HIROSHIMA MATH. J.
18 (1988), 373–397

Numerical interfaces in nonlinear diffusion equations with finite extinction phenomena

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(Received September 10, 1987)

1. Introduction

From a numerical point of view, we study the following nonlinear degenerate diffusion equation

(1.1) $$v_t = (v^m)_{xx} - cv^p, \quad (t, x) \in (0, \infty) \times \mathbf{R}^1$$

with an initial condition

(1.2) $$v(0, x) = v^0(x), \quad x \in \mathbf{R}^1,$$

where m (>1), c (>0) and p (>0) are all constants and v^0 is a nonnegative continuous function with a compact support.

Equations of the form (1.1) are known as models in the fields of fluid dynamics [13], plasma physics [1] and population dynamics [5]. For instance, (1.1) describes a one-dimensional nonlinear fluid-transfer process with an absorption, where $v = v(t, x)$ is the density of fluid at time t and place x. The most interesting phenomenon, which (1.1), (1.2) exhibits, is the occurrence of the finite propagation of the initial support. As is well known, in the absence of $-cv^p$, the support of the solution $S(t) \equiv \operatorname{supp} v(t, \cdot)$ expands and finally becomes unbounded as time increases. On the other hand, in the presence of $-cv^p$, which implies volumetric absorption, it is already shown that the behavior of $S(t)$ (in other words, *interfaces*) is qualitatively classified into the following three cases depending on m and p (see [2], [6], [7], [8], [9] and [10]):

(1) For $p \geqq m$, $S(t)$ expands as t increases and

$$S(t) \longrightarrow \mathbf{R}^1, \quad \text{as} \quad t \longrightarrow \infty.$$

(2) For $1 \leqq p < m$, $S(t)$ also expands and there exists a bounded set $\mathrm{B} \subset \mathbf{R}^1$ satisfying

$$S(t) \subset \mathrm{B}, \quad \text{for all} \quad t \geqq 0.$$

(3) For $0 < p < 1$, $S(t)$ is compact in $\mathbf{R}^1$ and there exists a positive number $T^* < \infty$ such that

$$S(t) \neq \emptyset \quad \text{on} \quad [0, T^*) \quad \text{and} \quad S(t) = \emptyset \quad \text{on} \quad (T^*, \infty).$$

We call T^* an *extinction time* of v and the behavior such as (3) *finite extinction phenomenon*. When $m+p=2$, Kersner [3] shows an explicit solution of (1.1), (1.2) as in Fig. 1. His solution shows that $S(t)$ is not always monotone, which is different from the cases (1) and (2). This result motivates us to study the finite extinction phenomena of (1.1), (1.2). The aim in this paper is to propose an interface tracking scheme to determine $S(t)$ in the case (3).

From the viewpoint of tracking interfaces, numerical schemes of (1.1), (1.2) in the cases (1) and (2) have been proposed (for instance, Mimura, Nakaki and Tomoeda [11]). However, as far as we know, we have not found any schemes for the case (3) except for Rosenau and Kamin [12] and Tomoeda [14]. It is shown in [12] that pulses are evolved into several sub-pulses within a finite time, by the effect of absorption. Unfortunately, theoretical results on their scheme are not discussed. On the other hand, in [14], an interface tracking scheme is proposed which gives good approximations to not only the solution but also interfaces. The stability of this scheme is proved but the convergence is not shown.

In this paper, we propose a modified version of Tomoeda's scheme and prove the stability as well as convergence when $m+p=2$. Moreover, we show that numerical extinction time converges to the exact one.

The outline of this paper is as follows. In Section 2, the difference scheme is presented. Stability and convergence of numerical approximations $v_h(t, x)$ are proved in Sections 3 and 4, respectively. Section 4 also contains the proof of the convergence of the numerical extinction time. The convergence of the support of $v_h(t, \cdot)$ is shown in Section 5. In Section 6, proofs of lemmas used in Section 3 are shown. Finally, in Section 7, several numerical simulations are demonstrated.

Our main results are as follows:

Suppose that $m+p=2$ and $(v^0)^{m-1}$ is concave on $S(0)$. Let $\ell_h(t)$ (resp. $r_h(t)$) be the front of the left (resp. right) hand side of the support of $v_h(t, x)$. Then $v_h(t, x)$ converges uniformly on $[0, \infty) \times \mathbf{R}^1$ to the solution $v(t, x)$ of (1.1), (1.2) as $h \to 0$ and there exist locally Lipschitz continuous functions $\ell(t)$ and $r(t)$ defined on $[0, T^*)$ satisfying

$$S(t) = [\ell(t), r(t)], \qquad \text{for} \quad 0 \leqq t < T^*$$

and $\ell_h(t)$ and $r_h(t)$ converge compact uniformly on $[0, T^*)$ to $\ell(t)$ and $r(t)$ as $h \to 0$, respectively. Moreover,

$$\lim_{h \to 0} T_h^* = T^*$$

holds, where T_h^* is the numerical extinction time of $v_h(t, x)$.

ACKNOWLEDGMENTS. The author expresses sincere thanks Professor Kenji Tomoeda of Osaka Institute of Technology for his helpful discussion and criticism. The author also would like to thank Professor Masayasu Mimura of Hiroshima University for many valuable suggestions and encouragements. This work has been partially done during the author's visit to Hiroshima University.

2. Difference schemes

2.1. Outline of difference schemes

By putting $u=v^{m-1}$, (1.1), (1.2) is rewritten as

$$u_t = muu_{xx} + a(u_x)^2 - c'u^q, \quad (t, x) \in (0, \infty) \times \mathbf{R}^1, \tag{2.1}$$

$$u(0, x) = u^0(x), \qquad x \in \mathbf{R}^1, \tag{2.2}$$

where $a=m/(m-1)$, $c'=c(m-1)$, $q=(m+p-2)/(m-1)$ and $u^0=(v^0)^{m-1}$. It is clear that the cases (1), (2) and (3) stated in Section 1 correspond to the cases $q \geqq 2$, $1 \leqq q<2$ and $q<1$, respectively. In this paper, for the case $q<1$, we construct a finite difference scheme for (2.1), (2.2) instead of (1.1), (1.2).

Let us define the operators P, H and D by

$$Pu = muu_{xx}, \quad Hu = a(u_x)^2 \quad \text{and} \quad Du = -c'u^q, \tag{2.3}$$

respectively, and rewrite (2.1) as

$$u_t = (P+H+D)u. \tag{2.4}$$

Let h (>0) be a space mesh width and denote by $u_h^n(x)$ the difference approximation to the solution $u(t_n, x)$ of (2.1). Our difference scheme is described in the following form:

Find the sequence $\{u_h^n\}_{n=0,1,2,\dots} \subset V_h$ such that

$$u_h^{n+1} = (I_h+(k/\mu)P_h)^\mu(I_h+kH_h)(I_h+kD_h)u_h^n. \tag{2.5}$$

Here V_h is a set of functions which will be defined later, I_h is the identity operator, P_h, H_h and D_h are difference operators approximating P, H and D, respectively, k $(=t_{n+1}-t_n)$ is a time mesh size and μ is some integer depending on n.

2.2. Admissible functions

We denote V_h by the set of nonnegative continuous functions u_h ($\not\equiv 0$) satisfying the following properties:

(i) u_h has a compact support;

(ii) u_h is linear on each interval $[x_i, x_{i+1}]$ $(i \in \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers and $x_i = x_i(\ell, r)$ $(i \in \mathbf{Z})$ is the nodal point defined by

$$x_i(\ell, r) = \begin{cases} ih, & i \in \mathbf{Z} \backslash \{L(\ell)-1, R(r)+1\}, \\ \ell, & i = L(\ell) - 1, \\ r, & i = R(r) + 1. \end{cases} \tag{2.6}$$

Here we used the following notations:

$$\ell = \ell(u_h) \equiv \sup\{y \in \mathbf{R}^1;\ u_h(x)=0 \text{ on } (-\infty, y)\},$$

$$r = r(u_h) \equiv \inf\{y \in \mathbf{R}^1;\ u_h(x)=0 \text{ on } (y, \infty)\},$$

$$L(\ell) \equiv \min\{i \in \mathbf{Z};\ ih > \ell\}, \quad R(r) \equiv \max\{i \in \mathbf{Z};\ ih < r\}.$$

We call $\ell(u_h)$ (resp. $r(u_h)$) a *left* (resp. *right*) *interface* of u_h, which means the front of the left (resp. right) hand side of support of u_h. We put $X(\ell, r) = \{x_i(\ell, r);\ i \in \mathbf{Z}\}$.

2.3. Difference operator P_h

For $u_h \in V_h$, we define $P_h u_h$ by a usual explicit difference

$$(P_h u_h)(x_i) = m u_i \delta^2 u_i, \qquad \text{for} \quad x_i \in X(\ell, r), \tag{2.7}$$

where $u_i = u_h(x_i)$, $\ell = \ell(u_h)$, $r = r(u_h)$,

$$\delta^2 u_i = (\delta u_i - \delta u_{i-1})/\{(h_i + h_{i-1})/2\}, \tag{2.8}$$

$$\delta u_i = (u_{i+1} - u_i)/h_i \quad \text{and} \quad h_i = x_{i+1} - x_i. \tag{2.9}$$

Since the diffusion coefficient mu_h of $P_h u_h$ is vanished at the point where $u_h = 0$, the support of $(I_h + k'P_h)u_h$ $(k' = k/\mu)$ coincides with that of u_h. Therefore, we use the nodal points of $(I_h + k'P_h)u_h$ as the same ones of u_h.
Suppose that k' satisfies the following conditions:

$$m\|u_h\|_\infty k'[1/h^2 + 2/\{h(h+h_i)\}] \leqq 1, \qquad \text{for} \quad i = L-1, \quad R, \tag{2.10}$$

$$4m\|(u_h)_x\|_\infty k'/(h+h_i) \leqq 1, \qquad \text{for} \quad i = L-1, \quad R, \tag{2.11}$$

where $L = L(\ell(u_h))$, $R = R(r(u_h))$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbf{R}^1)}$. Then we find that $(I_h + k'P_h)u_h$ also belongs to V_h (see Lemma 3.2).

2.4. Difference operator H_h

Since the definition of H_h is not simple to be stated, we consider the operator $H_{h,k} \equiv I_h + kH_h$ mapping from V_h into V_h. This definition is due to Mimura, Nakaki and Tomoeda [11].

Let $u_h \in V_h$. We define $H_{h,k}u_h$ as follows. First, define ℓ' and r', which will become the interfaces of $H_{h,k}u_h$ (see (2.19)), by

$$\ell' = \ell - a\delta u_{L-1}k \quad \text{and} \quad r' = r - a\delta u_R k, \tag{2.12}$$

respectively, where $\ell = \ell(u_h)$, $r = r(u_h)$, $L = L(\ell)$, $R = R(r)$ and $u_i = u_h(x_i)$ $(i \in \mathbf{Z})$.

Next, we define $(H_{h,k}u_h)(x_i')$ for $x_i' \in X(\ell', r')$ by

$$(H_{h,k}u_h)(x_i') = \begin{cases} u_i + a(\delta u_i)^2 k, & \text{if } i \in S_s^+ \cup S_r^+, \\ u_i + a(\delta u_{i-1})^2 k, & \text{if } i \in S_s^- \cup S_r^-, \\ u_i, & \text{if } i \in S^0, \\ (L'h - \ell')\delta u_{L-1}, & \text{if } i = L' = L-1, \\ (R'h - r')\delta u_R, & \text{if } i = R' = R+1, \\ 0, & \text{if } i \in \mathbf{Z}\backslash\{L', \cdots, R'\}, \end{cases} \tag{2.13}$$

where

$$S_s^+ = \{i \in \{L, \dots, R\};\ \delta u_{i-1} < \delta u_i \text{ and } \delta u_{i-1} > -\delta u_i\},$$

$$S_s^- = \{i \in \{L, \dots, R\};\ \delta u_{i-1} < \delta u_i \text{ and } \delta u_{i-1} \leqq -\delta u_i\},$$

$$S_r^+ = \{i \in \{L, \dots, R\};\ \delta u_{i-1} \geqq \delta u_i > 0\},$$

$$S_r^- = \{i \in \{L, \dots, R\};\ 0 > \delta u_{i-1} \geqq \delta u_i\},$$

$$S^0 = \{i \in \{L, \dots, R\};\ \delta u_{i-1} \geqq 0 \geqq \delta u_i\},$$

$$L' = L(\ell') \quad \text{and} \quad R' = R(r').$$

Assume that k satisfies the following three conditions:

$$4a\|(u_h)_x\|_\infty k \leqq h; \tag{2.14}$$

$$a\|(u_h)_x\|_\infty k \leqq Lh - \ell; \tag{2.15}$$

$$a\|(u_h)_x\|_\infty k \leqq r - Rh. \tag{2.16}$$

Then we have

$$L' = L - 1 \quad \text{or} \quad L' = L, \tag{2.17}$$

$$R' = R + 1 \quad \text{or} \quad R' = R, \tag{2.18}$$

$$\ell' = \ell(H_{h,k}u_h) \quad \text{and} \quad r' = r(H_{h,k}u_h). \tag{2.19}$$

Moreover, $H_{h,k}u_h \in V_h$ holds (see Lemma 3.1). We note that, from (2.17) and (2.18), $(H_{h,k}u_h)(x_i')$ is defined for all $x_i' \in X(\ell', r')$.

2.5. Difference operator D_h

Instead of the definition of D_h, we consider the operator $D_{h,k} \equiv I_h + kD_h$ mapping from V_h into V_h. To describe the operator $D_{h,k}$, we introduce the solution $u(t, x)$ of the ordinary differential equation

$$u_t = Du \equiv -c'u^q, \quad t > 0 \tag{2.20}$$

which is written as

$$u(t, x) = \{[\xi^{1-q} - c'(1-q)t]^+\}^{1/(1-q)}, \tag{2.21}$$

where $\xi = u(0, x)$ and $[f]^+ = \max\{f, 0\}$. Using this solution, we define

$$(D_{h,k}u_h)(x_i') = \{[u_h(x_i')^{1-q} - c'(1-q)k]^+\}^{1/(1-q)} \qquad \text{for} \quad x_i' \in X(\ell', r'), \tag{2.2}$$

where

$$\begin{cases} \ell' = \ell(\{[u_h^{1-q} - c'(1-q)k]^+\}^{1/(1-q)}), \\ r' = r(\{[u_h^{1-q} - c'(1-q)k]^+\}^{1/(1-q)}). \end{cases} \tag{2.23}$$

Then it can be easily shown that $\ell' = \ell(D_{h,k}u_h)$ and $r' = r(D_{h,k}u_h)$. Moreover, $D_{h,k}u_h \in V_h$ holds (see Lemma 3.3).

For the cases $q \geqq 1$, if $\tilde{x} \in \mathbf{R}^1$ satisfies $u(0, \tilde{x}) > 0$ (resp. $u(0, \tilde{x}) = 0$), it follows from (2.21) that

$$u(t, \tilde{x}) > 0 \quad (\text{resp.}\ u(t, \tilde{x}) = 0), \quad \text{for all} \quad t > 0.$$

This means that the interfaces never shrink nor expand. On the other hand, for the case $0 \leqq q < 1$, there exists $t^* > 0$ such that

$$u(t, x) = 0, \quad \text{for all} \quad x \in \mathbf{R}^1 \quad \text{and} \quad t > t^*.$$

Taking these properties into considerations, we may expect that the solution of (1.1), (1.2) is extinct in finite time for $0 \leqq q < 1$, but is not extinct in finite time for $q \geqq 1$.

2.6. Difference schemes

For $n = 0$, let $t_0 = 0$ and $u_h^0 \in V_h$ be defined by

$$\ell(u_h^0) = \ell(u^0), \quad r(u_h^0) = r(u^0) \quad \text{and} \quad u_h^0(ih) = u^0(ih) \quad (i \in \mathbf{Z}). \tag{2.24}$$

Suppose that u_h^n and t_n are given. We construct u_h^{n+1} and t_{n+1} as follows. First, we determine a time step k_{n+1} by the following ways: In the case

$$u_L^n = \max\{u_i^n;\ L \leqq i \leqq R\} \quad \text{or} \quad u_R^n = \max\{u_i^n;\ L \leqq i \leqq R\}, \tag{2.25}$$

where $L=L(\ell(u_h^n))$ and $R=R(r(u_h^n))$, we define k_{n+1} by

$$k_{n+1} = \min\{k>0;\ (I_h+kD_h)u_h^n \equiv 0\}, \tag{2.26}$$

which means the extinction of the difference solution in finite time. Therefore, putting $T_h^*=t_{n+1}=t_n+k_{n+1}$ and $u_h^{n+1}(x)=0$, we stop computing. To simplify the statements in the following sections, we put $t_m=T_h^*$ and $u_h^m(x)=0$ for $m>n+1$.

When (2.25) does not hold, we determine k_{n+1} as the largest possible time step k satisfying the following

CONDITION A. 1) k satisfies the stability conditions (2.14), (2.15) and (2.16) with $u_h=(I_h+kD_h)u_h^n$;

2) Every connected components of the set $[\operatorname{supp} u_h^n]\backslash[\operatorname{supp}(I_h+kD_h)u_h^n]$ has at most one point x such that x/h is an integer.

We note that Condition A–2) is imposed by a mathematical reason. For the time step k_{n+1} determined above, we put $t_{n+1}=t_n+k_{n+1}$.

Next, let us determine $\mu=\mu_{n+1}$ in a way that $k'=k_{n+1}/\mu_{n+1}$ satisfies the stability conditions (2.10) and (2.11) with $u_h\equiv(I_h+k_{n+1}H_h)(I_h+k_{n+1}D_h)u_h^n$. Thus we can obtain u_h^{n+1} by (2.5) with $k=k_{n+1}$. In general, we do not know whether (2.25) holds for some $n>0$ or not. Unless (2.25) holds for all $n>0$, we continue computing u_h^n for $0\leqq n<\infty$, and we define a numerical extinction time T_h^* by

$$T_h^* = \lim_{n\to\infty} t_n. \tag{2.27}$$

REMARK 2.1. In general, the time step k is determined implicitly. When $m+p=2$ and u^0 is concave on its support, k is explicitly represented (see Lemma 3.5).

3. Stability

From this section, we assume $m+p=2$. To state the stability of the scheme (2.5), we impose Condition B on the initial function u^0.

CONDITION B. $u^0(x)\equiv(v^0(x))^{m-1}$ is concave on (ℓ_0, r_0) and satisfies

$$u^0\in C^0(\mathbf{R}^1)\cap BV(\mathbf{R}^1),\ u_x^0\in L^\infty(\mathbf{R}^1)\cap BV(\mathbf{R}^1),$$

$$u^0(x)>0 \quad\text{on}\quad (\ell_0, r_0) \quad\text{and}\quad u^0(x)=0 \quad\text{on}\quad \mathbf{R}^1\backslash(\ell_0, r_0),$$

where $BV(\mathbf{R}^1)$ denotes the space of functions of bounded variation on $\mathbf{R}^1$.

Then we have

THEOREM 3.1 (*Stability of* (2.5)). *Assume Condition B and* $m+p=2$.

For $h>0$, *let* $k=k_{n+1}$ *be the time step determined in Subsection* 2.6 *and* $k'=k'_{n+1}=k_{n+1}/\mu_{n+1}$ *satisfy the stability conditions* (2.10) *and* (2.11). *Then*

$$T_h^* \leqq \|u^0\|_\infty/c', \tag{3.1}$$

and the difference solution u_h^n *belongs to* V_h *for* $t_n<T_h^*$ *and satisfies*

$$u_h^n(x) \text{ is concave on } (\ell_n, r_n), \tag{3.2}$$

$$u_h^n(x) > 0 \quad \text{on} \quad (\ell_n, r_n), \tag{3.3}$$

$$\|u_h^n\|_\infty \leqq \max\{\|u^0\|_\infty - c't_n, 0\}, \tag{3.4}$$

$$\|u_h^n\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \tag{3.5}$$

for $t_n<T_h^*$, *where* $\ell_n=\ell(u_h^n)$ *and* $r_n=r(u_h^n)$. *Moreover, there exists a constant* C, *which is independent of* h *and* n, *such that*

$$\|(u_h^n)_x\|_\infty \leqq C, \tag{3.6}$$

$$\|(u_h^n)_x\|_1 \leqq C, \tag{3.7}$$

$$V((u_h^n)_x) \leqq C, \tag{3.8}$$

$$\|(u_h^{n+1}-u_h^n)/k_{n+1}\|_1 \leqq C, \tag{3.9}$$

for $t_n<T_h^*$, *where* $V(f)$ *denotes the total variation of* f *on* $\mathbf{R}^1$ *and* $\|\cdot\|_1=\|\cdot\|_{L^1(\mathbf{R}^1)}$.

REMARK 3.1. When u^0 is not concave on (ℓ_0, r_0), (3.1), (3.4)–(3.9) of Theorem 3.1 follow, and instead of (3.3), $u_h^n(x)\geqq 0$ holds. However, we do not know whether (3.5) holds or not.

To prove Theorem 3.1, we prepare the following five lemmas:

LEMMA 3.1 (*Stability of* I_h+kH_h). *Let* $u_h\in V_h$ *satisfy* $u_h(x)>0$ *on* $(\ell(u_h), r(u_h))$ *and the time step* k *be the one stated in Subsection* 2.6. *Then* $u_h'\equiv(I_h+kH_h)u_h\in V_h$ *and*

$$0 < u_h'(x) \leqq \|u_h\|_\infty \quad \text{on} \quad (\ell(u_h'), r(u_h')), \tag{3.10}$$

$$\|(u_h')_x\|_\infty \leqq \|(u_h)_x\|_\infty, \tag{3.11}$$

$$\|(u_h')_x\|_1 \leqq \|(u_h)_x\|_1, \tag{3.12}$$

$$V((u_h')_x) \leqq V((u_h)_x), \tag{3.13}$$

$$\|(u_h'-u_h)/k\|_1 \leqq a\|(u_h)_x\|_\infty\|(u_h)_x\|_1. \tag{3.14}$$

Furthermore, if u_h *is concave on its support, so is* u_h' *and*

$$\delta u'_{L'-1} \leqq \delta u_{L-1} \quad \textit{and} \quad \delta u'_{R'} \geqq \delta u_R, \tag{3.15}$$

where

$$u_i = u_h(x_i), \quad u'_i = u'_h(x'_i), \quad L = L(\ell(u_h)),$$

$$L' = L(\ell(u'_h)), \quad R = R(r(u_h)), \quad R' = R(r(u'_h))$$

and $\{x_i\}$ *and* $\{x'_i\}$ *are sets of nodal points of* u_h *and* u'_h, *respectively.*

LEMMA 3.2 (*Stability of* $I_h + k'P_h$). *Let* $u_h \in V_h$ *satisfy* $u_h(x) > 0$ *on* $(\ell(u_h), r(u_h))$ *and* k' *satisfy* (2.10) *and* (2.11). *Then* (3.10)–(3.13) *hold for* $u'_h \equiv (I_h + k'P_h)u_h \in V_h$ *and*

$$\|(u'_h - u_h)/k'\|_1 \leqq m\|u_h\|_\infty V((u_h)_x). \tag{3.16}$$

Furthermore, if u_h *is concave on its support,* u'_h *is also concave and* (3.15) *holds.*

LEMMA 3.3 (*Stability of* $I_h + kD_h$). *Let* $m + p = 2$ *and let the time step* k *be an arbitrary positive number. For* $u_h \in V_h$ *satisfying* $u_h(x) > 0$ *on* $(\ell(u_h), r(u_h))$, *put* $u'_h \equiv (I_h + kD_h)u_h$. *Then either* $u'_h \equiv 0$ *or* $u'_h \in V_h$ *holds and* u'_h *satisfies* (3.11)–(3.13) *and*

$$\|u'_h\|_\infty \leqq \max\{\|u_h\|_\infty - c'k, 0\}, \tag{3.17}$$

$$\|(u'_h - u_h)/k\|_1 \leqq c'\{r(u_h) - \ell(u_h)\}. \tag{3.18}$$

Furthermore, if u_h *is concave on its support and* $u'_h \not\equiv 0$, u'_h *is also concave, and* (3.10) *and* (3.15) *hold.*

LEMMA 3.4 (*Monotonicity of numerical interfaces*). *Let the assumptions of Theorem* 3.1 *be satisfied. If* $\ell_{p+1} \geqq \ell_p$ (*resp.* $r_{p+1} \leqq r_p$) *for some* $p \geqq 0$, *then* $\ell_{n+1} \geqq \ell_n$ (*resp.* $r_{n+1} \leqq r_n$) *for all* $n \geqq p$ *satisfying* $t_{n+1} < T_h^*$.

LEMMA 3.5 (*Representation of the time step*). *Let the assumptions of Theorem* 3.1 *be satisfied. For the time step* k_{n+1},

$$k_{n+1} = \min\{k_{L,n}, k_{M,n}, k_{R,n}\} \tag{3.19}$$

holds for $n \geqq 0$ *such that* $t_{n+1} < T_h^*$, *where*

$$k_{M,n} = \sup K_M, \quad k_{i,n} = \sup(K_{1i} \cup K_{2i}) \quad (i = L, R), \tag{3.20}$$

$$K_M = \{k > 0;\ 4a\|(u_h^n)_x\|_\infty k \leqq h\}, \tag{3.21}$$

$$K_{1L} = \{k > 0;\ (a\|(u_h^n)_x\|_\infty \delta u_L^n + c')k \leqq u_{L+1}^n,\ c'k \geqq u_L^n\}, \tag{3.22}$$

$$K_{2L} = \{k > 0;\ (a\|(u_h^n)_x\|_\infty + c'/\delta u_{L-1}^n)k \leqq u_L^n,\ c'k < u_L^n\}, \tag{3.23}$$

(3.24) $$K_{1R} = \{k>0;\ (a\|(u_h^n)_x\|_\infty \delta u_{R-1}^n + c')k \leqq u_{R-1}^n,\ c'k \geqq u_R^n\},$$

(3.25) $$K_{2R} = \{k>0;\ (a\|(u_h^n)_x\|_\infty + c'/\delta u_R^n)k \leqq u_R^n,\ c'k < u_R^n\},$$

(3.26) $$L = L(\ell(u_h^n)) \quad and \quad R = R(r(u_h^n)).$$

Proofs of Lemmas 3.1–3.5 will be shown in Section 6.

PROOF OF THEOREM 3.1. It follows immediately from Lemmas 3.1–3.3 that u_h^n belongs to V_h and (3.2)–(3.4), (3.6)–(3.8) hold. (3.1) also follows from (3.4). Let us show (3.9). From (3.14), (3.16) and (3.18), we have

$$\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_1 \leqq m\|u_h^n\|_\infty V((u_h^n)_x) + a\|(u_h^n)_x\|_\infty \|(u_h^n)_x\|_1 + c'\{r_n - \ell_n\}.$$

Since $\|u_h^n\|_\infty$, $V((u_h^n)_x)$, $\|(u_h^n)_x\|_\infty$ and $\|(u_h^n)_x\|_1$ are uniformly bounded with respect to h and n, it suffices to show the uniform boundedness of $r_n - \ell_n$. By the determination of numerical interfaces (2.12) and (2.23), we find

(3.27) $$\ell_n - a\delta u_{L-1}^n k_{n+1} \leqq \ell_{n+1} < r_{n+1} \leqq r_n - a\delta u_R^n k_{n+1}$$

for $t_{n+1} < T_h^*$, where

$$u_i^n = u_h^n(x_i), \quad L = L(\ell_n), \quad R = R(r_n)$$

and $\{x_i\}$ is the set of nodal points of u_h^n. By using (3.27) repeatedly and by (3.6), we have

(3.28) $$\ell_0 - aCt_n \leqq \ell_n < r_n \leqq r_0 + aCt_n, \qquad \text{for} \quad t_n < T_h^*.$$

Then, it follows from (3.1) that

(3.29) $$0 < r_n - \ell_n \leqq M \equiv r_0 - \ell_0 + 2aC\|u^0\|_\infty/c', \qquad \text{for} \quad t_n < T_h^*.$$

Hence, $r_n - \ell_n$ is uniformly bounded, which completes the proof of (3.9).

Finally, we prove (3.5). Suppose that (3.5) does not hold. Then there exist constants N and k^* (>0) such that

(3.30) $$k_{L,n} \geqq k^* \quad \text{and} \quad k_{R,n} \geqq k^*, \qquad \text{for} \quad n \geqq N,$$

which will be shown later. Since

$$k_{M,n} = h/(4a\|(u_h^n)_x\|_\infty) \geqq h/(4aC), \qquad \text{for} \quad n \geqq 0$$

(see (3.20), (3.21) and (3.6)), we have by (3.19)

$$k_{n+1} \geqq \min\{k^*, h/(4aC)\} > 0, \qquad \text{for} \quad n \geqq N.$$

Therefore, it follows that

$$T_h^* \equiv \lim_{n\to\infty} t_n = \textstyle\sum_{n=0}^{\infty} k_{n+1} = \infty,$$

which contradicts to (3.1). Thus, (3.5) is proved.

Now, let us show the existence of N and k^* such that (3.30) holds. The assumption that (3.5) does not hold implies existence of a constant $\varepsilon>0$ satisfying

$$\|u_h^n\|_\infty \geqq \varepsilon, \qquad \text{for} \quad n \geqq 0. \tag{3.31}$$

By the concavity of u_h^n and (3.29), we have

$$\delta u_{L-1}^n \geqq \varepsilon/(r_n - \ell_n) \geqq \eta, \qquad \text{for} \quad n \geqq 0,$$

where $\eta=\varepsilon/M$. Since $\{\ell_n\}$ is a bounded by (3.28) and monotone increasing sequence for large n (see Lemma 3.4), we have

$$\ell_n \longrightarrow \ell_\infty \quad \text{as} \quad n \longrightarrow \infty$$

for some $\ell_\infty \in \mathbf{R}^1$. We consider the following two cases:

Case A. $L(\ell_n)=L(\ell_\infty)$, for $n \geqq N'$;

Case B. $L(\ell_n)=L(\ell_\infty)-1$, for $n \geqq N'$.

It is clear that one of *Cases A* and *B* occurs for sufficiently large N'.

In *Case A*, we can find an integer N ($\geqq N'$) such that

$$Lh - \ell_n \geqq \alpha \equiv (Lh - \ell_\infty)/2, \qquad \text{for} \quad n \geqq N, \tag{3.32}$$

where $L=L(\ell_\infty)$. Then it follows from (3.20) and (3.23) that

$$\begin{aligned} k_{L,n} &\geqq \min\{u_L^n/(a\|(u_h^n)_x\|_\infty + c'/\delta u_{L-1}^n),\ u_L^n/c'\} \\ &= (Lh-\ell_n)\delta u_{L-1}^n \min\{1/(a\|(u_h^n)_x\|_\infty + c'/\delta u_{L-1}^n),\ 1/c'\} \\ &\geqq k^*, \end{aligned}$$

where

$$k^* = \alpha\eta \min\{1/(aC + c'/\eta),\ 1/c'\} > 0,$$

which implies the first inequality of (3.30).

In *Case B*, it follows that

$$Lh - \ell_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \tag{3.33}$$

where $L=L(\ell_\infty)-1$. Since u_h^n is concave, we have

$$\begin{aligned} \varepsilon \leqq \|u_h^n\|_\infty = u_{p_n}^n &= (Lh-\ell_n)\delta u_{L-1}^n + h\delta u_L^n + \cdots + h\delta u_{p_n-1}^n \\ &\leqq (Lh-\ell_n)\delta u_{L-1}^n + h(p_n - L)\delta u_L^n \\ &\leqq (Lh-\ell_n)\delta u_{L-1}^n + (r_n - \ell_n)\delta u_L^n \\ &\leqq (Lh-\ell_n)\delta u_{L-1}^n + M\delta u_L^n, \end{aligned}$$

for some integer $p_n \in \{L, \cdots, R\}$, where M is the constant defined in (3.29). By (3.33), it follows that

$$\delta u_L^n \geqq \eta' \equiv \varepsilon/(2M), \qquad \text{for} \quad n \geqq N''$$

with some integer N'' $(\geqq N')$. Then, from (3.20), (3.22) and (3.33), there exists N $(\geqq N'')$ satisfying

$$\begin{aligned} k_{L,n} &\geqq \sup K_{1L} \\ &= u_{L+1}^n/(a\|(u_h^n)_x\|_\infty \delta u_L^n + c') \\ &= ((Lh - \ell_n)\delta u_{L-1}^n + h\delta u_L^n)/(a\|(u_h^n)_x\|_\infty \delta u_L^n + c') \\ &\geqq k^*, \qquad \text{for} \quad n \geqq N, \end{aligned}$$

where

$$k^* = h\eta'/(aC^2 + c') > 0.$$

Here we use the fact that

$$u_L^n/c' = (Lh - \ell_n)\delta u_{L-1}^n/c' \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence the first inequality of (3.30) follows. Similarly, the second inequality of (3.30) can be shown, and the proof of Theorem 3.1 is complete.

4. Convergence

In this section, we show the convergence of the difference approximation to the exact solution. First, we state the definition of a weak solution of (1.1), (1.2).

DEFINITION 4.1 (Herrero and Vazquez [6]). A function $v(t, x)$ defined on $\mathscr{H} = [0, \infty) \times \mathbf{R}^1$ is said to be a *weak solution* of (1.1), (1.2) if

(i) $v \in C^0(\mathscr{H}) \cap L^\infty(\mathscr{H})$ and $v \geqq 0$ on $\mathscr{H}$;

(ii) for any $x \in \mathbf{R}^1$, $v(0, x) = v^0(x)$;

(iii) for any function $\phi(t, x) \in C^{1,2}(\mathscr{H})$ with compact support in $\mathscr{H}$, the following integral relation holds:

$$\iint_{\mathscr{H}} (v^m \phi_{xx} + v\phi_t - cv^p \phi)\,dx\,dt + \int_{\mathbf{R}^1} v(0, x)\phi(0, x)\,dx = 0. \tag{4.1}$$

The existence, uniqueness and regularities of weak solutions have been studied by Kalashnikov [7], Kersner [8], Knerr [10] and Herrero and Vazquez [6]. By their results, a unique weak solution of (1.1), (1.2) exists under the assumptions that $m > 1$, $p > 0$, $m + p \geqq 2$ and v^0 is continuous, nonnegative and bounded.

Next, we define a weak solution of (2.1), (2.2).

DEFINITION 4.2. A function $u(t, x)$ defined on $\mathscr{H}$ is a *weak solution* of (2.1), (2.2) if

(i) $u \in C^0(\mathscr{H}) \cap L^\infty(\mathscr{H})$, $u_x \in L^\infty(\mathscr{H})$ and $u \geqq 0$ on $\mathscr{H}$;

(ii) for any $x \in \mathbf{R}^1$, $u(0, x) = u^0(x)$;

(iii) for any function $\phi(t, x) \in C^{1,2}(\mathscr{H})$ satisfying $\operatorname{supp} \phi \subset \operatorname{supp} u$, the following integral relation holds:

$$\iint_{\mathscr{H}} \{u\phi_t - muu_x\phi_x - (m-a)(u_x)^2\phi - c'u^q\phi\}dxdt + \int_{\mathbf{R}^1} u(0, x)\phi(0, x)dx = 0. \tag{4.2}$$

REMARK 4.1. Let $\mathscr{E}$ be the space of all continuous functions with compact support in $\mathscr{H}$, and $\mathscr{E}'$ be its dual. If a weak solution u of (2.1), (2.2) with $1<m<2$ satisfies

$$u_{xx} \in \mathscr{E}' \quad \text{and} \quad u_t \in \mathscr{E}', \tag{4.3}$$

then we find that $v = u^{1/(m-1)}$ is a unique weak solution of (1.1), (1.2) (see Theorem 7.1 in Graveleau and Jamet [4]).

To show the convergence of the difference approximation, we extend the region of definition of u_h^n computed by (2.5) to $\mathscr{H}$ in a way that

$$u_h(t, x) = u_h^n(x), \qquad \text{for} \quad t \in [t_n, t_{n+1}), \quad n \geqq 0, \tag{4.4}$$

$$u_h(t, x) = 0, \qquad \text{for} \quad t \geqq T_h^*. \tag{4.5}$$

Then we have

THEOREM 4.1. *Let Condition B and $m+p=2$ be satisfied. For an arbitrary sequence $\{h\}$ tending to zero, assume that Condition A and the stability conditions* (2.10) *and* (2.11) *hold for each h, k_{n+1} and $k'_{n+1} = k_{n+1}/\mu_{n+1}$ $(n \geqq 0)$. Then there exists a weak solution u of* (2.1), (2.2) *satisfying* (4.3), *and as $h \to 0$*

$$\|u_h - u\|_{L^\infty(\mathscr{H})} \longrightarrow 0, \tag{4.6}$$

$$\|(u_h)_x - u_x\|_{L^p(\mathscr{H})} \longrightarrow 0 \quad (1 \leqq p < \infty). \tag{4.7}$$

By the above theorem and Remark 4.1, we have

THEOREM 4.2. *Under the same assumptions as stated in Theorem* 4.1, *let $v_h = u_h^{1/(m-1)}$. Then*

$$\|v_h - v\|_{L^\infty(\mathscr{H})} \longrightarrow 0 \quad as \quad h \longrightarrow 0, \tag{4.8}$$

where v is the unique weak solution of (1.1), (1.2).

The proof of Theorem 4.1 is stated later in this section, but the proof of Theorem 4.2 is omitted (see Theorem 7.1 in [4]).

For the convergence of the numerical extinction time T_h^*, we obtain

THEOREM 4.3. *Under the same assumptions as stated in Theorem* 4.1,

$$\lim_{h\to 0} T_h^* = T^*, \tag{4.9}$$

holds, where T^* is the extinction time of the weak solution v of (1.1), (1.2).

PROOF. For an arbitrary number $t<T^*$, there exists $\tilde{x}\in\mathbf{R}^1$ such that $v(t, \tilde{x})>0$. From (4.8), it follows that

$$v_h(t, \tilde{x}) \geqq v(t, \tilde{x})/2 > 0, \qquad \text{for} \quad h < h',$$

where $h'>0$ is some constant, which implies $T_h^*>t$. Thus

$$\liminf_{h\to 0} T_h^* \geqq T^*. \tag{4.10}$$

On the other hand, from (3.10) in Lemmas 3.1 and 3.2 and (3.17), it follows that

$$T_h^* \leqq t + \|u_h(t,\cdot)\|_\infty/c', \qquad \text{for} \quad t\geqq 0.$$

For any $\varepsilon>0$, let $t=T^*+\varepsilon$. Since

$$\|u_h(T^*+\varepsilon,\cdot)\|_\infty \longrightarrow \|u(T^*+\varepsilon,\cdot)\|_\infty = 0 \quad \text{as} \quad h\longrightarrow 0,$$

we obtain

$$\limsup_{h\to 0} T_h^* \leqq T^* + \varepsilon,$$

which yields

$$\limsup_{h\to 0} T_h^* \leqq T^*. \tag{4.11}$$

Hence, (4.9) follows from (4.10) and (4.11), and the proof is complete.

PROOF OF THEOREM 4.1. Using Theorem 6.1 in Graveleau and Jamet [4] and Theorem 3.1, we can find a subsequence $\{h'\}$ of $\{h\}$ and a function u satisfying (i), (ii) in Definition 4.2 and (4.3) such that (4.6) and (4.7) holds. Here, we used the fact that the support of u_h is uniformly bounded in $\mathscr{H}$ with respect to h (see (3.1) and (3.29)).

To complete the proof, it suffices to show that u satisfies the integral relation (4.2), because (4.6) and (4.7) with the whole sequence $\{h\}$ follow from the uniqueness of the weak solution. In the following, to simplify notations, we write $\{h\}$ instead of $\{h'\}$.

Let us define u_i^n by $u_i^n = u_h(t_n, ih)$ for $i \in \mathbf{Z}$ and $n \geqq 0$. Then

$$(4.12) \qquad \begin{aligned} (u_i^{n+1} - u_i^n)/k_{n+1} = {} & (1/\mu_{n+1}) \textstyle\sum_{r=0}^{\mu_{n+1}-1} (u_i^{n,r+1} - u_i^{n,r})/k'_{n+1} \\ & + (u_i^{n,0} - \bar{u}_i^n)/k_{n+1} \\ & + (\bar{u}_i^n - u_i^n)/k_{n+1} \end{aligned}$$

for $i \in \mathbf{Z}$ and $n \geqq 0$, where

$$\begin{aligned} \bar{u}_i^n &= ((I_h + k_{n+1} D_h) u_h^n)(ih), \\ u_i^{n,0} &= ((I_h + k_{n+1} H_h)(I_h + k_{n+1} D_h) u_h^n)(ih), \\ u_i^{n,r} &= ((I_h + k'_{n+1} P_h)^r (I_h + k_{n+1} H_h)(I_h + k_{n+1} D_h) u_h^n)(ih). \end{aligned}$$

Let $\phi \in C^{1,2}(\mathscr{H})$ satisfy $\operatorname{supp} \phi \subset \operatorname{supp} u$. Multiplying both sides of (4.12) by $hk_{n+1}\phi_i^n$, where $\phi_i^n = \phi(t_n, ih)$, and summing for $i \in \mathbf{Z}$ and $n \geqq 0$, we have by summation by parts

$$(4.13) \qquad \textstyle\sum_i hu_i^0 \phi_i^0 - \sum_{n,i} hk_{n+1} u_i^{n+1} (\phi_i^{n+1} - \phi_i^n)/k_{n+1} = A_h + B_h + C_h,$$

where

$$\begin{aligned} A_h &= \textstyle\sum_{n,i} hk_{n+1} \phi_i^n (1/\mu_{n+1}) \{ \sum_{r=0}^{\mu_{n+1}-1} (u_i^{n,r+1} - u_i^{n,r})/k'_{n+1} \}, \\ B_h &= \textstyle\sum_{n,i} hk_{n+1} \phi_i^n (u_i^{n,0} - \bar{u}_i^n)/k_{n+1}, \\ C_h &= \textstyle\sum_{n,i} hk_{n+1} \phi_i^n (\bar{u}_i^n - u_i^n)/k_{n+1}. \end{aligned}$$

To prove (4.2), we have only to show the following results: As $h \to 0$,

$$(4.14) \qquad A_h \longrightarrow - \iint_{\mathscr{H}} \{ muu_x \phi_x + m(u_x)^2 \phi \} dxdt,$$

$$(4.15) \qquad B_h \longrightarrow \iint_{\mathscr{H}} a(u_x)^2 \phi dxdt,$$

$$(4.16) \qquad C_h \longrightarrow \iint_{\mathscr{H}} (-c'\phi) dxdt,$$

$$(4.17) \qquad \textstyle\sum_{n,i} hk_{n+1} u_i^{n+1} (\phi_i^{n+1} - \phi_i^n)/k_{n+1} \longrightarrow \displaystyle\iint_{\mathscr{H}} u\phi_t dxdt,$$

$$(4.18) \qquad \textstyle\sum_i hu_i^0 \phi_i^0 \longrightarrow \displaystyle\int_{\mathbf{R}^1} u(0, x)\phi(0, x) dx.$$

Since (4.14), (4.15) and (4.17) are proved in Lemma 4.1 in Mimura, Nakaki and Tomoeda [11] and (4.18) is obvious, it suffices to prove (4.16).

Let us define $\phi_h(t, x)$ by

$$\phi_h(t, (i+\theta)h) = (1-\theta)\phi_i^n + \theta\phi_{i+1}^n,$$

for $t \in [t_n, t_{n+1})$, $n \geqq 0$ and $\theta \in [0, 1]$, and let χ_h be the characteristic function of $\operatorname{supp} u_h$ defined by

$$\chi_h(t, x) = \begin{cases} 1, & \text{if} \quad u_h(t, x) > 0, \\ 0, & \text{if} \quad u_h(t, x) = 0. \end{cases}$$

Then

$$\begin{aligned} \left| C_h - \iint_{\mathscr{X}} (-c'\phi) dx dt \right| &\leqq |C_h - \textstyle\sum_{n,i} hk_{n+1}\phi_i^n(-c'\chi_h(t_n, ih))| \\ &+ \left| \textstyle\sum_{n,i} hk_{n+1}\phi_i^n(-c'\chi_h(t_n, ih)) - \displaystyle\iint_{\mathscr{X}} (-c'\phi_h\chi_h) dx dt \right| \\ &+ \left| \iint_{\mathscr{X}} (-c'\phi_h\chi_h) dx dt - \iint_{\mathscr{X}} (-c'\phi) dx dt \right| \\ &\leqq \|\phi\|_\infty \textstyle\sum_n k_{n+1} C_{1h}^n + c' C_{2h} + c' C_{3h}, \end{aligned}$$

where

$$C_{1h}^n = \textstyle\sum_i h|(u_i^n)_t - (-c'\chi_h(t_n, ih))|,$$

$$C_{2h} = |\textstyle\sum_{n,i} hk_{n+1}\phi_i^n\chi_h(t_n, ih)) - \displaystyle\iint_{\mathscr{X}} \phi_h\chi_h dx dt|,$$

$$C_{3h} = \iint_{\mathscr{X}} |\phi_h\chi_h - \phi| dx dt \quad \text{and} \quad (u_i^n)_t = (\bar{u}_i^n - u_i^n)/k_{n+1}.$$

From $\operatorname{supp}\phi \subset \operatorname{supp} u$ and the convergence of u_h, it is easily to verify that C_{2h} and C_{3h} tend to zero as $h \to 0$.

Since $\bar{u}_i^n = [u_i^n - c'k_{n+1}]^+$ (see (2.22)), we have

$$(u_i^n)_t = -c'\chi_h(t_h, ih), \quad \text{if} \quad u_i^n - c'k_{n+1} \geqq 0 \quad \text{or} \quad u_i^n = 0.$$

In the case $t_{n+1} < T_h^*$, from Condition A-2), the number of an integer i satisfying

$$u_i^n - c'k_{n+1} < 0 \quad \text{and} \quad u_i^n > 0 \tag{4.19}$$

does not exceed two. When $t_{n+1} = T_h^*$, (4.19) holds for all $i \in \{L(\ell(u_h^n)), \cdots, R(r(u_h^n))\}$. Therefore,

$$C_{1h}^n \leqq \begin{cases} 2hc', & \text{if} \quad t_{n+1} < T_h^*, \\ (R-L+1)hc', & \text{if} \quad t_{n+1} = t_{N+1} \equiv T_h^*, \end{cases}$$

where $R = R(r(u_h^N))$ and $L = L(\ell(u_h^N))$. Here, we used the fact

$$|(u_i^n)_t - (-c'\chi_h(t_n, ih))| \leqq c'.$$

Since

$$(R-L+1)h \leqq r_N - \ell_N + h \leqq M + h$$

(see (3.29)), we have

$$\begin{aligned}\textstyle\sum_n k_{n+1} C^n_{1h} &\leqq 2hc' \textstyle\sum_n k_{n+1} + (M+h)c'k_{N+1} \\ &\leqq 2hc'T^*_h + (M+h)c'k_{N+1}.\end{aligned}$$

Thus,

$$\textstyle\sum_n k_{n+1} C^n_{1h} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.$$

Hence, (4.16) is shown, and the proof of Theorem 4.1 is complete.

5. Convergence of numerical interfaces

For the numerical solutions $u^n_h(x)$ $(n \geqq 0)$ computed by the scheme (2.5), we define a numerical left (resp. right) interface $\ell_h(t)$ (resp. $r_h(x)$) by piecewise-linearly interpolating $(t_n, \ell(u^n_h))$ (resp. $(t_n, r(u^n_h))$ $(n \geqq 0)$. Let T be an arbitrary number satisfying $0 < T < T^*$, where T^* is the extinction time of the weak solution of (1.1), (1.2). Then, by Theorem 4.3, $\ell_h(t)$ and $r_h(t)$ are defined on $[0, T]$ for sufficiently small $h > 0$.

THEOREM 5.1. *Let the assumptions of Theorem* 4.1 *be satisfied and let T be an arbitrary number satisfying* $0 < T < T^*$. *Then there exist positive constants* C_T *and C, which are independent of h, satisfying*

$$-C \leqq (\ell_h(t') - \ell_h(t))/(t'-t) \leqq C_T, \tag{5.1}$$

$$-C_T \leqq (r_h(t') - r_h(t))/(t'-t) \leqq C \tag{5.2}$$

for $t, t' \in [0, T]$.

PROOF. We prove (5.1). Since ℓ_h is a piecewise-linearly function, it suffices to show (5.1) for $t = t_n$ and $t' = t_{n+1}$ belonging to $[0, T]$. For simplicity, we use the following notations:

$$u_i = u^n_h(x_i), \quad \ell = \ell(u^n_h), \quad L = L(\ell),$$
$$\ell^+ = \ell((I_h + kD_h)u^n_h), \quad L^+ = L(\ell^+) \quad \text{and} \quad k = k_{n+1},$$

where $\{x_i\}$ is the set of nodal points of u^n_h. We consider the following two cases:

Case 1. $L^+ = L$;

Case 2. $L^+ = L + 1$.

We note that one of these two cases occurs by Condition A–2).

In *Case* 1, we have from (2.12) and (2.23)

$$\ell_h(t_{n+1}) - \ell_h(t_n) = \ell_h(t_{n+1}) - \ell^+ + \ell^+ - \ell \tag{5.3}$$
$$= - a\delta u_{L-1}k + c'k/\delta u_{L-1}.$$

Since $\|u(t,\cdot)\|_\infty$ decreases monotonously with respect to t and u_h is concave, it follows that

$$\delta u_{L-1} \geqq \|u_h^n\|_\infty/(r_n - \ell_n) \geqq \varepsilon_T \equiv \|u(T,\cdot)\|_\infty/M > 0,$$

where M is the constant defined in (3.29). Hence

$$- aCk \leqq \ell_h(t_{n+1}) - \ell_h(t_n) \leqq c'k/\varepsilon_T \tag{5.4}$$

holds, which implies (5.1). Here C is the constant in (3.6).

In *Case* 2, the concavity of u_h^n, (2.23) and (2.12) yield

$$0 \leqq \ell^+ - \ell = (c'k - u_L)/\delta u_L + Lh - \ell \tag{5.5}$$
$$= (ck' - u_L)/\delta u_L + u_L/\delta u_{L-1}$$
$$\leqq c'k/\delta u_L,$$

$$0 > \ell_h(t_{n+1}) - \ell^+ = - a\delta u_L k. \tag{5.6}$$

Hence

$$- a\delta u_L k \leqq \ell_h(t_{n+1}) - \ell_h(t_n) \leqq c'k/\delta u_L. \tag{5.7}$$

Since (3.15) holds in Lemmas 3.1–3.3, we have

$$\delta u_L = \delta u_{L^+-1} \geqq \delta u_{L_{n+1}-1}^{n+1} \geqq \varepsilon_T, \tag{5.8}$$

where $L_{n+1} = L(\ell(u_h^{n+1}))$. Therefore, by (5.7) and (5.8), the same inequality as (5.4) holds. Thus (5.1) is proved. Similarly, (5.2) can be shown, and the proof of Theorem 5.1 is complete.

By using the above theorem, we obtain the convergence theorem of numerical interfaces.

THEOREM 5.2. *Under the same assumptions as stated in Theorem* 4.1, *there exist locally Lipschitz continuous functions* $\ell(t)$ *and* $r(t)$ *defined on* $[0, T^*)$ *such that*

(5.9) $\ell_h \longrightarrow \ell$ *and* $r_h \longrightarrow r$, *compact uniformly on* $[0, T^*)$ *as* $h \longrightarrow 0$,

(5.10) $v(t, x) > 0$ *on* $0 \leqq t < T^*$ *and* $\ell(t) < x < r(t)$,

(5.11) $v(t, x) = 0$ *on* $0 \leqq t < T^*$ *and* $x \leqq \ell(t)$, $x \geqq r(t)$,

where v *is the unique weak solution of* (1.1), (1.2) *and* T^* *is its extinction time.*

PROOF. Let T be an arbitrary number satisfying $0<T<T^*$. Then, by Theorem 5.1 and Ascoli-Arzela's Theorem, there exist Lipschitz continuous functions $\ell(t)$ and $r(t)$ defined on $[0, T]$ and a subsequence $\{h'\}$ of $\{h\}$ such that

(5.12) $\ell_{h'} \longrightarrow \ell$ and $r_{h'} \longrightarrow r$, uniformly on $[0, T]$ as $h' \longrightarrow 0$.

Now we show (5.10) and (5.11). (5.11) follows from the convergence of $\ell_{h'}$ and $r_{h'}$.

For each fixed t $(0 \leqq t \leqq T)$, let x^* be a point satisfying $u(t, x^*)=\|u(t, \cdot)\|_\infty$. Then, by the concavity of $u_{h'}$, we have

(5.13) $$u_{h'}(t, x) \geqq (u_{h'}(t, x^*)-u_{h'}(t, \ell_{h'}(t)))(x-\ell_{h'}(t))/(x^*-\ell_{h'}(t))$$
$$\text{for} \quad x \in [\ell_{h'}(t), x^*],$$

(5.14) $$u_{h'}(t, x) \geqq (u_{h'}(t, x^*)-u_{h'}(t, r_{h'}(t))(x-r_{h'}(t))/(x^*-r_{h'}(t))$$
$$\text{for} \quad x \in [x^*, r_{h'}(t)].$$

For each fixed $\tilde{x} \in (\ell(t), r(t))$, it follows that

$$\ell_{h'}(t) \leqq \tilde{x} \leqq r_{h'}(t) \qquad \text{for} \quad h' < h_1,$$

where h_1 is some positive constant. Replacing x by $\tilde{x}$ in (5.13) and (5.14), and letting $h \to 0$, we have $u(t, \tilde{x})>0$. Thus (5.10) holds.

Since $\ell(t)$ and $r(t)$ satisfying (5.10) and (5.11) are uniquely determined, (5.12) holds for the whole sequence $\{h\}$, and the proof of Theorem 5.2 is complete.

REMARK 5.1. In general, the interfaces do not satisfy Lipschitz condition on $[0, T^*)$ (see Kersner's explicit solution (Fig. 2)).

6. Proofs of Lemmas in Section 3

6.1. Proof of Lemma 3.1

The facts that u_h^j belongs to V_h and (3.10)–(3.14) are shown in Lemma 3.1 in Mimura, Nakaki and Tomoeda [11].

To show the remainder of Lemma 3.1, let us consider a Cauchy problem

(6.1) $$w_t = a(w^2)_x, \quad (t, x) \in (0, \infty) \times \mathbf{R}^1,$$

(6.2) $$w(0, x) = w^0(x) \equiv (u_h)_x(x), \quad x \in \mathbf{R}^1,$$

which is obtained by differentiating $u_t = Hu$ with respect to x. $w^0(x)$ becomes a piecewisely constant function. The problem (6.1), (6.2) is called the Riemann problem of the Burgers equation. It is already known that, for a time step $k>0$

satisfying (2.14)–(2.16), a solution $w(k, x)$ of (6.1), (6.2) at $t=k$ consists of constant states 0, $w^0(x_{L-1}+0)$, $w^0(x_L+0),\cdots, w^0(x_R+0)$, 0, which are separated by shock waves and connected by rarefaction waves. In particular, the states 0 and $w^0(x_{L-1}+0)$ (resp. $w^0(x_R+0)$ and 0) are separated by a shock wave on the line

$$(6.3)\qquad y_\ell(t) = \ell - aw^0(x_{L-1}+0)t \quad (\text{resp. } y_r(t)=r-aw^0(x_R+0)t),$$

where $\ell=\ell(u_h)$ and $r=r(u_h)$. (6.3) is well known as the Rankine-Hugoniot jump condition. Putting

$$\tilde{u}(x) = \int_{-\infty}^{x} w(k, \eta)d\eta,$$

we have from the conservation law of the Burgers equation and simple calculations

$$(6.4)\qquad \tilde{u}(ih) = u_i' \equiv u_h'(x_i'), \qquad \text{for} \quad i\in\mathbf{Z},$$

$$(6.5)\qquad \ell(\tilde{u}) = y_\ell(k) = \ell' \equiv \ell(u_h'),$$

$$(6.6)\qquad r(\tilde{u}) = y_r(k) = r' \equiv r(u_h').$$

Since u_h is concave on its support, the states $w^0(x_{L-1}+0)$, $w^0(x_L+0),\cdots, w^0(x_R+0)$ are connected by rarefaction waves. From this fact, the solution of the Riemann problem possesses the following properties:

$$(6.7)\qquad w(k, \xi) \geqq w(k, \eta), \qquad \text{for} \quad \ell' < \xi < \eta < r',$$

$$(6.8)\qquad w(k, \ell'+0) \leqq w(x_{L+1}+0) \quad \text{and} \quad w(k, r'-0) \geqq w(x_R+0).$$

Moreover, from (6.4),

$$\delta u_i' = w(k, y_i), \qquad \text{for} \quad i\in\{L'-1,\cdots, R'\}$$

holds for some number $y_i \in (x_i', x_{i+1}')$. Hence, the concavity of u_h' follows from (6.7), and (6.8) yields (3.15). Thus, the proof of Lemma 3.1 is complete.

6.2. Proof of Lemma 3.2

It is shown in Lemma 3.2 in Mimura, Nakaki and Tomoeda [11] that u_h' belongs to V_h and satisfies (3.10)–(3.13) and (3.16).

First, we show (3.15). We note that $L'=L$ and $R'=R$ hold, because the operator P_h does not change the interfaces. From (2.8) and the concavity of u_h, we have

$$u'_L = u_L + k'mu_L\delta^2 u_L \leqq u_L,$$

which implies $\delta u'_{L-1} \leqq \delta u_{L-1}$. Similarly, we can prove $\delta u_R' \geqq \delta u_R$. Thus, (3.15) is shown.

Next, we show

$$\delta^2 u_i' \leqq 0 \quad (L \leqq i \leqq R), \tag{6.9}$$

which means the concavity of u_h'. By (2.7), we have

$$\delta^2 u_L' = \{1 - 2k' m \delta u_{L-1}/h\}\delta^2 u_L + 2k' m u_{L+1}/\{h(h + h_{L-1})\}\delta^2 u_{L+1}. \tag{6.10}$$

Since (2.11) yields

$$1 - 2k' m \delta u_{L-1}/h \geqq 0,$$

the inequality (6.9) with $i = L$ follows from (6.10) and $\delta^2 u_i \leqq 0$ $(i \in \mathbf{Z})$. Similarly, (6.9) can be shown for $i = R$. Let us consider the case $L+1 \leqq i \leqq R-1$. We have from (2.7)

$$\delta^2 u_i' = \{1 - 2\lambda u_i\}\delta^2 u_i + \lambda u_{i+1}\delta^2 u_{i+1} + \lambda u_{i+1}\delta^2 u_{i-1}, \tag{6.11}$$

where $\lambda = k' m / h^2$. Each coefficient of $\delta^2 u_j$ $(j = i-1, i, i+1)$ on the right hand side of (6.11) is nonnegative by (2.10) and (2.11). Hence, we obtain (6.9), and the proof of Lemma 3.2 is complete.

6.3. Proof of Lemma 3.3

Since the assumption $m + p = 2$ implies $q = 0$, (2.22) is rewritten as

$$u_h' = [u_h(x) - c' k]^+,$$

from which, Lemma 3.3 can be easily shown.

6.4. Proof of Lemma 3.4

We show $\ell_{n+1} \geqq \ell_n$ $(n \geqq p)$ by induction on n. It can be similarly shown that $r_{n+1} \leqq r_n$ $(n \geqq p)$. Suppose

$$\ell_{j+1} \geqq \ell_j, \qquad \text{for some} \quad j(\geqq p). \tag{6.12}$$

We now prove

$$\ell_{j+2} \geqq \ell_{j+1}. \tag{6.13}$$

For simplicity, we use the notations

$$u_i = u_h^j(x_i), \quad u_i' = u_h^{j+1}(x_i'),$$
$$L_n = L(\ell(u_h^n)) \quad \text{and} \quad L_n^+ = L(\ell((I_h + k_{n+1} D_h) u_h^n)) \quad (n = j, j+1),$$

where $\{x_i\}$ and $\{x_i'\}$ are sets of nodal points of u_h and u_h', respectively. We consider the following four cases:

Case A: $L_j^+ = L_j + 1$ and $L_{j+1}^+ = L_{j+1}$;
Case B: $L_j^+ = L_j + 1$ and $L_{j+1}^+ = L_{j+1} + 1$;
Case C: $L_j^+ = L_j$ and $L_{j+1}^+ = L_{j+1}$;
Case D: $L_j^+ = L_j$ and $L_{j+1}^+ = L_{j+1} + 1$.

We will prove (6.13) in *Case A*.

Since u_h^m $(m \geqq 0)$ are concave on its support by Lemmas 3.1–3.3, we have from (5.5) and (5.6)

$$\ell_{j+1} - \ell_j \leqq (-a\delta u_L + c'/\delta u_L)k_{j+1}, \tag{6.14}$$

and we have from (5.3)

$$\ell_{j+2} - \ell_{j+1} = (-a\delta u'_{L'-1} + c'/\delta u'_{L'-1})k_{j+2}, \tag{6.15}$$

where $L = L_j$ and $L' = L_{j+1}$. Moreover, by (3.15) stated in Lemmas 3.1 and 3.2,

$$a(\delta u'_{L'-1})^2 - c' \leqq a(\delta u_L)^2 - c' \tag{6.16}$$

holds. Since the right hand side of (6.16) is nonpositive by (6.12) and (6.14), (6.13) follows from (6.15). Similarly, (6.13) is proved in *Cases B*, *C* and *D*. Thus, the proof of Lemma 3.4 is complete.

6.5. Proof of Lemma 3.5

It is easy to show that the set of k satisfying (2.14)–(2.16) with $\ell = \ell'$, $L = L'$, $r = r'$ and $R = R'$ becomes K_M, $K_{1L} \cup K_{2L}$ and $K_{1R} \cup K_{2R}$, respectively, where $\ell' = \ell(u_h')$, $L' = L(\ell')$, $r' = r(u_h')$, $R' = R(r')$ and $u_h' = (I_h + D_h)u_h$. Moreover, any number $k \in K_{1L} \cup K_{1R}$ satisfies Condition A–2). By using these facts, Lemma 3.5 can be easily proved.

7. Numerical simulations

In this section, we display some numerical examples of finite extinction phenomena. First, to check accuracy of our numerical scheme, we compare the numerical approximations with the exact solutions.

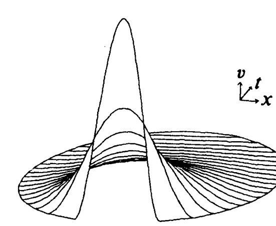

Fig. 1. Kersner's exact solution of (1.1) with $m = 1.5$, $c = 1$ and $p = 0.5$.

Fig. 1 shows Kersner's exact solution with $m = 1.5$, $c = 1$ and $p = 0.5$. We note that his initial function satisfies all of our assumptions.

Fig. 2 displays the interfaces of Kersner's solution and the numerical interfaces of our scheme with $h = 0.005$, where the initial function is the exactly same as that of Kersner's solution. It is observed that our scheme gives good approximations to his solution.

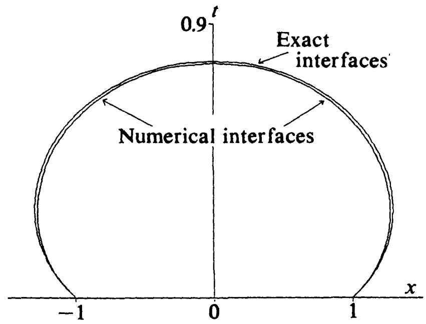

Fig. 2. Exact interfaces of Kersner's solution and numerical interfaces of our scheme with $h=0.05$.

Let us show another example which suggests that there exists a constnat $\alpha>0$ satisfying

$$\text{mes}\,(S(t)) > \alpha \quad \text{on} \quad 0 \leqq t < T^*, \tag{7.1}$$

where mes(A) denotes the measure of $A\subset\mathbf{R}^1$. Figs. 3 and 4 display the numerical solution and its interfaces, respectively, where $m=1.5$, $c=10$, $p=0.5$ and

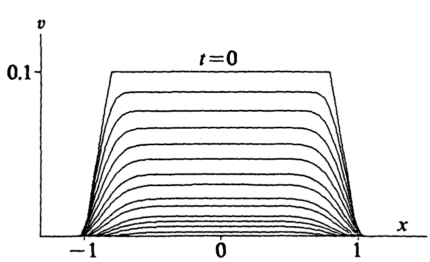

Fig. 3. Numerical solution with $m=1.5$, $c=10$, $p=0.5$ and $h=0.02$. The initial function takes (7.2). $t=0$, 0.004, 0.008, 0.012, 0.016, 0.020, 0.025, 0.028, 0.033, 0.036, 0.041, 0.044, 0.048, 0.053.

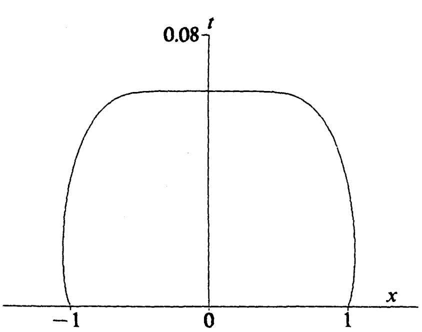

Fig. 4. Numerical interfaces of the solution in Fig. 3.

$$v^0(x) = \begin{cases} 0.1, & |x| \leqq 0.8, \\ (1-|x|)/2, & 0.8 < |x| \leqq 1, \\ 0, & |x| > 1. \end{cases} \tag{7.2}$$

Fig. 3 shows that the flatness of the numerical solution is kept until the numerical

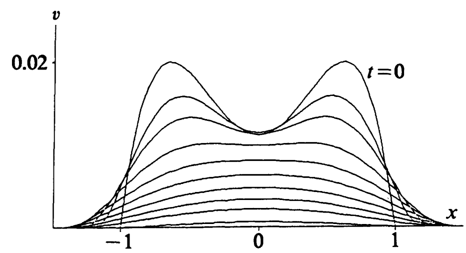

Fig. 5. Numerical solution with $m=1.5$, $c=0.1$, $p=0.5$ and $h=0.05$. The initial function takes (7.3). $t=0$, 0.1, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.8, 2.3.

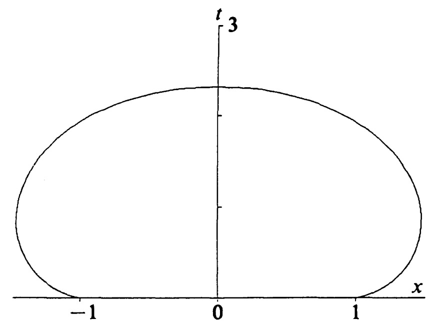

Fig. 6. Numerical interfaces of the solution in Fig. 5.

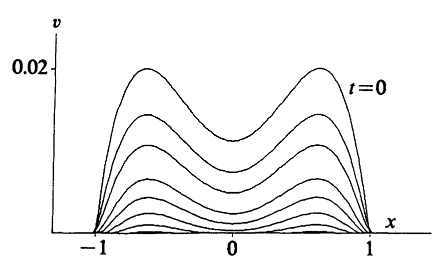

Fig. 7. Numerical solution with $m=1.5$, $c=5$, $p=0.5$ and $h=0.02$. The initial function takes (7.3). $t=0$, 0.08, 0.014, 0.023, 0.029, 0.035, 0.042, 0.050.

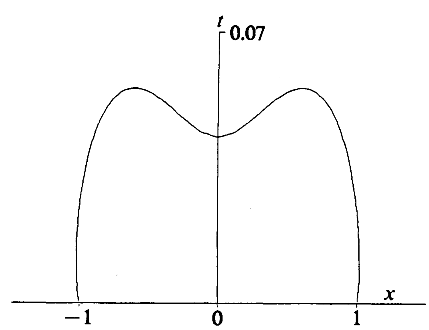

Fig. 8. Numerical interfaces of the solution in Fig. 7.

extinction time comes. Therefore, it seems that (7.1) holds (see Fig. 4).

Finally, we try to calculate the case where $m=1.5$, $p=0.5$ and

$$(7.3) \qquad v^0(x) = \begin{cases} 0.077(1-x^2)(x^2+0.02), & |x| \leqq 1, \\ 0, & |x| > 1, \end{cases}$$

though our result is not valid because $(v^0)^{m-1}$ is not concave (see Remark 3.1). With $c=0.1$, a number of peaks of $v_h(t, \cdot)$ changes from 2 to 1 and $S(t)$ is connected on $[0, T_h^*]$ (see Figs. 5 and 6). On the other hand, with $c=5$, Figs. 7 and 8 show that a number of peaks is invariant and $S(t)$ is splitted.

Until now, we have not found out numerical solutions such that their support are monotonously expanding on $[0, T_h^*)$. While Kersner's explicit solution shows that the support is contracting monotonously on some time interval (T, T^*).

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