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Comparison of powers of a class of tests for covariance matrices

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1. Introduction

This paper is concerned with problems of testing the hypotheses (i) for the equality of covariance matrix to a given matrix, (ii) for the sphericity and (iii) for the equality of two covariance matrices. When the underlying distribution is normal, the commonly used tests for testing these hypotheses are the likelihood ratio (=LR) tests. Anderson [1], Sugiura ([10], [11], [12]) and Nagao ([6], [8]) derived the asymptoic expansions of their null and non-null distributions. Nagao ([7], [9]) proposed certain test statistics for testing the above hypotheses and derived the asymptotic expansions of their null and non-null distributions. Hayakawa [3] proposed a modified Wald statistic for a simple hypothesis when underlying distribution is more general. He made the comparison of some tests for the problem (i) under local alternatives.

Let the $p \times 1$ vectors $X_1, \dots, X_N$ be a random sample from a normal distribution with mean vector μ and covariance matrix Σ. The modified LR criterion for testing the hypothesis $\mathscr{H}: \Sigma = \Sigma_0$ against the alternatives $\mathscr{K}: \Sigma \neq \Sigma_0$ for some given positive definite matrix Σ_0, is given by

$$(1.1) \qquad \lambda = |\Sigma_0^{-1} S|^{n/2} \operatorname{etr}\{-(n/2)(\Sigma_0^{-1} S - I)\},$$

where $S = n^{-1} \sum_{j=1}^{N} (X_j - \bar{X})(\bar{X}_j - \bar{X})'$, $\bar{X} = N^{-1} \sum_{J=1}^{N} X_j$ and $n = N-1$. Wald statistic is given by

$$(1.2) \qquad T_1 = (n/2) \operatorname{tr}(S^{-1} \Sigma_0 - I)^2.$$

The test statistic proposed by Nagao [7] is given by

$$(1.3) \qquad T_2 = (n/2) \operatorname{tr}(\Sigma_0^{-1} S - I)^2.$$

These three statistics are the symmetric functions of latent roots of $\Sigma_0^{-1} S$. Let $d_1, \dots, d_p$ be the latent roots of $\Sigma_0^{-1} S$. It is seen that $(d_1, \dots, d_p)$ is a maximal invariant under a certain group of transformations (see e.g., Muirhead [5]). We consider a class C of ststistics

$$(1.4) \qquad T = n \sum_{j=1}^{p} Q(d_j),$$

where the critical region based on T is "$T>k$" and $Q(d)$ is a function on $d>0$ satisfying the following assumptions $\mathscr{A}1 \sim \mathscr{A}4$.

$\mathscr{A}1$. $Q(d)$ has a continuous fifth order derivative in a neighborhood of $d=1$,

$\mathscr{A}2$. $Q''(1)=1$,

$\mathscr{A}3$. $Q(1)=Q'(1)=0$,

$\mathscr{A}4$. $Q'(d)<0$ if $0<d<1$,
$Q'(d)>0$ if $1<d$.

If $\mathscr{H}$: $\Sigma=\Sigma_0$ is true, the roots d_j should be close to 1. T may be regarded as a measure of the deviation from the hypothesis by $\mathscr{A}3$ and $\mathscr{A}4$. Without loss of generality we can impose $\mathscr{A}2$ under $\mathscr{A}1$, $\mathscr{A}3$ and $\mathscr{A}4$. $\mathscr{A}1$ is necessary for obtaining an asymptotic expansion of the distribution of T. The modified LR test $T_0=-2\log\lambda$, Wald test T_1 and Nagao's T_2 belong to C.

In this paper we shall derive the asymptotic expanison of the null and non-null distributions of test statistics in C. Our purpose is to compare the local powers of tests on the basis of the expansions. The differences in the powers of all the tests in C can be explained in terms of $q=Q^{(3)}(1)$. The comparisons reveal no uniform superiority properties.

2. Asymptotic expansions of the null distribution for $\Sigma=\Sigma_0$

We shall give an asymptotic expansion of the null distribution of T given by (1.4). Without loss of generality we may assume that $\Sigma_0=\boldsymbol{I}$. Then $n\boldsymbol{S}$ has a Wishart distribution $\mathscr{W}_p(n, \boldsymbol{I})$. Let $\boldsymbol{Y}=n^{1/2}(\boldsymbol{S}-\boldsymbol{I})$, then $\boldsymbol{Y}$ is asymptotically normal and the p. d. f. of the distribution of $\boldsymbol{Y}$ can be expressed (see, Siotani, Hayakawa and Fujikoshi [13]) as

$$f(\boldsymbol{Y})=c_p^{-1}\operatorname{etr}\{-(1/4)\boldsymbol{Y}^2\}\{\boldsymbol{I}+n^{-1/2}Q_1(\boldsymbol{Y})+n^{-1}Q_2(\boldsymbol{Y})\}+o(n^{-1}), \tag{2.1}$$

where $c_p=\pi^{p(p+1)/4}2^{p(p+3)/4}$ and

$$\begin{aligned} Q_1(\boldsymbol{Y}) &= -(1/2)(p+1)\operatorname{tr}\boldsymbol{Y}+(1/6)\operatorname{tr}\boldsymbol{Y}^3, \\ Q_2(\boldsymbol{Y}) &= (1/2)\{Q_1(\boldsymbol{Y})\}^2-(1/24)p(2p^2+3p-1) \\ &\quad +(1/4)(p+1)\operatorname{tr}\boldsymbol{Y}^2-(1/8)\operatorname{tr}\boldsymbol{Y}^4. \end{aligned} \tag{2.2}$$

Considering a Taylor expansion of $Q(d)$ at $d=1$, we can express the statistic T in terms of $\boldsymbol{Y}$ for large n as

$$T=(1/2)\operatorname{tr}\boldsymbol{Y}^2+(1/6)qn^{-1/2}\operatorname{tr}\boldsymbol{Y}^3+(1/24)rn^{-1}\operatorname{tr}\boldsymbol{Y}^4+o_p(n^{-1}), \tag{2.3}$$

where $q=Q^{(3)}(1)$ and $r=Q^{(4)}(1)$. We shall find the asymptotic expansion of the characteristic function of T up to the order n^{-1} and invert it. The characteristic function of T can be written as

$$(2.4)\qquad c(t)=\mathscr{E}[\exp(1/2)\ (it)\ \mathrm{tr}\ Y^2)\ \{1+(1/6)\ (qit)\ n^{-1/2}\mathrm{tr}\ Y^3 + n^{-1}[(1/72)q^2(it)^2(\mathrm{tr}\, Y^3)^2+(1/24)\ (rit)\ \mathrm{tr}\, Y^4]\}+o_p(n^{-1})].$$

Using the p.d.f. of Y given by (2.1) and (2.2), we can obtain

$$(2.5)\qquad c(t)=(1-2it)^{-f/2}\mathscr{E}[1+n^{-1/2}\{Q_1(\tilde{Y})+(1/6)\ (qit)\ \mathrm{tr}\,\tilde{Y}^3\} + n^{-1}\{(1/27)\ (qit)^2(\mathrm{tr}\,\tilde{Y}^3)^2+(1/24)\ (rit)\ \mathrm{tr}\,\tilde{Y}^4+Q_2(\tilde{Y}) + (1/6)\ (qit)\ \mathrm{tr}\,\tilde{Y}^3Q_1(\tilde{Y})\}]+o(n^{-1}),$$

where $f=(1/2)p(p+1)$, $Q_1(\cdot)$ and $Q_2(\cdot)$ are given by (2.2) and $\tilde{Y}=(\tilde{y}_{ij})$ has a $p(p+1)/2$ variate normal distribution with mean zero and $\mathrm{cov}(\tilde{y}_{ij},\tilde{y}_{kl})=(1-2it)^{-1}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Calculating the expectations in (2.5), we have

$$(2.6)\qquad c(t)=(1-2it)^{-f/2}\ \{1+n^{-1}\textstyle\sum_{j=0}^{3}a_j(1-2it)^{-j}\}+o(n^{-1}),$$

where

$$(2.7)\qquad \begin{aligned} a_0 &= -(1/24)p(2p^2+3p-1),\\ a_1 &= (1/24)p(2p^2+3p-1)+(1/48)\ (q+2)^2p(4p^2+9p+7)\\ &\quad -(1/12)\ (q+2)p(p^2+3p+4)-(1/48)\ (r-6)p(2p^2+5p+5),\\ a_2 &= -(1/24)\ (q+2)^2p(4p^2+9p+7)+(1/12)\ (q+2)p(p^2+3p+4)\\ &\quad +(1/48)\ (r-6)p(2p^2+5p+5),\\ a_3 &= (1/48)\ (q+2)^2p(4p^2+9p+7). \end{aligned}$$

This implies the following theorem.

THEOREM 2.1. *Let $d_1, \dots, d_p$ be the latent roots of $\Sigma_0^{-1}\boldsymbol{S}$ and let $T=T(d_1, \dots, d_p)$ be the statistic given by* (1.4). *Then the null distribution of T can be expanded for large n as*

$$(2.8)\qquad \mathrm{P}(T\leqslant x)=\mathrm{P}(x_f^2\leqslant x) + n^{-1}\textstyle\sum_{j=0}^{3}a_j\mathrm{P}(x_{f+2j}^2\leqslant x)+o(n^{-1}),$$

where $f=p(p+1)/2$ and the coefficients a_j's are given by (2.7).

The asymptotic expansions of T_0, T_1 and T_2 are obtained from (2.8) by putting $(q, r)=(-2, 6)$, $(-6, 36)$ and $(0, 0)$, respectively. The asymptotic expansions of the null and non-null distributions of T_0, T_1 and T_2 were obtained by Sugiura ([10], [11]), Hayakawa [3] and Nagao ([7], [9]), respectively.

Next we consider Bartlett's [2] adjustment for T. From (2.8) we have

$$\mathscr{E}(T)=f+n^{-1}\textstyle\sum_{j=0}^{3}a_j(f+2j)+o(n^{-1}) \tag{2.9}$$
$$=f(1+h/n)+o(n^{-1}),$$

where

$$h=(1/24)f^{-1}\{2p(2p^2+3p-1)+4(q+2)p(p^2+3p+4) \tag{2.10}$$
$$+(r-6)p(2p^2+5p+5)\}.$$

Therefore the Bartlett's adjustment factor is given by $\rho=1-h/n$ and ρT has an expected value closer to that of χ_f^2 than T has, in the sense of $\mathscr{E}(\rho T)=f+o(n^{-1})$. Further it holds that under the null hypothesis

$$\mathrm{P}(\rho T\leqslant x)=\mathrm{P}(\chi_f^2\leqslant x) \tag{2.11}$$
$$+n^{-1}\textstyle\sum_{j=0}^{3}\tilde{a}_jP(\chi_{f+2j}^2\leqslant \mathrm{x})+o(n^{-1}),$$

where

$$\tilde{a}_0=(1/12)\ (q+2)p(p^2+3p+4)+(1/48)\ (r-6)p(2p^2+5p+5), \tag{2.12}$$
$$\tilde{a}_1=(1/48)\ (q+2)^2p(4p^2+9p+7)-(1/6)\ (q+2)p(p^2+3p+4)$$
$$-(1/24)\ (r-6)p(2p^2+5p+5),$$
$$\tilde{a}_2=a_2,\ \tilde{a}_3=a_3$$

This shows that Bartlett's adjustment implies $\mathrm{P}(\rho T\leqslant x)=\mathrm{P}(\chi_f^2\leqslant x)+o(n^{-1})$ if and only if $q=-2$ and $r=6$. The LR test T_0 satisfies this property. (2.11) also shows that the χ^2-approximation χ_f^2 to T will get worse as $|q+2|$ and $|r-6|$ are large.

3. Asymptotic expansion of the non-null distribution and the comparison of powers for $\Sigma=\Sigma_0$

The asymptotic non-null distribution of T depends on the type of alternative being considered. Here we consider a sequence of alternatives $\mathscr{K}_n$: $\Sigma=\Sigma_0(I+n^{-1/2}\Theta)$. Under the alternatives $\mathscr{K}_n$, $W=\Sigma_0^{-1/2}S\Sigma_0^{-1/2}$ has a Wishart distribution $\mathscr{W}_p(n,\ I+n^{-1/2}\Theta)$. Let $Y=n^{1/2}(W-I-n^{-1/2}\Theta)$, then Y is asymptotically normal and the p.d.f. of the distribution of Y can be expanded (see, Siotani, Hayakawa and Fujikoshi [13]) as

$$f(Y)=c_p^{-1}\mathrm{etr}\{-(1/4)Y^2\}\ \{1+n^{-1/2}Q(Y,\ \Theta)\}+o(n^{-1/2}), \tag{3.1}$$

where c_p is given by (2.1) and

(3.2)
$$Q(Y, \Theta) = -(1/2)\ (p+1)\mathrm{tr}\,\Theta + (1/2)\mathrm{tr}\,Y^2\Theta - (1/2)\ (p+1)\mathrm{tr}\,Y + (1/6)\mathrm{tr}\,Y^3.$$

By the same consideration as in the null case, the characteristic function of T is given by

(3.3)
$$\begin{aligned} c(t) = (1-2it)^{-f/2}&\exp\{[it/2(1-2it)]\mathrm{tr}\,\Theta^2\} \\ \cdot\,\mathscr{E}[1+n^{-1/2}&\{(1/2)[1+(q+2)ita_t]\mathrm{tr}\,Y^2\Theta \\ &+ (1/6)[qita_t^3 + 8(it)^3a_t^3 + 2(it)^2a_t^2]\mathrm{tr}\,\Theta^3 \\ &-(1/2)\ (p+1)a_t\mathrm{tr}\,\Theta] + o(n^{-1/2}),\end{aligned}$$

where $f=p(p+1)/2$, $a_t=(1-2it)^{-1}$ and $Y=(y_{ij})$ has a $p(p+1)/2$ variate normal distribution with mean zero and $\mathrm{cov}\,(y_{ij}, y_{kl})=a_t(\delta_{ik}\delta_{jk}+\delta_{il}\delta_{jk})$. Calculating the expectations in (3.3), we have

(3.4)
$$c(t) = (1-2it)^{-f/2}\{1+n^{-1/2}\textstyle\sum_{j=0}^{3}b_j(1-2it)^{-j}\} + o(n^{-1/2}),$$

where the coefficient b_j's are given by

(3.5)
$$\begin{aligned} b_0 &= (1/3)\theta_3, \\ b_1 &= -(1/4)\ (p+1)\ (q+2)\theta_1 - (1/2)\theta_3, \\ b_2 &= (1/4)\ (p+1)\ (q+2)\theta_1 - (1/12)\ \{q+2)-2\}\theta_3, \\ b_3 &= (1/12)\ (q+2)\theta_3,\end{aligned}$$

with $\theta_j = \mathrm{tr}\,\Theta^j$. This implies the following theorem.

THEOREM 3.1 *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma=\Sigma_0(I+n^{-1/2}\Theta)$, *the distribution of T given by* (1.4) *can be expanded for large n as*

(3.6)
$$\mathrm{P}(T\leqslant x) = \mathrm{P}(\chi_f^2(\delta)\leqslant x) + n^{-1/2}\textstyle\sum_{j=0}^{3}b_j\mathrm{P}(\chi_{f+2j}^2(\delta)\leqslant x) + o(n^{-1/2}),$$

where $\delta=\mathrm{tr}\,\Theta^2/2$ *and the coefficient b_j's are given by* (3.5).

Using the relation

(3.7)
$$\mathrm{P}(\chi_{f+2}^2(\delta)>x) - \mathrm{P}\,(\chi_f^2(\delta)>x) = 2g_{f+2}(x;\ \delta),$$

where $g_f(x;\ \delta)$ is the p.d.f. of a noncentral χ^2 variate with f degrees of freedom and noncentrality parameter δ, the power of the test based on T in C follows from theorem 3.1.

THEOREM 3.2. *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma=\Sigma_0(I+n^{-1/2}\Theta)$, *the power* β_T *of the test with a level* α *based on* T *can be expanded as*

$$\beta_T=\beta_0+(1/2)n^{-1/2}(q+2)d(\Theta, u)+o(n^{-1/2}), \tag{3.8}$$

where β_0 *is the power of the test with* $q=-2$, u *is the upper* $100\alpha\%$ *point of the* χ_f^2 *distribution and*

$$d(\Theta, u)=(p+1)\theta_1 g_{f+4}(u;\ \delta)+(1/3)\theta_3 g_{f+6}(u;\ \delta). \tag{3.9}$$

Further β_0 *is given by*

$$\begin{aligned}\beta_0=\mathrm{P}(\chi_f^2(\delta)>u)&+(1/6)n^{-1/2}\theta_3\{2\mathrm{P}(\chi_f^2(\delta)>u)\\ &-3\mathrm{P}(\chi_{f+2}^2(\delta)>u)+\mathrm{P}(\chi_{f+4}^2(\delta)>u)\}+o(n^{-1/2}).\end{aligned} \tag{3.10}$$

We note that the power of the modified LR test is expanded as β_0 in (3.10). The sign of $d(\Theta, u)$ depends on Θ and u, and no one statistic is uniformly superior to the remainder in the sense of the comparisons of powers up to the order $n^{-1/2}$. If $d(\Theta, u)>0$, the test with larger q than -2 is preferable, but such a test is very poor for the alternatives such that $d(\Theta, u)<0$. Similarly if $d(\Theta, u)<0$, the test with smaller q than -2 is preferable, but such a test is very poor for the alternatives such that $d(\Theta, u)>0$. In practice, we will hesitate to recommend such a tests with larger values of $|q+2|$ because the approximation to the distribution of T will be worse as $|q+2|$ is large, and we do not know that $d(\Theta, u)$ is positive or negative. Some sufficient conditions for $d(\Theta, u)>0$ and $d(\Theta, u)<0$ are given as follows.

(a) If Θ is positive (negative) semidefinite, then $d(\Theta, u)>0$ (<0), for all u.

(b) If $\theta_3<0$ and $3(f+4)$ $(p+1)\theta_1+u\theta_3>0$, then $d(\Theta,u)>0$.

(c) If $\theta_3>0$ and $3(f+4)$ $(p+1)\theta_1+u\theta_3<0$, then $d(\Theta, u)<0$.

The conditions (b) and (c) are obtained by expressing the noncentral χ^2-density as a Poisson mixture of central χ^2-densities.

4. A class of test statistics for sphericity

Let the $p\times 1$ vectors $X_1,\dots, X_N$ be a random sample from a normal distribution with mean μ and covariance matrix Σ, and let $d_1,\dots, d_p$ be the latent roots of $pS/\mathrm{tr}\,S$, where $S=n^{-1}\sum_{j=0}^N(X_j-\bar{X})(X_j-\bar{X})^t$, $\bar{X}=N^{-1}\sum_{j=0}^N X_j$ and $n=N-1$. $(d_1,\dots, d_p)$ is a maximal invariant under a certain group of transformations (see, Muirhead [5]). For the sphericity hypothesis $\mathscr{H}$: $\Sigma=\sigma^2 I$ against the alternatives $\mathscr{K}$: $\Sigma\neq\sigma^2 I$, where σ^2 is unspecified, we consider the same class C of test statistics as in the problem (i), i.e.,

$$T = n\textstyle\sum_{j=1}^{p} Q(d_j), \tag{4.1}$$

where $Q(d)$ satisfies the same assumptions $\mathscr{A}1 \sim \mathscr{A}4$ as in (1.4). The LR test statistic T_0 and a test statistic T_1 proposed by Nagao [7] are defined by $Q(d) = d - 1 - \log d$ and $(d-1)^2/2$, respectively, but Wald statistic

$$T_2 = (1/2)n\{p - (\mathrm{tr}\,\boldsymbol{S}^{-1})^2/\mathrm{tr}\,\boldsymbol{S}^{-2}\} \tag{4.2}$$

dose not belong to C. In this section we shall give the asymptotic expansion of the null distribution of T. Without loss of generality we may assume that $\sigma^2 = 1$. Then $n\boldsymbol{S}$ has a Wishart distribution $\mathscr{W}_p(n, \boldsymbol{I})$. By expressing T with $\boldsymbol{Y} = n^{1/2}(\boldsymbol{S} - \boldsymbol{I})$, we have

$$\begin{aligned} T = {} & (1/2)\,\{\mathrm{tr}\,\boldsymbol{Y}^2 - p^{-1}(\mathrm{tr}\,\boldsymbol{Y})^2\} \\ & + n^{-1/2}R_1(\boldsymbol{Y}) + n^{-1}R_2(\boldsymbol{Y}) + o_p(n^{-1}), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} R_1(\boldsymbol{Y}) = {} & (1/6)q\,\mathrm{tr}\,\boldsymbol{Y}^3 - (1/2)\,(q+2)p^{-1}(\mathrm{tr}\,\boldsymbol{Y})\,(\mathrm{tr}\,\boldsymbol{Y}^2) \\ & + (1/3)\,(q+3)p^{-2}(\mathrm{tr}\,\boldsymbol{Y})^3, \\ R_2(\boldsymbol{Y}) = {} & (1/24)r\,\mathrm{tr}\,\boldsymbol{Y}^4 - (1/6)\,(r+3q)p^{-1}(\mathrm{tr}\,\boldsymbol{Y})\,(\mathrm{tr}\,\boldsymbol{Y}^3) \\ & + (1/4)\,(r+6q+6)p^{-2}(\mathrm{tr}\,\boldsymbol{Y})^2(\mathrm{tr}\,\boldsymbol{Y}) \\ & - (1/8)\,(r+8q+12)p^{-3}(\mathrm{tr}\,\boldsymbol{Y})^4. \end{aligned} \tag{4.4}$$

with $q = Q^{(3)}(1)$ and $r = Q^{(4)}(1)$. Using the p.d.f. of $\boldsymbol{Y}$ given by (2.1) and (2.2), the characteristic function of T can be expanded as

$$c(t) = (1-2it)^{-f/2}\{1 + n^{-1}\textstyle\sum_{j=0}^{3} a_j(1-2it)^{-j}\} + o(n^{-1}) \tag{4.5}$$

where $f = (p-1)\,(p+2)/2$ and

$$\begin{aligned} a_0 = {} & -\,(1/24)\,(2p^3 + 3p^2 - p - 4p^{-1}), \\ a_1 = {} & (1/24)\,(2p^3 + 3p^2 - p - 4p^{-1}) \\ & - (1/48)\,(r-6)\,(2p^3 + 5p^2 - 7p - 12 + 12p^{-1}) \\ & + (1/48)\,(q+2)\,(q-2)\,(p^3 + 3p^2 - 8p - 12 + 16p^{-1}), \\ a_2 = {} & (1/48)\,(r-6)\,(2p^3 + 5p^2 - 7p - 12 + 12p^{-1}) \\ & - (1/24)q(q+2)\,(p^3 + 3p^2 - 8p - 12 + 16p^{-1}), \\ a_3 = {} & (1/48)\,(q+2)^2(p^3 + 3p^2 - 8p - 12 + 16p^{-1}). \end{aligned} \tag{4.6}$$

This implies the following theorem.

THEOREM 4.1. *Let $d_1,\dots, d_p$ be the latent roots of $pS/\mathrm{tr}\,S$ and let $T=T(d_1,\dots, d_p)$ be the statistic given by* (4.1). *Then the null distribution of T can be expanded for large n as*

$$\text{(4.7)} \qquad \mathrm{P}(T\leqslant x)=\mathrm{P}(\chi_f^2\leqslant x) + n^{-1}\textstyle\sum_{j=0}^{3} a_j \mathrm{P}(\chi_{f+2j}^2\leqslant x)+o(n^{-1}),$$

where $f=(p-1)(p+2)/2$ and the coefficient a_j's are given by (4.6).

From (4.7), the Bartlett's adjustment factor is given by

$$\text{(4.8)} \qquad \begin{aligned}\rho = 1 &- (1/24)\mathrm{n}^{-1}\mathrm{f}^{-1}\{2(2\mathrm{p}^3+3\mathrm{p}^2-\mathrm{p}-4\mathrm{p}^{-1})\\ &+(r-6)(2p^3+5p^2-7p-12+12p^{-1})\\ &+4(q+2)(p^3+3p^2-8p-12+16p^{-1})\},\end{aligned}$$

and under the null hypothesis

$$\text{(4.9)} \qquad P(\rho T\leqslant x)=\mathrm{P}(x_f^2\leqslant x)+n^{-1}\textstyle\sum_{j=0}^{3}\tilde{a}_j\mathrm{P}(x_{f+2j}^2\leqslant x)+o(n^{-1}),$$

where

$$\text{(4.10)} \qquad \begin{aligned}\tilde{a}_0 &= (1/48)(r-6)(2p^3+5p^2-7p-12+12p^{-1})\\ &\quad+(1/12)(q+2)(p^3+3p^2-8p-12+16p^{-1}).\\ \tilde{a} &= -(1/24)(r-6)(2p^3+5p^2-7p-12+12p^{-1})\\ &\quad+(1/48)(q+2)(q-6)(p^3+3p^2-8p-12+16p^{-1}),\\ \tilde{a}_2 &= a_2,\ \tilde{a}_3=a_3.\end{aligned}$$

This shows that Bartlett's adjustment implies $\mathrm{P}(\rho T\leqslant x)=\mathrm{P}(\chi_f^2\leqslant x)+o(n^{-1})$ if and only if $q=-2$ and $r=6$. The LR test T_0 satisfies this property.

5. Asymptotic expansion of the non-null distribution and the comparison of powers for sphericity

We shall compare the powers of test statistics in C under the sequence of alternatives $\mathscr{K}_n$: $\Sigma=\sigma^2(I+n^{-1/2}\Theta)$, using the asymptotic expansions of the non-null distributions. Under the alternatives $\mathscr{K}_n$, $W=\sigma^{-2}S$ has a Wishart distribution $\mathscr{W}_p(n, I+{}^{-1/2}\Theta)$. Let $Y=n^{1/2}(W-I-n^{-1/2}\Theta)$, then Y is asymptotically normal and the p.d.f. of Y is given by (3.1) and (3.2). By the same method as in Section 3, we obtain the following theorem.

THEOREM 5.1. *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma=\sigma^2(I+n^{-1/2}\Theta)$, *the distribution of* T *given by* (4.1) *can be expanded for large* n *as*

$$\text{(5.1)}\qquad \mathrm{P}(T\leqslant x)=\mathrm{P}(\chi_f^2(\delta)\leqslant x) + n^{-1/2}\textstyle\sum_{j=0}^{3} b_j \mathrm{P}(\chi_{f+2j}^2(\delta)\leqslant x)+o(n^{-1/2}),$$

where $f=(p-1)\ (p+2)/2$, $\delta=\mathrm{tr}\,\tilde{\Theta}^2/2$ *and the coefficient* b_j*'s are given by*

$$\text{(5.2)}\qquad \begin{aligned} b_0 &= (1/6)\mathrm{tr}[(3\Theta-\tilde{\Theta})\tilde{\Theta}^2],\\ b_1 &= -(1/2)\mathrm{tr}(\Theta\tilde{\Theta}^2),\\ b_2 &= -(1/12)q\,\mathrm{tr}\,\tilde{\Theta}^3,\\ b_3 &= (1/12)\ (q+2)\mathrm{tr}\,\tilde{\Theta}^3,\end{aligned}$$

withh $\tilde{\Theta}=\Theta-(\mathrm{tr}\,\Theta)p^{-1}I$.

The asymptotic expansions of the distributions of T_0 and T_1 are obtained from (4.7) and (5.1) by putting $(q, r)=(-2,6)$ and (0, 0), respectively.

The power function of the test T can be expanded as in the following theorem, by using (4.7), (5.1) and (3.7).

THEOREM 5.2. *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma=\sigma^2(I+n^{-1/2}\Theta)$, *the power* β_T *of the test with a level* α *based on* T *in* C *can be expanded as*

$$\text{(5.3)}\qquad \beta_T=\beta_0+(1/6)n^{-1/2}(q+2)\mathrm{tr}\,\tilde{\Theta}g_{f+6}(u,\delta)+o(n^{-1/2}),$$

where β_0 *is the power of the test with* $q=Q^{(3)}(1)=-2$, u *is the upper* 100α% *point of the* χ_f^2 *distribution.* β_0 *is given by*

$$\text{(5.4)}\qquad \begin{aligned}\beta_0=\mathrm{P}(\chi_f^2(\delta)>u)+(1/3)n^{-1/2}\{&\mathrm{tr}\,\tilde{\Theta}g_{f+4}(u,\delta)\\ &-\mathrm{tr}[3\Theta-\tilde{\Theta})\tilde{\Theta}^2]g_{f+2}(u,\delta)\}+o(n^{-1/2}).\end{aligned}$$

Theorem 5.2 shows that if $\mathrm{tr}\,\tilde{\Theta}>0$, the test with larger q than -2 is preferable and that if $\mathrm{tr}\,\tilde{\Theta}<0$, the test with smaller q than -2 is preferable. This agrees with the results of Harris and Peers [4].

6. A class of test statistics for the equality of two covariance matrices

In this section we consider the problem (iii) of testing the equality of two covariance matrices. Let the $p\times 1$ vectors $X_{i1},\ldots, X_{iN(i)}$ be a random sample from a normal distribuution with mean μ_i and covariance matrix $\Sigma_i\,(i=1, 2)$, and let $d_1,\ldots, d_p$ be the latent roots of $S_2^{-1}S_1$ where $S_i=n_i^{-1}\sum_{j=0}^{N(i)}(X_{ij}-\bar{X}_i)(X_{ij}-\bar{X}_i)^t$, $X_i=N(i)^{-1}\sum_{j=0}^{N(i)}X_{ij}$ and $n_i=N(i)-1$. We consider a class C of test statistics

$$T = nk_1k_2\sum_{j=1}^{p} Q(d_j), \tag{6.1}$$

where $n = n_1 + n_2$, $k_i = n_i/n$, the critical region based on T is "$T > k$" and $Q(d)$ satisfies the same assumptions $\mathscr{A}1 \sim \mathscr{A}4$ as in (1.4). The modified LR test statistic T_0 and Nagao's [7] satistic T_1 are defined by $Q(d) = \{\log(k_1 d + k_2) - k_1 \log d\}/k_1k_2$ and $(d-1)^2/2(k_1 d + k_2)^2$, respectively.

By the same consideration as in the problem (i) and (ii), the asymptotic expansions of the null and the non-null distribution of T are obtained. We state the results and proofs are omitted. The expansion of the null distribution is given in the following theorem.

THEOREM 6.1. *Let $d_1, \dots, d_p$ be the latent roots of $S_2^{-1}S_1$ and let $T = T(d_1, \dots, d_p)$ be the statistic given by* (6.1). *Then the null distribution of T can be expanded for large n as*

$$\begin{aligned} \mathrm{P}(T \leqslant x) &= \mathrm{P}(\chi_f^2 \leqslant x) \\ &\quad + n^{-1}\sum_{j=0}^{3} a_j \mathrm{P}(\chi_{f+2j}^2 \leqslant x) + o(n^{-1}), \end{aligned} \tag{6.2}$$

where $f = p(p+1)/2$ and the coefficients a_j's are given by

$$\begin{aligned} a_0 &= (1/24)\{1 - (k_1k_2)^{-1}\}p(2p^2 + 3p - 1) \\ a_1 &= -(1/24)\{1 - (k_1k_2)^{-1}\}p(2p^3 + 3p - 1) \\ &\quad - (1/12)(k_1k_2)^{-1}(q + 2k_1 + 2)\{p(p^2 + 3p + 4) + k_1 p(4p^2 + 9p + 7)\} \\ &\quad + (1/48)(k_1k_2)^{-1}(q + 2k_1 + 2)^2 p(4p^2 + 9p + 7) \\ &\quad - (1/48)(k_1k_2)^{-1}[r - 6(k_1^2 + k_1 + 1)]p(2p^2 + 5p + 5), \\ a_2 &= (1/12)(k_1k_2)^{-1}(q + 2k_1 + 2)\{p(p^2 + 3p + 4) + k_1 p(4p^2 + 9p + 7)\} \\ &\quad - (1/24)(k_1k_2)^{-1}(q + 2k_1 + 2)^2 p(4p^2 + 9p + 7) \\ &\quad + (1/48)(k_1k_2)^{-1}[r - 6(k_1^2 + k_1 + 1)]p(2p^2 + 5p + 5), \\ a_3 &= (1/48)(k_1k_2)^{-1}(q + 2k_1 + 2)^2 p(4p^2 + 9p + 7), \end{aligned} \tag{6.3}$$

with $q = Q^{(3)}(1)$ and $r = Q^{(4)}(1)$.

From (6.2), the Bartlett's adjustment factor is given by

$$\begin{aligned} \rho = 1 - (1/24)n^{-1}(fk_1k_2)^{-1}&2[1 - (k_1k_2)^{-1}]p(2p^2 + 3p - 1) \\ &+ 4(q + 2k_1 + 2)[p(p^2 + 3p + 4) + k_1 p(4p^2 + 9p + 7)] \\ &+ [r - 6(k_1^2 + k_1 + 1)]p(2p^2 + 5p + 5)\}, \end{aligned} \tag{6.4}$$

and under the null hypothesis $\mathscr{H}$: $\Sigma_1 = \Sigma_2$,

$$\text{(6.5)} \qquad \mathrm{P}(\rho T \leqslant x) = \mathrm{P}(\chi_f^2 \leqslant x) + n^{-1}\textstyle\sum_{j=0}^{3} \tilde{a}_j \mathrm{P}(\chi_{f+2j}^2 \leqslant x) + o(n^{-1}),$$

where

$$\text{(6.6)} \qquad \begin{aligned} \tilde{a}_0 &= (1/12)\ (k_1k_2)^{-1}(q+2k_1+2)\ \{p(p^2+3p+4)+k_1p(4p^2+9p+7)\} \\ &\quad + (1/48)\ (k_1k_2)^{-1}[r-6(k_1^2+k_1+1)]p(2p^2+5p+5), \\ \tilde{a}_1 &= -\ (1/6)\ (k_1k_2)^{-1}(q+2k_1+2)\ \{p(p^2+3p+4)+k_1p(4p^2+9p+7)\} \\ &\quad + (1/48)\ (k_1k_2)^{-1}(q+2k_1+2)^2p(4p^2+9p+7) \\ &\quad -\ (1/24)\ (k_1k_2)^{-1}[r-6(k_1^2+k_1+1)]p(2p^2+5p+5), \\ \tilde{a}_2 &= a_2,\ \tilde{a}_3 = a_3. \end{aligned}$$

This shows that Bartlett's adjustment implies $\mathrm{P}(\rho T \leqslant x) = \mathrm{P}(\chi_f^2 \leqslant x) + o(n^{-1})$ if and only if $q = -2(k_1+1)$ and $r = 6(k_1^2+k_1+1)$. The modified LR test T_0 satisfies this property.

For expanding the non-null distribution of T, we cosnsider the sequence of alternatives $\mathscr{K}_n$: $\Sigma_2^{-1/2}\Sigma_1\Sigma_2^{-1/2} = \boldsymbol{I} + n^{-1/2}\boldsymbol{\Theta}$. The asymptotic expansion of the distribution of T under $\mathscr{K}_n$ is given in the following theorem.

THEOREM6.2 *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma_2^{-1/2}\Sigma_1\Sigma_2^{-1/2} = \boldsymbol{I} + n^{-1/2}\boldsymbol{\Theta}$, *the distribution of T given by* (6.1) *can be expanded for large n as*

$$\text{(6.7)} \qquad \mathrm{P}(T \leqslant x) = \mathrm{P}(\chi_f^2(\delta) \leqslant x) + n^{-1/2}\textstyle\sum_{j=0}^{3} b_j \mathrm{P}(\chi_{f+2j}^2(\delta) \leqslant x) + o(n^{-1/2}),$$

where $f = p(p+2)/2$, $\delta = k_1k_2\theta_2/2$ and the coefficient b_j's are given by

$$\text{(6.8)} \qquad \begin{aligned} b_0 &= (1/6)k_1k_2(2-k_1)\theta_3, \\ b_1 &= -\ (1/4)\ (q+2k_1+2)\ (p+1)\theta_1 - (1/2)k_1k_2^2\theta_3, \\ b_2 &= (1/4)\ (q+2k_1+2)\ (p+1)\theta_1 \\ &\quad + (1/12)\ \{2k_1k_2(1-2k_1) - k_1k_2(q+2k_1+2)\theta_3, \\ b_3 &= (1/12)k_1k_2(q+2k_1+2)\theta_3, \end{aligned}$$

with $\theta_j = \mathrm{tr}\,\boldsymbol{\Theta}^j$.

The asymptotic expansions of the null and the non-null distributions of T_0 *and* T_1 are obtained from (6.2) and (6.7) by putting $(q, r) = (-2k_1-2, 6[k_1^2+k_1+1])$ and $(-6k_1, 36k_1^2)$, respectively. These special cases have been treated by Anderson [1] and Nagao ([7], [8]).

The power function of the test T can be expanded as in the following theorem, by using (6.2), (6.7) and (3.7).

THEOREM 6.3 *Under the sequence of alternatives* $\mathscr{K}_n$: $\Sigma_2^{-1}\Sigma_1\Sigma_2^{-1/2}=I+n^{-1/2}\Theta$, *the power* β_T *of the test with a level* α *based on* T *can be expanded as*

$$\beta_T=\beta_0+(1/2)n^{-1/2}k_1k_2(q+2k_1+2)d(\Theta, u)+o(n^{-1/2}), \tag{6.9}$$

where $d(\Theta, u)$ *is the same as in* (3.9), *and* β_0 *is the power of the test with* $q=-2(k_1+1)$ *and is given by*

$$\beta_0=\mathrm{P}(\chi_f^2(\delta)>u)+(1/3)k_1\theta_3n^{-1/2}\{(1-2k_1)g_{f+6}(u, \delta) - (2-k_1)g_{f+4}(u, \delta)\}+o(n^{-1/2}). \tag{6.10}$$

Theorem 6.3 shows that the results on the power comparisons of tests in C are the same as the ones in Section 3.

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