm{dist}\,(\varphi(T'+T_0;\varepsilon),\Gamma) &\leq \mathrm{dist}\,(\pi(T_0;\varphi(T';\varepsilon)),\Gamma)\\ &\quad+|\varphi(T'+T_0;\varepsilon)-\pi(T_0;\varphi(T';\varepsilon))|\\ &\leq M_0e^{-\beta_0T_0}\,\mathrm{dist}\,(\varphi(T';\varepsilon),\Gamma)+C\varepsilon\\ &\leq \frac{1}{3}\cdot\mathrm{dist}\,(\varphi(T';\varepsilon),\Gamma)+\frac{\eta}{3}\end{aligned}$$

for any $T'\geq T_1$ if $\varepsilon\leq\min\{\varepsilon_0, \eta/\{3C)\}$. Hence, we obtain inductively

$$\begin{aligned}\mathrm{dist}\,(\varphi(T'+nT_0;\varepsilon),\Gamma) &\leq \frac{1}{3^n}\cdot\mathrm{dist}(\varphi(T';\varepsilon),\Gamma)+\eta\cdot\sum_{i=1}^{n}\frac{1}{3^i}\\ &\leq\frac{1}{3^n}\cdot C_1+\frac{1}{2}\cdot\eta\end{aligned} \tag{6.23}$$

for any $n\in\mathbf{N}$, any $T'\geq T_1$ and any $\varepsilon\leq\min\{\varepsilon_0, \eta/(3C)\}$, where C_1 is a constant satisfying dist $(\varphi(T;\varepsilon),\Gamma)\leq C_1$ for any $T\geq T_1$ and any $0<\varepsilon\leq\varepsilon_0$. (6.23) implies that

$$\mathrm{dist}\,(\varphi(T;\varepsilon),\Gamma)\leq\eta \tag{6.24}$$

for any $T\geq T_\eta$ if $\varepsilon\leq\min\{\varepsilon_0, \eta/(3C)\}$ and $T_\eta=T_1+\{[\log_3(2C_1/\eta)]+1\}T_0$, where [ ] is a Gaussian symbol. If we set $\eta=3C\varepsilon$ for $0<\varepsilon\leq\varepsilon_0$, then $\varepsilon=\eta/(3C)$ satisfies $\varepsilon\leq\min\{\varepsilon_0, \eta/(3C)\}$, so that (6.24) holds, namely,

$$\mathrm{dist}\,(\varphi(T;\varepsilon),\Gamma)\leq 3C\varepsilon \tag{6.25}$$

for any $T\geq T_{3C\varepsilon}$. (6.25) shows that dist $(Qu(t;\varepsilon),\Gamma)\leq 3C\varepsilon$ for any $t\geq t_\varepsilon=T_{3C\varepsilon}/\varepsilon$. Finally, it is obvious from Lemma 6.6 that $\|Pu(t;\varepsilon)-e^{-tA}Pu_0\|_\alpha\leq C_2\varepsilon$ for some C_2. Thus, the proof of Theorem 4.2 is complete. ∎

6.3. Proofs of Corollaries 4.1 and 4.2

Proof of Corollary 4.1

It suffices to show that there exist $C>0$ and $\varepsilon_0>0$ such that $|\varphi(T;\varepsilon)-\pi(T; Qu_0)|\leq C\varepsilon$ for any $0<\varepsilon\leq\varepsilon_0$ and any $0\leq T<\infty$, because it has already been shown that $\|Pu(t;\varepsilon)-e^{-tA}Pu_0\|_\alpha=O(\varepsilon)$ uniformly on $t\in[0,\infty)$. By assumptions of Corollary 4.1 we can take some positive constants r, β_1, β and M_0, δ such that $|\pi(T;v)-\xi|\leq M_0e^{-\beta T}|v-\xi|$ for any $v\in V=U_r(\xi)$ and $\mathrm{Re}\,\sigma(QF'(v)|_{B_1})<-\beta_1$ for any $v\in U_\delta(V)$. Let $Y(T;\varepsilon)=\varphi(T;\varepsilon)-\pi(T;Qu_0)$. Then, from Theorems 4.1 and 4.2 there exist $T_1>0$, $\varepsilon_1>0$ and $C_1>0$ such that $\varphi(T;\varepsilon)\in U_{C_1\varepsilon}(V)\subset U_\delta(V)$ for $T\geq T_1$ and $0<\varepsilon\leq\varepsilon_1$ and that $Y(T;\varepsilon)$ satisfies for $0<\varepsilon\leq\varepsilon_1$;

$$|Y(T;\varepsilon)|\leq C_1\varepsilon \tag{6.26}$$

for $0\leq T\leq T_1$ and

$$\frac{dY}{dT}=\int_0^1 QF'(\theta Y+\pi(T;Qu_0))d\theta Y+g_\varepsilon(T) \tag{6.27}$$

for some $g_\varepsilon(T)$ and $T\geq T_1$ with $|g_\varepsilon(T)|\leq C_1\varepsilon$.

Denote by $U(T,S;\varepsilon)$ the solution of the operator equation

$$\begin{cases} \dfrac{dU}{dT}=\displaystyle\int_0^1 QF'(\theta Y(T;\varepsilon)+\pi(T;Qu_0))d\theta U, \\ U(S,S;\varepsilon)=Id. \end{cases} \tag{6.28}$$

Then (6.27) is reduced to

$$Y(T;\varepsilon)=U(T,T_1;\varepsilon)Y(T_1;\varepsilon)+\int_{T_1}^T U(T,S;\varepsilon)g_\varepsilon(S)dS \tag{6.29}$$

for $T\geq T_1$. Since $\theta Y(T;\varepsilon)+\pi(T;Qu_0)\in U_\delta(V)$ for $T\geq T_1$, it follows that

$$\mathrm{Re}\,\sigma(QF'(\theta Y(T;\varepsilon)+\pi(T;Qu_0))|_{B_1}<-\beta_1 \tag{6.30}$$

for $T\geq T_1$. Here we make r, δ and ε_1 small so that $|d\varphi/dT|$ is sufficiently small for $T\geq T_1$. Then, (6.30) implies that $|U(T,S;\varepsilon)|_{\mathscr{L}(B_1)}\leq M_1\exp\{-\beta_1(T-S)\}$ for some M_1 and any $T\geq S\geq T_1$, though we omit the details (Potier-Ferry [37, Proposition 5]). Consequently, it follows from (6.26) and (6.29) that

$$\begin{aligned} |Y(T;\varepsilon)| &\leq M_1e^{-\beta_1(T-T_1)}\cdot C_1\varepsilon+\int_{T_1}^T M_1e^{-\beta_1(T-S)}\cdot C_1\varepsilon dS \\ &\leq M_2\varepsilon \end{aligned} \tag{6.31}$$

for any $T \geq T_1$, any $0<\varepsilon \leq \varepsilon_1$ and some M_2. (6.26) and (6.31) complete the proof. ∎

PROOF OF COROLLARY 4.2

From the assumption of Corollary 4.2, there exist $\varepsilon_0>0$, $C>0$ such that $|Qu(t;\varepsilon)| \leq C$ for any $t \geq 0$ and any $\varepsilon \in (0, \varepsilon_0]$. If we regard C and ε_0 as the constants M_2 and ε_1 in the proof of (6.10) respectively, then Lemma 6.6 directly holds with $I_2^\varepsilon = [0, \infty)$ and we obtain

$$w(t;\varepsilon) = J_\varepsilon(w)(t) \tag{6.32}$$

and

$$\|w(t;\varepsilon) - e^{-tA}Pu_0\|_\alpha \leq M_5\varepsilon \tag{6.33}$$

on $t \in [0, \infty)$ for any $0<\varepsilon<\varepsilon_2$. Here, we note that we use same symbols as in the proof of (6.10). Since F maps B_1 into itself, we have $F(\varphi(T;\varepsilon)) \in B_1$, namely, $PF(\varphi(T;\varepsilon))=0$. Moreover, (6.33) shows that $\|w(t;\varepsilon)\|_\alpha \leq 1+M_0M_1$. Therefore, we obtain

$$\begin{aligned}\|PF(\varphi(\varepsilon s;\varepsilon)+w(s;\varepsilon))\| &\leq \int_0^1 \|PF(\varphi(\varepsilon s;\varepsilon)+\theta w(s;\varepsilon))\|_{\mathscr{L}(B^\alpha,B)}d\theta \|w(s;\varepsilon)\|_\alpha \\ &\leq M_6\|w(s;\varepsilon)\|_\alpha,\end{aligned}$$

which implies from (6.32) that

$$\|w(t;\varepsilon)\|_\alpha \leq M_0e^{-at}\|Pu_0\|_\alpha + \varepsilon\int_0^t M_0M_6\frac{e^{-a(t-s)}}{(t-s)^\alpha}\|w(s;\varepsilon)\|_\alpha ds. \tag{6.34}$$

If we define $|||w||| = \sup_{t\geq 0} e^{\tilde{a}t}\|w(t;s)\|_\alpha$, it follows from (6.34) that

$$|||w||| \leq M_0\|Pu_0\| + \varepsilon M_7|||w|||, \tag{6.35}$$

where $0<\tilde{a}<a$ and $M_7 = M_0M_6\int_0^\infty \frac{e^{-(a-\tilde{a})s}}{s^\alpha}ds$. (6.35) shows that if we fix ε^* as $0<\varepsilon^*<1/M_7$,

$$|||w||| \leq \frac{M_0}{1-\varepsilon M_7}\|Pu_0\|_\alpha \tag{6.36}$$

holds for any $\varepsilon \in (0, \varepsilon^*]$. (6.36) and the proof of Lemma 6.7 give the direct proof of this corollary. ∎

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