HIROSHIMA MATH. J.
19 (1989), 141–166

Integration in mixed topological spaces*)

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(Received January 20, 1988)

In the present paper we are concerned with integration of functions with values in mixed topological spaces. The theory of Lebesgue integral on a general measure space has been extended to the case of functions taking their values in Banach spaces by Birkhoff [1], Bochner [3], Pettis [17] and others [12]. These vector integration theories have been extended further to the case of locally convex spaces by Phillips [18] and Rickart [20]. Mixed topological spaces form an important class of locally convex spaces. These spaces have many interesting properties and are very abundant. The mixed topological structures often appear in various problems from analysis as well as the theory of partial differential equations, and it is expected that the theory of integration in mixed topological spaces is not only significant from the theoretical point of view, but also it has considerable practical applicability.

A mixed topological space is a locally convex space (E, τ) equipped with a bornology on E. A subset B of E is called a ball in E if it is an absolutely convex subset which does not contain a nontrivial subspace. By a bornology on E we mean a family $\mathscr{B}$ of balls in E with the four properties below: (a) $\mathscr{B}$ is a covering of E, (b) $\lambda B \in \mathscr{B}$ for $B \in \mathscr{B}$ and $\lambda > 0$, (c) for B, $C \in \mathscr{B}$ there exists $D \in \mathscr{B}$ with $B + C \subset D$, and (d) if $B \in \mathscr{B}$ and C is a ball contained in B then $C \in \mathscr{B}$. If in particular there exists a countable subfamily $\{B_n\}$ of $\mathscr{B}$ such that any element $B \in \mathscr{B}$ is contained in some B_n, then $\mathscr{B}$ is said to be of countable type. To the locally convex space (E, τ) there corresponds a bornology $\mathscr{B}_\tau$ called the von Neumann bornology on E that is the family of all τ-bounded, absolutely convex subsets of E. In this paper we restrict ourselves to a bornology $\mathscr{B}$ on E satisfying the compatibility condition

$$\mathscr{B} \subset \mathscr{B}_\tau ,$$

and assume that there exists a countable subfamily $\{B_n\}$ of $\mathscr{B}$ such that any element $B \in \mathscr{B}$ is contained in some B_n and any B_n is τ-closed. Now to the triplet $(E, \mathscr{B}, \tau)$ one can introduce a new locally convex topology that is finer than the original topology τ and denote it by $\gamma \equiv \gamma[\mathscr{B}, \tau]$. This topology γ is

*) This research is partially supported by Grant-in-Aid for Scientific Research, Ministry of Science and Culture, Japan.

defined as the finest locally convex topology on E which coincides with τ on each set belonging to $\mathscr{B}$. The γ is called the mixed topology and, in this sense, $(E, \mathscr{B}, \tau)$ is called a mixed topological space. The space (E, γ) inherits various topological properties from (E, τ), while there are interesting differences between γ and τ. One of the significant properties of γ is for instance property (B) in the sense of Pietsch, and these properties make the mixed topological spaces abundant.

This work is strongly affected by the recent results due to Thomas [23] and Blondia [2]. In general, it is hardly possible to develop an integration theory for weakly measurable functions. However, in the case of locally convex Souslin spaces, the concept of weak measurability is equivalent to that of strong measurability. Noting this fact, Thomas gave useful criteria for Pettis integrability of functions with values in locally convex Souslin spaces. He also introduced a new notion of integrability called total summability and showed that Fubini's theorem is valid for the class of such integrable functions. We shall advance our integration theory from the same point of view as in his work. On the other hand, Blondia considered a notion of integral by seminorm and developed an integration theory in connection with the works of Schmets [8], Grothendieck [10] and Saab [21]. He studied in [2] the relationships between the strong (Bochner type) integrals, the integrals by seminorm and the Pettis integrals. We shall also treat this problem in mixed topological spaces.

The objective of this paper is therefore threefold. First we advance an integration theory in mixed topological spaces. Secondly, we investigate the relationships between the above-mentioned three kinds of integrals in both of mixed topological spaces and mixed topological Souslin spaces. Thirdly, we exhibit how the three kinds of notions of integrability as well as measure theoretic properties of the integrals in mixed spaces can be interpreted in terms of the original topology τ and the bornology $\mathscr{B}$.

Section 1 contains preliminaries and some fundamental facts which are used in the subsequent sections. In particular, it is shown that every mixed topological space has property (B) in the sense of Pietsch; this fact plays an important role in our argument. In Section 2 we state some fundamental theorems such as Nikodým's boundedness theorem and the Vitali-Hahn-Saks theorem for locally-convex-space-valued measures. Here we also study vector measures with values in mixed topological spaces. Section 3 presents a Vitali type convergence theorem and its consequences for integrals by seminorm and Pettis integrals in locally convex spaces. Our results here extend the results obtained by Musiał [16] for vector measures with values in Banach spaces. Section 4 is the main section of this paper. Here, we develop an integration theory in a mixed topological space $(E, \gamma[\mathscr{B}, \tau])$ and investigate properties of the three kinds of integrals in $(E, \gamma[\mathscr{B}, \tau])$ under various assumptions on the system $(E, \mathscr{B}, \tau)$.

1. Preliminaries

Throughout this paper, only standard terminologies in the theory of locally convex spaces are used. Also, we assume without further mention that locally convex spaces under consideration are Hausdorff.

Let E be a vector space. A *ball* in E means an absolutely convex subset of E which does not contain a nontrivial subspace. If B is a ball in E, we write E_B for the linear space $\bigcup_{n=1}^{\infty} nB$ of B in E. On E_B one can define a natural norm $\|\cdot\|$ by

$$\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\} \quad \text{for} \quad x \in E_B .$$

If in particular $(E_B, \|\cdot\|_B)$ is a Banach space, B is called a *Banach ball.* If B is a closed ball in E with a locally convex topology, the Hahn-Banach theorem implies that $\|x\|_B = \sup\{|\langle x, x'\rangle| : x' \in B^0\}$ for all $x \in E$, where B^0 denotes the polar of B. Hence we see that the function $x \to \|x\|_B$ is lower semi-continuous on E.

For a vector space E, a (*convex*) *bornology* on E is a family $\mathscr{B}$ of balls in E such that (a) $\mathscr{B}$ is a covering of E; (b) for $\mathbf{B} \in \mathscr{B}$ $\lambda > 0$, $\lambda B \in \mathscr{B}$; (c) for B, $C \in \mathscr{B}$ there exists $D \in \mathscr{B}$ with $B + C \subset D$; (d) if $B \in \mathscr{B}$ and C is a ball contained in B, then $C \in \mathscr{B}$. We call such a pair $(E, \mathscr{B})$ a *bornological space.* A subset B of E is *$\mathscr{B}$-bounded* if it is contained in some ball in $\mathscr{B}$. A *basis* for $\mathscr{B}$ is a subfamily $\mathscr{B}_1$ of $\mathscr{B}$ such that each $B \in \mathscr{B}$ is contained in some $B_1 \in \mathscr{B}_1$. A bornological space $(E, \mathscr{B})$ is said to be *complete* if $\mathscr{B}$ has a basis consisting of Banach balls. $\mathscr{B}$ is said to be *of countable type* if $\mathscr{B}$ has a countable basis. If (E, τ) is a locally convex space, then the family $\mathscr{B}_\tau$ of all τ-bounded, absolutely convex subsets of E forms a bornology on E. This $\mathscr{B}_\tau$ is called the *von Neumann bornology.* In many applications $\mathscr{B}$ is taken as the von Neumann bornology defined by an appropriate norm on E. This $\mathscr{B}$ is of countable type, since the family $(nB)_{n \in \boldsymbol{N}}$, B being the unit ball of E, gives a basis.

We often denote by E_τ a vector space E equipped with a locally convex topology τ. In this paper we consider a vector space E with a locally convex topology τ which is compatible with a bornology $\mathscr{B}$ of countable type in the following sense:

$$(*) \qquad \mathscr{B} \subset \mathscr{B}_\tau \text{ and } \mathscr{B} \text{ has a basis of } \tau\text{-closed sets} .$$

In this case we can choose a basis (B_n) for $\mathscr{B}$ with the following two properties:

$$\text{(a)} \quad B_n + B_n \subset B_{n+1} \text{ for each } n; \qquad \text{(b)} \quad \text{each } B_n \text{ is } \tau\text{-closed} .$$

Let $\mathscr{U} = (U_n)_{n \in N}$ be a sequence of absolutely convex τ-neighbourhoods of zero and let

$$\gamma(\mathscr{U}) := \bigcup_{n=1}^{\infty} (U_1 \cap B_1 + \cdots + U_n \cap B_n).$$

Then the set of all such sets $\gamma(\mathscr{U})$ forms a base of neighbourhoods of zero for a locally convex structure on E and this is denoted by $\gamma[\mathscr{B}, \tau]$ (or simply by γ if no confusion arises). In the case where $\mathscr{B}$ is the bornology defined by a norm on E, we write $\gamma[\|\cdot\|, \tau]$ for the structure $\gamma[\mathscr{B}, \tau]$. In this paper a triplet $(E, \mathscr{B}, \tau)$ satisying the compatibility condition $(*)$ is called a *mixed space* and the locally convex topology $\gamma[\mathscr{B}, \tau]$ is called a *mixed topology.* The following statements give characteristic and useful properties of γ (as for the proof and more about the mixed topology, we refer to the book of Cooper [5]): (1) γ is the finest linear topology on E which coincides with τ on the sets of $\mathscr{B}$; (2) a subset B of E is $\mathscr{B}$-bounded if and only if it is γ-bounded; (3) a sequence (x_n) in E converges to x in E_γ if and only if (x_n) is $\mathscr{B}$-bounded and $x_n \to x$ in E_τ.

A *Saks space* is a triplet $(E, \|\cdot\|, \tau)$ of a vector space, a locally convex topology τ on E, and a norm $\|\cdot\|$ on E such that the unit ball $B_{\|\cdot\|}$ of $(E, \|\cdot\|)$ is τ-bounded and τ-closed. A Saks space $(E, \|\cdot\|, \tau)$ is complete if $B_{\|\cdot\|}$ is τ-complete; in this case $(E, \|\cdot\|)$ is a Banach space. Let $(E, \|\cdot\|, \tau)$ be a Saks space and let $\mathscr{S}$ be a defining family of seminorms for τ which is closed for finite suprema and is such that $\|\cdot\| = \sup \mathscr{S}$ (See [5, Lemma 3.1]). Then for any pair of sequence (p_n) in $\mathscr{S}$ and (λ_n) in $(0, \infty)$ with $\lambda_n \uparrow \infty$, $p(x) = \sup_n p_n(x)/\lambda_n$ is a seminorm on E. The family of all such seminorms defines a locally convex topology $\tilde{\gamma}[\|\cdot\|, \tau]$ on E. The following result states the relationship between the topologies $\gamma[\|\cdot\|, \tau]$ and $\tilde{\gamma}[\|\cdot\|, \tau]$.

PROPOSITION 1.1 ([5, *Proposition* I.4.4]). *Let $(E, \|\cdot\|, \tau)$ be a Saks space and suppose that either*

(a) *for every $x \in E$, $\varepsilon > 0$, $p \in \mathscr{S}$, there are elements $y, z \in E$ so that $x = y + z$, $p(z) = 0$ and $\|y\| \leqq p(x) + \varepsilon$, or*

(b) *$B_{\|\cdot\|}$ is τ-compact.*

Then $\tilde{\gamma}[\|\cdot\|, \tau] = \gamma[\|\cdot\|, \tau]$.

We next consider the duality theory for (E, γ). A mixed space E has three dual spaces: the topological dual of the locally convex space (E, τ), E'_τ; the topological dual of the locally convex space (E, γ), E'_γ; the space of linear forms on E which are bounded on the sets of $\mathscr{B}$, $E'_{\mathscr{B}}$. It is obvious that $E'_\tau \subset E'_\gamma \subset E'_{\mathscr{B}}$ and these spaces are regarded as a locally convex space with the topology of uniform convergence on the τ-bounded sets, that on the γ-bounded sets, and that on the sets of $\mathscr{B}$, respectively. Since $\mathscr{B}$ is of countable type, $E'_{\mathscr{B}}$ is metrizable and it is also complete. Hence it is a Fréchet space. Moreover we have the following result:

PROPOSITION 1.2 ([5, *Proposition* I.1.17]).
(i) *E'_γ is a locally convex subspace of $E'_{\mathscr{B}}$;*
(ii) *E'_γ is the closure of E'_τ in $E'_{\mathscr{B}}$ and so is a Fréchet space.*

A locally convex space (E, τ) is said to be a *(df)-space* if it is sequentially evaluable (i.e., every null sequence in $(E', \beta(E', E))$ is equicontinuous) and admits a fundamental sequence of bounded sets. It is known [15, Theorem 12.4.1] that the strong duals of such spaces are Fréchet spaces. A locally convex space (E, τ) is said to be *$\aleph_0$-evaluable*, if every bornivorous barrel in E that can be represented as the intersection of a sequence of closed and abolutely convex 0-neighbourhoods in (E, τ) is itself a 0-neighbourhood in (E, τ). A (df)-space which is also $\aleph_0$-evaluable is traditionally said to be *(DF)-space.* Let (E, τ) be a locally convex space possessing a fundamental sequence $\mathscr{C} = (B_n)$ of bounded sets. The symbol $\tau^{\mathscr{C}}$ stands for the finest locally convex topology on E which coincides with τ on every B_n, $n \in N$. If in particular $\tau = \tau^{\mathscr{C}}$, then (E, τ) is called a *(gDF)-space* (for "generalized (DF)-space"). Every (DF)-space is a (gDF)-space and it is easily verified [15, p. 257] that every (gDF)-space is also a (df)-space. Let $(E, \mathscr{B}, \tau)$ be a mixed space. Since $\gamma[\mathscr{B}, \tau]$ is the finest locally convex topology on E which coincides with τ on the sets of $\mathscr{B}$, we see that $\gamma = \gamma^{\mathscr{C}}$, where $\mathscr{C}$ denotes a countable basis of $\mathscr{B}$. Hence γ is a (gDF)-space, and so a (df)-space.

Let (E, τ) be a locally convex space and $\mathscr{P}(\tau)$ the family of τ-continuous seminorm on E. We denote by $l^1_N\{E_\tau\}$ the space of absolutely summable sequences in E, regarded as a locally convex space with the family of seminorms $\{\tilde{p} : p \in \mathscr{P}(\tau)\}$, where $\tilde{p} : (x_n) \to \sum_{n \in N} p(x_n)$. A locally convex space (E, τ) is said to have *property* (B) if for each bounded subset $\mathfrak{B}$ of the space $l^1_N\{E_\tau\}$ there exists an absolutely convex closed bounded subset B in E such that $\sum_{n=1}^\infty \|x_n\|_B \leqq 1$, for each $(x_n)_{n \in N} \in \mathfrak{B}$. It is known ([19, Theorem 1.5.8]) that metrizable locally convex spaces and (df)-spaces have property (B). Since mixed topological spaces are (df)-spaces, we obtain the following result which plays an important role in the subsequent discussions.

PROPOSITION 1.3. *Let $(E, \mathscr{B}, \tau)$ be a mixed space. Then E_γ has property (B).*

For Saks spaces, it is already known in [5, Proposition II.6.9] that the mixed topology γ has property (B).

A topological space P is said to be *Polish* if there is a metric on P defining the topology of P and P, equipped with this metric, forms a complete separable metric space. A Hausdorff topological space E is said to be *Souslin* if there is a Polish space P and a continuous mapping from P onto E. The following are Souslin spaces (see L. Schwartz [22] and M. De Wilde [6] for the detailed arguments: (a) Borel subsets of a Souslin space, (b) a product (hence a projective limit) of a sequence of Souslin spaces, (c) countable intersections or countable unions of Souslin subspaces of a Hausdorff topological space, (d) countable topological sums (hence an inductive limit) of a sequence of Souslin spaces, and (e) continuous images of a Souslin space (hence a quotient of a Souslin space). Since a Polish space is separable, every Souslin space is separable. In fact, many of well-known separable linear topological spaces are Souslin spaces. Separable Banach or Fréchet spaces are Polish. The topological dual E' with the topology of uniform convergence on compact sets of a separable Fréchet space E is a Souslin space. In particular, $(E', \sigma(E', E))$ is Souslin. Most of function spaces and their strong duals which appear in the distribution theory are Souslin. Souslin spaces have the following useful properties: (1) If E is a Souslin space and E_τ is the space E equipped with a weaker Hausdorff topology τ, E_τ (which incidentally is a Souslin space) have the same Borel sets as those in E; (2) any finite positive Borel measures on a Souslin space are Radon measures; (3) any separating family $(f_i)_{i \in I}$ of continuous functions on a Souslin space has a countable subfamily which still separate the points of the space.

The following fact is elementary but important for our subsequent discussions.

Proposition 1.4. *Let $(E, \mathscr{B}, \tau)$ be a mixed space. Then the following are equivalent*:

(1) *$\mathscr{B}$ has a basis (B_n) such that B_n equipped with the relative topology induced from τ is a Souslin space for $n \in N$*;

(2) *E_τ is Souslin*;

(3) *E_γ is Souslin.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious from the third stability property (c) of Souslin spaces mentioned above. We then prove the implication $(2) \Rightarrow (3)$. Let E_γ be a Souslin space and $(B_n)_{n \in N}$ a countable basis of the bornology $\mathscr{B}$. Since B_n are all τ-closed, each one is also a Souslin space with the topology induced by τ. Since γ and τ coincide on each B_n, B_n equipped with the relative topology induced from γ is a Souslin space for $n \in N$. Since (B_n) is a covering of E, E_γ is also a Souslin space. Finally, we show that (3) implies (1). Since B_n are τ-closed, they are γ-closed, and hence Souslin with respect to the relative topology induced on B_n by γ. Again, using the fact that γ and τ coincide on each B_n, we see that each B_n is Souslin with respect to the relative topology induced by τ. q.e.d.

2. Vector measures in mixed topological spaces

Fundamental results in measure theory can be extended to vector measures in locally convex spaces. In this section we make an attempt to investigate some basic properties of mixed-space-valued measures in terms of the bornology and the original topology.

Let (E, γ) be a locally convex space and E'_γ its topological dual. We denote by $\mathscr{P}(\gamma)$ the family of all γ-continuous seminorms on E. Let U_p be the p-unit γ-closed ball, that is, $U_p = \{x \in E : p(x) \leqq 1\}$ for $p \in \mathscr{P}(\gamma)$. The polar of a set V of E is denoted by V^0, namely, $V^0 = \{x' \in E'_\gamma : |\langle x, x'\rangle| \leqq 1 \text{ for all } x \in V\}$.

A function v from a field $\mathscr{F}$ of subsets of a set S to a locally convex space E is said to be a *finitely additive* vector measure, or simply a vector measure, if A_1 and A_2 are disjoint members of $\mathscr{F}$ then $v(A_1 \cup A_2) = v(A_1) + v(A_2)$. If in addition $v(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} v(A_n)$ holds in the original topology γ of E for all sequences (A_n) of disjoint members of $\mathscr{F}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$, then we say that v is a *γ-countably additive* vector measure, or simply, v is γ-countably additive.

Let $A \in \mathscr{F}$ and let $\Pi(A)$ denote the set of all finite measurable partitions of A. If $A = S$, we simply write Π for $\Pi(S)$. Furthermore, let $v: \mathscr{F} \to E$ be a vector measure and let $p \in \mathscr{P}(\gamma)$. Then the *$p$-variation* of v is the extended nonnegative function $|v|_p(\cdot)$ whose value at $A \in \mathscr{F}$ is defined by

$$|v|_p(A) = \sup_{(A_i) \in \Pi(A)} \textstyle\sum_i p(v(A_i)) .$$

Moreover, v is said to be of *γ-bounded variation* if $|v|_p(S) < \infty$ for all $p \in \mathscr{P}(\gamma)$. The *$p$-semivariation* of v is the extended nonnegative function $\|v\|_p(\cdot)$ whose value at $A \in \mathscr{F}$ is given by

$$\|v\|_p(A) = \sup \{|\langle v, x'\rangle|(A) : x' \in U_p^0\} ,$$

where $|\langle v, x'\rangle|(\cdot)$ is the variation of the scalar-valued measure $\langle v, x'\rangle$. Moreover, v is said to be of *γ-bounded semivariation* if $\|v\|_p(S) < +\infty$ for all $p \in \mathscr{P}(\gamma)$.

Let $\mathscr{F}$ be a field of subsets of the set S and let $v: \mathscr{F} \to E$ be a vector measure. The measure v is said to be *γ-strongly additive* if for any sequence (A_n) of disjoint members of $\mathscr{F}$ the series $\sum_{n=1}^{\infty} v(A_n)$ converges with respect to the topology γ. A family $\{v_\alpha : \alpha \in \Lambda\}$ of γ-strongly additive vector measures from $\mathscr{F}$ to E is said to be *uniformly γ-strongly additive*, if for any sequence (A_n) of disjoint members of $\mathscr{F}$ and any $p \in \mathscr{P}(\gamma)$ one has $\lim_{n\to\infty} p(\sum_{m=n}^{\infty} v_\alpha(A_m)) = 0$ uniformly for $\alpha \in \Lambda$.

Let $v: \mathscr{F} \to E$ be a vector measure and μ a finite nonnegative real-valued measure on $\mathscr{F}$. If $\lim_{\mu(A)\to 0} |v|_p(A) = 0$ for each $p \in \mathscr{P}(\gamma)$, then v is said to be *absolutely μ-continuous with respect to γ*; if $\lim_{\mu(A)\to 0} p(v(A)) = 0$, then v is said to be *μ-continuous with respect to γ*.

We now give some results concerning (E, γ)-valued measures which extend basic results for the case of normed spaces given for instance in [7] and [13].

PROPOSITION 2.1. *Let $v: \mathscr{F} \to E$ be a vector measure. Then for $A \in \mathscr{F}$ and for $p \in \mathscr{P}(\gamma)$, one has*

$$\|v\|_p(A) = \sup\{p(\textstyle\sum_i \varepsilon_i v(A_i)) : (A_i) \in \Pi(A), |\varepsilon_i| \leqq 1\},$$

and

$$\sup\{p(v(B)) : A \supset B \in \mathscr{F}\} \leqq \|v\|_p(A)$$
$$\leqq 4 \sup\{p(v(B)) : A \supset B \in \mathscr{F}\}.$$

Consequently a vector measure is of γ-bounded semivariation on S if and only if its range is γ-bounded in E.

PROOF. Since $p(x) = \sup\{|\langle x, x'\rangle| : x' \in U_p^0\}$ for $x \in E$ and $p \in \mathscr{P}(\gamma)$, we can apply the same argument as in [4, Proposition I.1.11] to get the desired assertion. q.e.d.

In view of this proposition a vector measure of γ-bounded semivariation may be called a γ-*bounded* vector measure.

PROPOSITION 2.2. *Any one of the following statements about a collection $\{v_\alpha : \alpha \in \Lambda\}$ of E-valued measures defined on a field $\mathscr{F}$ implies the others.*

(i) *The set $\{v_\alpha : \alpha \in \Lambda\}$ is uniformly γ-strongly additive.*

(ii) *For every equicontinuous subset H of E'_γ the set $\{\langle v_\alpha, x'\rangle : \alpha \in \Lambda, x' \in H\}$ is uniformly γ-strongly additive.*

(iii) *For every sequence (A_n) of disjoint members of $\mathscr{F}$, $\lim_n p(v_\alpha(A_n)) = 0$ holds uniformly in $\alpha \in \Lambda$ for every $p \in \mathscr{P}(\gamma)$.*

(iv) *For every sequence (A_n) of disjoint members of $\mathscr{F}$, $\lim_n \|v_\alpha\|_p(A_n) = 0$ uniformly in $\alpha \in \Lambda$ for every $p \in \mathscr{P}(\gamma)$.*

(v) *For every equicontinuous subset H of E'_γ the set $\{|\langle v_\alpha, x'\rangle| : \alpha \in \Lambda, x' \in H\}$ is uniformly γ-strongly additive.*

PROOF. It is obvious that (i) implies (ii) and (ii) implies (iii). To prove that (iii) implies (iv), suppose (iv) fails under (iii). Then there exists $p \in \mathscr{P}(\gamma)$, $\delta > 0$ and a sequence (A_n) of pairwise disjoint members of $\mathscr{F}$ for which $\sup_\alpha \|v_\alpha\|_p(A_n) > 5\delta > 0$ holds for all n. Hence for each n there is $\alpha(n) \in \Lambda$ such that

$$\|v_{\alpha(n)}\|_p(A_n) > 5\delta > 0. \tag{1}$$

On the other hand, for each n, there is $B_n \in \mathscr{F}$ such that $A_n \supset B_n$ and

$$4 \sup\{p(v_{\alpha(n)}(B) : B \subset A_n\} - \delta < 4p(v_{\alpha(n)}(B_n)). \tag{2}$$

By Proposition 2.1, the above relations (1) and (2), we have $\delta < \sup_\alpha p(v_\alpha(B_n))$, which contradicts (iii). This shows that (iii) implies (iv).

We next prove that (iv) implies (v). Suppose that $\{|\langle v_\alpha, x'\rangle|(\cdot):\alpha \in \Lambda, x' \in U_p^0\}$ is not uniformly strongly additive for some $p \in \mathscr{P}(\gamma)$. Then there exists a disjoint sequence (A_n) in $\mathscr{F}$ and a $\delta > 0$ such that for each m one has

$$\sup\{\textstyle\sum_{n=m}^{\infty} |\langle v_\alpha, x'\rangle|(A_n):\alpha \in \Lambda, x' \in U^0\} \geqq 2\delta > 0\,.$$

Thus there is an increasing sequence $(m(j))$ of positive integers such that for all j

$$\begin{aligned}&\sup\{\textstyle\sum_{n=m(j)+1}^{m(j+1)} |\langle v_\alpha, x'\rangle|(A_n):\alpha \in \Lambda, x' \in U_p^0\}\\ &\quad= \sup\{|\langle v_\alpha, x'\rangle|(\bigcup_{n=m(j)+1}^{m(j+1)} A_n):\alpha \in \Lambda, x' \in U_p^0\} \geqq \delta > 0\,.\end{aligned}$$

Therefore the sequence (B_j) of pairwise disjoint members of $\mathscr{F}$ defined by

$$B_j = \textstyle\bigcup_{n=m(j)+1}^{m(j+1)} A_n \text{ for } j = 1, 2, \dots,$$

satisfies

$$\sup\{\|v_\alpha\|_p(B_j):\alpha \in \Lambda\} = \sup\{|\langle v_\alpha, x'\rangle|(B_j):\alpha \in \Lambda, x' \in U_p^0\} \geqq \delta > 0 \text{ for } j = 1, \dots$$

This contradicts (iv), and thus (iv) implies (v). It is obvious from Proposition 2.1 that (v) implies (i). q.e.d.

COROLLARY 2.3. *The following statements about a vector measure v defined on a field $\mathscr{F}$ are equivalent*:

(i) *v is γ-strongly additive.*

(ii) *For every equicontinuous subset H of E'_γ, $\{\langle v, x'\rangle : x' \in H\}$ is uniformly γ-strongly additive.*

(iii) *v is γ-strongly bounded, i.e., $\lim_n v(A_n) = 0$ for any sequence (A_n) of disjoint members of $\mathscr{F}$.*

(iv) *$\|v\|_p$ is γ-strongly bounded for every $p \in \mathscr{P}(\gamma)$. Namely, if (A_n) is a sequence of mutually disjoint members of $\mathscr{F}$, then $\lim_n \|v\|_p(A_n) = 0$ for every $p \in \mathscr{P}(\gamma)$.*

(v) *For every equicontinuous subset H of E'_γ, $\{|\langle v, x'\rangle| : x' \in H\}$ is uniformly γ-strongly additive.*

(vi) *The limit $\lim_n v(A_n)$ exists for every nondecreasing monotone sequence (A_n) of members of $\mathscr{F}$.*

(vii) *The limit $\lim_n v(A_n)$ exists for every nonincreasing monotone sequence (A_n) of members of $\mathscr{F}$.*

PROOF. The equivalence (i) through (v) is clear from Proposition 2.2. The equivalence between (vi) and (vii) follows from the identity $v(A) + v(S\backslash A) = v(S)$. To see that (i) implies (vi), let (A_n) be a nondecreasing sequence of members of $\mathscr{F}$. Then $\lim_n v(A_n) = v(A_1) + \lim_n \sum_{m=1}^{n-1} v(A_{m+1}\backslash A_m)$ exists since the sequence $(A_{m+1}\backslash A_m)$ consists of disjoint members of $\mathscr{F}$. This proves that (i) implies (vi). Finally, we demonstrate that (vi) implies (i). Suppose (vi) holds. Let (A_n) be any sequence of disjoint members of $\mathscr{F}$. Then $\lim_n \nu(\bigcup_{m=1}^n A_m)$ exists by (vi). Thus $\lim_n \nu(A_n) = \lim_n [\nu(\bigcup_{m=1}^n A_m) - \nu(\bigcup_{m=1}^{n-1} A_m)] = 0$. This proves that (i) holds. q.e.d.

COROLLARY 2.4. *A γ-strongly additive vector measure on a field $\mathscr{F}$ is γ-bounded.*

PROOF. Let $\mathscr{F}$ be a field of sets and $\nu : \mathscr{F} \to E$ a γ-strongly additive measure. Suppose $\|\nu\|_p(S) = +\infty$ for some $p \in \mathscr{P}(\gamma)$. Then one can choose $B_1 \in \mathscr{F}$ such that $p(\nu(B_1)) \geqq 1 + 2p(\nu(S))$. Since $\nu(B_1) = \nu(S) - \nu(S \backslash B_1)$, it follows that $p(\nu(B_1)) - p(\nu(S)) \leqq p(\nu(S \backslash B_1))$. Thus $p(\nu(S \backslash B_1)) \geqq 1$. Now $\|\nu\|_p$ is subadditive on disjoint sets so either $\|\nu\|_p(B_1)$ or $\|\nu\|_p(S \backslash B_1)$ is infinite. If $\|\nu\|_p(B_1) = +\infty$, put $A_1 = B_1$; otherwise, let $A_1 = S \backslash B_1$. In either case, $\|\nu\|_p(A_1) = +\infty$ and $p(\nu(A_1)) \geqq 1$. Replacing S by A_1 in the above line of reasoning, we see that there is an element A_2 of $\mathscr{F}$ contained in A_1 such that $\|\nu\|_p(A_2) = +\infty$ and $p(\nu(A_2)) \geqq 2$. Iterating this procedure, we obtain a non-increasing sequence (A_n) of member of $\mathscr{F}$ such that $\|\nu\|_p(A_n) = +\infty$ and $p(\nu(A_n)) \geqq n$. Thus $\lim_n \nu(A_n)$ does not exist, and an appeal to Corollary 2.3(vii) shows that ν is not γ-strongly additive. q.e.d.

THEOREM 2.5. *Suppose $\{\nu_\alpha : \alpha \in \Lambda\}$ is a uniformly γ-bounded and uniformly γ-countably additive family of E-valued measures defined on a σ-field Σ. If $\mu : \Sigma \to [0, \infty)$ is a countably additive measure and ν_α is μ-continuous with respect to γ for each $\alpha \in \Lambda$, then for every $p \in \mathscr{P}(\gamma)$ we have*

$$\lim_{\mu(A) \to 0} \sup_{\alpha \in \Lambda} p(\nu_\alpha(A)) = 0 .$$

PROOF. First we note that $\{\nu_\alpha : \alpha \in \Lambda\}$ is uniformly γ-countably additive if and only if the family $\{\langle \nu_\alpha, x' \rangle : x' \in U_p^0, \alpha \in \Lambda\}$ is uniformly γ-countably additive for every $p \in \mathscr{P}(\gamma)$. Hence it suffices to prove the statement on scalar-valued countably additive measures. To this end, assume that $\{\mu_\alpha : \alpha \in \Lambda\}$ is a bounded family of uniformly countably additive scalar-valued measures defined on Σ. Define $\nu : \Sigma \to l^\infty(\Lambda)$ by the equation

$$\nu(A)(\alpha) = \mu_\alpha(A) , \qquad \text{for } A \in \Sigma \text{ and } \alpha \in \Lambda .$$

By the uniform countable additivity of $\{\mu_\alpha : \alpha \in \Lambda\}$, it is readily seen that ν is a countably additive vector measure. Moreover, $\nu(A) = 0$ whenever $\mu(A) = 0$. Hence by [7, Theorem I.2.1] ν is μ-continuous, i.e.,

$$\lim_{\mu(A) \to 0} \sup_\alpha |\mu_\alpha(A)| = 0 ,$$

which is the desired result. q.e.d.

THEOREM 2.6 (*Nikodým's Boundedness Theorem*). *Let* $\{v_\alpha : \alpha \in \Lambda\}$ *be a family of E-valued bounded vector measures defined on a* σ*-field* Σ. *If* $\sup_{\alpha \in \Lambda} p(v_\alpha(A)) < +\infty$ for $A \in \Sigma$ *and* $p \in \mathscr{P}(\gamma)$, *then the family* $\{v_\alpha : \alpha \in \Lambda\}$ *is uniformly* γ*-bounded, i.e.,*

$$\sup_{\alpha \in \Lambda} \|v_\alpha\|_p(S) < +\infty \quad \textit{for each } p \in \mathscr{P}(\gamma)\,.$$

PROOF. For $p \in \mathscr{P}(\gamma)$, $\alpha \in \Lambda$ and $A \in \Sigma$ the identity $\sup\{|\langle v_\alpha(A), x'\rangle| : x' \in U_p^0\} = p(v_\alpha(A))$ holds. Hence we can apply the same argument as in [7, Theorem I.3.1] to the family $\{\langle v_\alpha, x'\rangle : \alpha \in \Lambda, x' \in U_p^0\}$ of scalar-valued measures on Σ. q.e.d.

THEOREM 2.7 (*Vitali-Hahn-Saks-Nikodým*). *Let* Σ *be a* σ*-field of a set S and* (v_n) *a sequence of* γ*-strongly additive E-valued measures on* Σ. *If* $\lim_n v_n(A)$ *exists in* γ*-topology for each* $A \in \Sigma$, *then the sequence* (v_n) *is uniformly* γ*-strongly additive.*

PROOF. Since $\lim_n v_n(A)$ exists for each $A \in \Sigma$, an appeal to Theorem 2.6 implies that the sequence (v_n) is uniformly γ-bounded. Assume for the moment that $\lim_n v_n(A) = 0$ for all $A \in \Sigma$. If (v_n) is not uniformly γ-strongly additive, then there exists an equicontinuous sequence (x_n') in E_γ' such that the sequence of scalar measures $(\langle v_n, x_n'\rangle)$ is not uniformly strongly additive. Moreover, (x_n') is a $\sigma(E_\gamma', E)$-bounded sequence, and so $\lim_n \langle v_n(A), x_n'\rangle = 0$ for $A \in \Sigma$. We then define $v : \Sigma \to c_0$ by

$$v(A) = (\langle v_n(A), x_n'\rangle)$$

for all $A \in \Sigma$. The set function v is a c_0-valued bounded measure on Σ. From [7, Theorem I.4.2] it follows that the measure v is strongly additive. We see from the definition of the norm of c_0 that $(\langle v, x_n'\rangle)$ is a uniformly strongly additive sequence, a contradiction. We now consider the general case in which $\lim_n v_n(A)$ exists for all $A \in \Sigma$. If the sequence (v_n) is not uniformly γ-strongly additive, then by Proposition 2.2 there exist a sequence (A_n) of disjoint members of Σ and $p \in \mathscr{P}(\gamma)$ such that

$$\lim_n \sup_m p(v_m(A_n)) > 0\,.$$

By choosing an appropriate subsequence and relabeling, one may assume that $p(v_n(A_n)) > \delta$ for all n and some $\delta > 0$. Furthermore, by making use of the fact that each v_n is γ-strongly additive, we have (by choosing another subsequence if necessary)

$$p(v_n(A_n)) > \delta \quad \text{and} \quad p(v_n(A_{n+1})) < \delta/2$$

for all n. Now set $\sigma_n = v_{n+1} - v_n$. Since $\lim_n v_n(A)$ exists for all $A \in \Sigma$, one has $\lim_n \sigma_n(A) = 0$ for all $A \in \Sigma$. On the other hand,

$$\begin{aligned} p(\sigma_n(A_{n+1})) &\geqq p(\nu_{n+1}(A_{n+1})) - p(\nu_n(A_{n+1})) \\ &> \delta - \delta/2 = \delta/2 . \end{aligned}$$

Hence $\lim_n \sup_m (\sigma_m(A_n)) > 0$ and (σ_n) is not uniformly γ-strongly additive. But according to the first past of the proof, (σ_n) must be uniformly γ-strongly additive because it tends setwise to 0. This contradiction completes the proof.

q.e.d.

Corollary 2.8 (*Vitali-Hahn-Saks*). *Let (ν_n) be a sequence of E-valued γ-countably additive measures such that $\lim_n \nu_n(A) = \nu(A)$ exists for each $A \in \Sigma$. If μ is a nonnegative real-valued countably additive measure such that each ν_n is μ-continuous with respect to γ, then the sequence (ν_n) is uniformly μ-continuous with respect to γ in the sense that $\lim_{\mu(A)\to 0} p(\nu_n(A)) = 0$ uniformly in $n \in N$ for each $p \in \mathscr{P}(\gamma)$. Consequently ν is μ-continuous with respect to γ.*

Proof. By Theorem 2.6 the sequence (ν_n) is uniformly γ-strongly additive. Since ν_n is γ-countably additive, the sequence (ν_n) is uniformly γ-countably additive. Theorem 2.5 can then be applied to obtain the desired assertion.

q.e.d.

Now we assume that the locally convex topology γ on E is the mixed topology associated with a mixed space $(E, \mathscr{B}, \tau)$. We aim to characterize various properties of E_γ-valued vector measures in terms of $\mathscr{B}$ and τ.

Proposition 2.9. *Let $\nu : \mathscr{F} \to E$ be a vector measure defined on a field $\mathscr{F}$. Then ν is γ-strongly additive if and only if $\nu(\mathscr{F})$ is $\mathscr{B}$-bounded and ν is τ-strongly additive.*

Proof. Assume that ν is γ-strongly additive. Then ν is τ-strongly additive and we see from Corollary 2.4 that $\nu(\mathscr{F})$ is γ-bounded. Therefore $\nu(\mathscr{F})$ is $\mathscr{B}$-bounded in virtue of the property (2) of mixed topologies explained in Section 1. Conversely, assume that $\nu(\mathscr{F})$ is $\mathscr{B}$-bounded and ν is τ-strongly additive. Let (A_n) be any sequence of disjoint members of $\mathscr{F}$. Then in virtue of Corollary 2.3 we have $\lim_n \nu(A_n) = 0$ with respect to τ. Since $\nu(\mathscr{F})$ is $\mathscr{B}$-bounded, $\lim_n \nu(A_n) = 0$ with respect to γ by the property (3) of mixed topologies. Hence, we see from Corollary 2.3 again that ν is γ-strongly additive. q.e.d.

Similarly, we obtain the following result.

Proposition 2.10. *Let $\nu : \mathscr{F} \to E$ be a vector measure defined on a field $\mathscr{F}$. Then ν is γ-countably additive if and only if $\nu(\mathscr{F})$ is $\mathscr{B}$-bounded and ν is τ-countably additive.*

PROPOSITION 2.11. *Let $\{\nu_\alpha : \alpha \in \Lambda\}$ be a family of E-valued vector measures defined on a field $\mathscr{F}$. Then $\{\nu_\alpha : \alpha \in \Lambda\}$ is uniformly γ-bounded if and only if $\bigcup_{\alpha\in\Lambda} \nu_\alpha(\mathscr{F})$ is $\mathscr{B}$-bounded.*

PROOF. Since the sufficiency is obvious, we prove the necessity. To the contrary, assume that $\bigcup_{\alpha\in\Lambda} \nu_\alpha(\mathscr{F})$ is not $\mathscr{B}$-bounded. Then for every n there exists $\alpha(n) \in \Lambda$ and $A_n \in \mathscr{F}$ such that $\nu_{\alpha(n)}(A_n) \notin B_n$. Thus $\{\nu_{\alpha(n)}(A_n)\}$ is not $\mathscr{B}$-bounded, i.e., it is not γ-bounded and hence $\sup_n p(\nu_{\alpha(n)}(A_n)) = +\infty$ for some $p \in \mathscr{P}(\gamma)$. Consequently we have $\sup_\alpha \|\nu_\alpha\|_p(S) = +\infty$. This is a contradiction.
q.e.d.

PROPOSITION 2.12. *Let $\nu : \mathscr{F} \to E$ be a vector measure defined on a field $\mathscr{F}$. Then ν is of γ-bounded variation if and only if there exists an absolutely convex τ-closed and $\mathscr{B}$-bounded subset B of E such that B contains $\nu(\mathscr{F})$ and $\nu : \mathscr{F} \to E_B$ is of $\|\cdot\|_B$-bounded variation. If in particular $(E, \|\cdot\|, \tau)$ is a Saks space, then ν is of γ-bounded variation if and only if ν is of $\|\cdot\|$-bounded variation.*

PROOF. Assume that ν is of γ-bounded variation. Then the set $\{(\nu(A))_{A\in\pi} : \pi \in \Pi\}$ is bounded in $l^1_N\{E_\gamma\}$. Since E_γ has property (B) by Proposition 1.3, there exists a τ-closed $B \in \mathscr{B}$ such that $\sum_{A\in\pi} \|\nu(A)\|_B \leqq 1$ for each $\pi \in \Pi$. This means that $\nu : \mathscr{F} \to E_B$ is of $\|\cdot\|_B$-bounded variation. Conversely, suppose that there exists an absolutely convex τ-closed and $\mathscr{B}$-bounded subset B of E such that B contains $\nu(\mathscr{F})$ and $\nu : \mathscr{F} \to E_B$ is of $\|\cdot\|_B$-bounded variation. Then the inclusion map $I : (E_B, \|\cdot\|_B) \to (E, \gamma)$ is bounded. Hence for each $p \in \mathscr{P}(\gamma)$ there exists $M_p > 0$ such that $p(x) \leqq M_p\|x\|_B$ for each $x \in E_B$. This means that if ν is of $\|\cdot\|_B$-bounded variation, then ν is of γ-bounded variation.
q.e.d.

3. Integration and convergence theorems

In this section, we treat the Vitali type convergence theorems from the point of view of the three types (defined below) of integrals of functions which take values in locally convex spaces. Throughout this section (S, Σ, μ) is supposed to be a fixed complete nonnegative finite measure space and (E, γ) a locally convex space.

We begin by introducing four notions of measurability of functions from S into E.

(I) A function $f : S \to E$ is said to be *γ-strongly measurable* if there exists a sequence $(f_n)_{n\in N}$ of measurable simple functions such that $\lim_{n\to\infty} f_n(s) = f(s)$ μ-a.e. in γ.

(II) (1) Let $p \in \mathscr{P}(\gamma)$. A function $f : S \to E$ is said to be *measurable by p* if there exist $S_{0,p} \subset S$ with $\mu(S_{0,p}) = 0$, and a sequence $(f_n^p)_{n\in N}$ of measurable simple functions such that $\lim_{n\to\infty} p(f_n^p(s) - f(s)) = 0$ for all $s \in S \backslash S_{0,p}$.

(2) A function $f: S \to E$ is said to be *measurable by γ-seminorm* if f is measurable by each $p \in \mathscr{P}(\gamma)$.

(III) A function $f: S \to E$ is said to be *γ-weakly measurable* if for each $x' \in E'_\gamma$, $\langle f(s), x' \rangle$ is measurable.

The integral of a measurable simple function $f = \sum_i x_i \chi_{A_i}$, $A_i \in \Sigma$ is defined as usual by

$$\int_A f\, d\mu = \sum_i x_i \mu(A \cap A_i).$$

We then introduce three kinds of definitions of integrability corresponding to the notions of measurability introduced above.

(I) A function $f: S \to E$ is said to be *γ-strongly integrable* if there exists a sequence $(f_n)_{n\in N}$ of measurable simple functions such that

1. $f_n(s) \to f(s)$ μ-a.e. in γ, i.e., f is γ-strongly measurable;
2. $p(f_n(s) - f(s)) \in L^1(\mu)$ for each $n \in N$, and

$$\lim_{n\to\infty} \int_S p(f_n(s) - f(s))\, d\mu = 0 \quad \text{for each} \quad p \in \mathscr{P}(\gamma);$$

3. $\int_A f_n\, d\mu$ converges in (E, γ) for each $A \in \Sigma$.

In this case we write $(B)_\gamma$-$\int_A f\, d\mu$ for the limit $\lim_{n\to\infty} \int_A f_n\, d\mu$ and call it the *γ-strong integral* of f over A.

(II) A function $f: S \to E$ is said to be *integrable by γ-seminorm* if for each $p \in \mathscr{P}(\gamma)$ there exists a subset $S_{0,p}$ of S with $\mu(S_{0,p}) = 0$ and a sequence $(f_n^p)_{n\in N}$ of measurable simple functions such that

1. for each $s \in S \backslash S_{0,p}$, $\lim_{n\to\infty} p(f_n^p(s) - f(s)) = 0$;
2. $\lim_{n\to\infty} \int_S p(f_n^p(s) - f(s))\, d\mu = 0$ for each $p \in \mathscr{P}(\gamma)$;
3. for each $A \in \Sigma$, there exists $x_A \in E$ such that x_A is independent of p

and

$$\lim_{n\to\infty} p\left(\int_A f_n^p(s)\, d\mu - x_A\right) = 0 \quad \text{for each} \quad p \in \mathscr{P}(\gamma).$$

In this case we write $x_A = (\gamma)$-$\int_A f\, d\mu$ and call x_A the integral by γ-seminorm of f over A.

(III) A function $f: S \to E$ is said to be *γ-Pettis integrable* if

1. f is γ-weakly integrable in the sense that $\langle f(s), x' \rangle \in L^1(\mu)$ for every $x' \in E'_\gamma$;

2. for each $A \in \Sigma$, there exists $x_A \in E$ such that $\langle x_A, x' \rangle = \int_A \langle f(s), x' \rangle \, d\mu$ for every $x' \in E'_\gamma$.

The value x_A is written as $x_A = (P)_\gamma\text{-}\int_A f \, d\mu$ and is called the γ-Pettis integral of f on A.

If in the definition (I) (E, γ) is sequentially quasi-complete, then the third condition in (I) is not necessary. As seen from the above definitions of integrals that the γ-strong integrability implies the integrability by γ-seminorm and the integrability by γ-seminorm also implies the γ-Pettis integrability. In [23] Thomas treats Pettis integral in quasi-complete locally convex Souslin spaces, and gave interesting criteria for the integrability of vector valued functions. Obviously, if T is a continuos linear map from a locally convex space (E, τ) to another locally convex space (F, γ), and if $f : S \to E$ is τ-Pettis integrable, then $T \circ f$ *is also* γ-Pettis integrable and $(P)_\gamma\text{-}\int_A T \circ f \, d\mu = T((P)_\tau\text{-}\int_A f \, d\mu)$ for all $A \in \Sigma$. Blondia [2] investigated the relationship between the above-mentioned three types of integrability.

The following three theorems are directly derived from the results due to Blondia [2]. The first two theorems give crucial relationships between Pettis integrability and integrability by seminorm.

THEOREM 3.1. *Let f be measurable by γ-seminorm. Then f is integrable by γ-seminorm if and only if f is γ-Pettis integrable and $p(f(s)) \in L^1(\mu)$ for each $p \in \mathscr{P}(\gamma)$. Moreover $(\gamma)\text{-}\int_A f \, d\mu = (P)_\gamma\text{-}\int_A f \, d\mu$ for each $A \in \Sigma$.*

THEOREM 3.2. *Let f be γ-Pettis integrable and measurable by γ-seminorm. Then the induced measure ν, defined by $\nu(A) = (P)_\gamma\text{-}\int_A f \, d\mu$ for $A \in \Sigma$, has γ-bounded variation if and only if f is integrable by γ-seminorm. Moreover, $|\nu|_p(A) = \int_A p(f(s)) \, d\mu$ for each $A \in \Sigma$.*

THEOREM 3.3. *Let (E, γ) be a complete locally convex space and let f: $S \to E$ be measurable by γ-seminorm. If $p(f(s)) \in L^1(\mu)$ for each $p \in \mathscr{P}(\gamma)$, then f is integrable by γ-seminorm.*

We now establish a convergence theorem for Pettis integrals in locally convex spaces.

THEOREM 3.4 (*Vitali Convergence Theorem for Pettis Integrals*). *Let (E, γ) be a complete locally convex space and let $f : S \to E$. If there exists a sequence $(f_n)_{n \in N}$ of E-valued γ-Pettis integrable functions on S satisfying*

(a) *for every equicontinuous subset H of E'_γ, the set $\{\langle f_n, x' \rangle : x' \in H, n \in N\}$ is uniformly integrable;*

(b) $\lim_{n \to \infty} \langle f_n, x' \rangle = \langle f, x' \rangle$ *in measure for every $x' \in E'_\gamma$,*

then f is γ-Pettis integrable and $\lim_{n \to \infty} (P)_\gamma\text{-}\int_A f_n \, d\mu = (P)_\gamma\text{-}\int_A f \, d\mu$ *γ-weakly in E for every $A \in \Sigma$.*

The above theorem was given by Musiał [16] for the case of Banach spaces. We here prove the theorem based on his argument. First, we need the following lemma.

LEMMA 3.5. *Let (E, γ) be a locally convex space. Let H be a subset of E'_γ which is absolutely convex and $\sigma(E'_\gamma, E)$-compact. Assume that $f: S \to E$ is a function such that $\langle f, x' \rangle \in L^1(\mu)$ for every $x' \in H$. Then we have*

$$\sup_{x' \in H} \int_A |\langle f, x' \rangle| \, d\mu < +\infty \quad \textit{for every } A \in \Sigma .$$

PROOF. First we consider a normed space E'_H generated in E'_γ by H. Namely, E'_H is the linear space generated by H and the norm on E'_H is defined by $\|x'\|_H = \inf\{\lambda > 0 : x' \in \lambda H\}$ for each $x' \in E'_H$. Hence H is exactly the unit ball of the normed space E'_H. Since H is $\sigma(E'_\gamma, E)$-compact, E'_H is a Banach space. Let $A \in \Sigma$. Then it follows from the closed graph theorem that the map $x' \to \langle f, x' \rangle$ from E'_H to $L^1(\mu)$ is continuous. Thus there exists $M(A) > 0$ such that $\int_A |\langle f, x' \rangle| \, d\mu \leqq M(A)\|x'\|_H$ for every $x' \in E'_H$. This means that $\sup_{x' \in H} \int_A |\langle f, x' \rangle| \, d\mu < +\infty$. q.e.d.

PROOF OF THEOREM 3.4. First, assume that E is a complete locally convex space over the real field $\boldsymbol{R}$. Fix any $A \in \Sigma$, and let C be the weak closure of the set $\{(P)_\gamma\text{-}\int_A f_n \, d\mu : n \in \boldsymbol{N}\}$. Since Vitali's convergence theorem guarantees that $\lim_{n \to \infty} \int_A \langle f_n, x' \rangle \, d\mu = \int_A \langle f, x' \rangle \, d\mu$ for every $x' \in E'_\gamma$, we see that C is bounded and $C \setminus \{(P)_\gamma\text{-}\int_A f \, d\mu : n \in \boldsymbol{N}\}$ consists of at most one point. In order to prove our assertion it is sufficient to show that C is weakly compact. In fact, if C would be weakly compact, then there would exist a weak limit of $((P)_\gamma\text{-}\int_A f_n \, d\mu)_{n \in \boldsymbol{N}}$ in E. Clearly the limit must coincide with $(P)_\gamma\text{-}\int_A f \, d\mu$, and so we could conclude that f is γ-Pettis integrable on A. To the contrary suppose that C is not weakly compact. Then, by a well-known result due to James ([14, Theorem 1]), there exist an equicontinuous subset $\{x_n : n \in \boldsymbol{N}\}$, a set $\{x_n : n \in \boldsymbol{N}\} \subset C$, and $\theta > 0$, such that $x'_k(x_n) = 0$ for $k > n$ and $x'_k(x_n) > \theta$ for $k \leqq n$. Consequently, we can then choose a subsequence $(g_m)_{m \in \boldsymbol{N}}$ of $(f_n)_{n \in \boldsymbol{N}}$ and a subsequence $(y'_m)_{m \in \boldsymbol{N}}$ of $(x'_n)_{n \in \boldsymbol{N}}$, such that

$$\int_A \langle g_m, y'_k \rangle \, d\mu = 0 \quad \text{for } k > m , \tag{1}$$

$$\int_A \langle g_m, y'_k \rangle \, d\mu > \theta \quad \text{for } k \leqq m , \tag{2}$$

$$\lim_{m \to \infty} \int_A \langle g_m, x' \rangle \, d\mu = \int_A \langle f, x' \rangle \, d\mu \quad \text{for every } x' \in E'_\gamma . \tag{3}$$

We then consider the set $\{\langle f, y'_m\rangle : m \in \boldsymbol{N}\}$. Since the set $\{y'_m : m \in \boldsymbol{N}\}$ is equicontinuous, it follows from Lemma 3.5 that

$$\sup_{m\in \boldsymbol{N}} \int_S |\langle f, y'_m\rangle|\, d\mu < +\infty\, .$$

From this and (a) we see that $\{\langle f, y'_m\rangle : m \in \boldsymbol{N}\}$ is uniformly integrable (i.e., $\lim_{\mu(B)\to 0} \sup_{m\in \boldsymbol{N}} \int_B |\langle f, y_m\rangle|\, d\mu = 0$) and bounded. Hence it is relatively weakly compact. This yields the existence of a function $h \in L^1(\mu)$ and a subsequence $(z'_j)_{j\in \boldsymbol{N}}$ of $(y'_m)_{m\in \boldsymbol{N}}$ such that $\lim_{j\to\infty} \langle f, z'_j\rangle = h$ weakly in $L^1(\mu)$. Applying (3) to every z'_j we get an inequality $\int_A \langle f, z'_j\rangle\, d\mu \geqq \theta$ and hence $\int_A h\, d\mu \geqq \theta$. We now appeal to the theorem of Mazur. Let $a_1^m, \ldots, a_{k(m)}^m$, $m \in \boldsymbol{N}$, be non-negative numbers such that $\sum_j a_j^m = 1$ and $\lim_m \sum_j a_j^m \langle f, z'_{j+m}\rangle = h$ in $L^1(\mu)$. Without loss of generality, we may assume that this convergence holds μ-a.e. Let z'_0 be a $\sigma(E'_\gamma, E)$-cluster point of the sequence $(\sum_j a_j^m z'_{j+m})_{m\in \boldsymbol{N}}$, then $h = \langle f, z_0\rangle$ μ-a.e. In particular, we have

$$\int_A \langle f, z'_0\rangle\, d\mu \geqq \theta\, . \tag{4}$$

On the other hand, each g_n is γ-Pettis integrable, and also the function $x' \to \int_A \langle g_n, x'\rangle\, d\mu$ is $\sigma(E'_\gamma, E)$-continuous. Hence, if $(w'_{n,\alpha})$ is a subnet of $(\sum_j a_j^m z'_{j+m})_{m>n}$ converging to z'_0 in $\sigma(E'_\gamma, E)$, then the application of (1) implies

$$\begin{aligned} 0 = \lim_\alpha \int_A \langle g_n, w'_{n,\alpha}\rangle\, d\mu &= \lim_\alpha \left\langle (P)_\gamma\text{-}\int_A g_n\, d\mu, w'_{n,\alpha}\right\rangle \\ &= \left\langle (P)_\gamma\text{-}\int_A g_n\, d\mu, z'_0\right\rangle = \int_A \langle g_n, z'_0\rangle\, d\mu\, . \end{aligned}$$

Since this holds for every $n \in \boldsymbol{N}$, we see from (3) that $\int_A \langle f, z'_0\rangle\, d\mu = 0$. But this contradicts the inequality (4). Thus it follows that C is weakly compact and so the real case of the theorem is proved.

Next assume that E is a complete locally convex space over the complex field. Let $E_{\boldsymbol{R}}$ be a locally convex space E restricted over the real field $\boldsymbol{R}$. If $f : S \to E_{\boldsymbol{R}}$ is γ-Pettis integrable in $(E_{\boldsymbol{R}}, \gamma)$, then f is also γ-Pettis integrable in (E, γ) and $\langle \nu(A), x'\rangle = \int_A \langle f, x'\rangle\, d\mu$ for every $x' \in E'$ and $A \in \Sigma$, where $\nu(A)$ is the indefinite γ-Pettis integrable of f on A in $(E_{\boldsymbol{R}}, \gamma)$. To show this, we have only to note that if $f : S \to E_{\boldsymbol{R}}$ is γ-Pettis integrable in $(E_{\boldsymbol{R}}, \gamma)$, then $i\cdot f(\cdot)$ ("i" means the imaginary unit) is also γ-Pettis integrable in $(E_{\boldsymbol{R}}, \gamma)$ and

$$(P)_\gamma\text{-}\int_A i\cdot f(s)\, d\mu = i\cdot (P)_\gamma\text{-}\int_A f\, d\mu \text{ in } E_{\boldsymbol{R}}$$

for every $A \in \Sigma$. But this is clear since the multiplier $i\cdot$ is continuous on $(E_{\boldsymbol{R}}, \gamma)$. q.e.d.

As a direct consequence of Theorem 3.4 we get the following generalization of the Lebesgue dominated convergence theorem for Bochner integrals:

THEOREM 3.6 (*Lebesgue Dominated Convergence Theorem for Pettis Integrals*). *Let $f: S \to E$ be a function satisfying the following two conditions*:

(α) *There exists a sequence of γ-Pettis integrable functions $f_n: S \to E$, $n \in N$, such that $\lim_n \langle f_n, x' \rangle = \langle f, x' \rangle$ in measure, for every $x' \in E'_\gamma$.*

(β) *There exists a γ-Pettis integrable function $g: S \to E$ such that $|\langle f_n, x' \rangle| \leqq |\langle g, x' \rangle|$ μ-a.e. for each $x' \in E'_\gamma$ and $n \in N$ (the exceptional set may depend on x').*

Then f is γ-Pettis integrable and

$$\lim_n (P)_\gamma\text{-}\int_A f_n\, d\mu = (P)_\gamma\text{-}\int_A f\, d\mu \quad \gamma\text{-weakly in } E \text{ for all } A \in \Sigma\,.$$

PROOF. It suffices to show that condition (a) of Theorem 3.4 is induced from (β). Let H be any equicontinuous subset of E'_γ. Then there exists a closed and absolutely convex neighborhood U of 0 so that $H \subset U^0$. Put $p(x) = \inf\{\lambda > 0 : x \in \lambda U\}$. Then p defines a γ-continuous seminorm on E. Given $x' \in U^0$ and $A \in \Sigma$, (β) implies

$$\begin{aligned}\int_A |\langle f_n, x' \rangle|\, d\mu &\leqq \int_A |\langle g, x' \rangle|\, d\mu = |\langle v, x' \rangle|(A) \\ &\leqq 4 \sup\{|\langle v(B), x' \rangle| : B \subset A, B \in \Sigma\} \\ &\leqq 4 \sup\{p(v(B)) : B \subset A, B \in \Sigma\}\,,\end{aligned}$$

where v is the indefinite γ-Pettis integral of g and $|\langle v, x' \rangle|(A)$ means the variation on A of the measure $\langle v, x' \rangle$. Thus

$$\sup_{x' \in U^0} \int_A |\langle f_n, x' \rangle|\, d\mu \leqq 4 \sup\{p(v(B)) : B \subset A, B \in \Sigma\}\,.$$

Consequently, it follows from the absolute continuity of v (see [23]) that $\lim_{\mu(A)\to 0} \sup_{x' \in U^0} \int_A |\langle f_n, x' \rangle|\, d\mu = 0$. q.e.d.

Next we state a convergence theorem for integrals by seminorm.

THEOREM 3.7 (*Vitali Convergence Theorem for Integrals by Seminorm*). *Let (E, γ) be a complete locally convex space. Let $f: S \to E$ and suppose that* there exists a sequence $(f_n)_{n \in N}$ *of E-valued functions on S which are integrable by γ-seminorm and satisfies*

(c) *for every $p \in \mathscr{P}(\gamma)$, the set $\{p(f_n) : n \in N\}$ is uniformly integrable*;

(d) $\lim_{n\to\infty} p(f_n - f) = 0$ *in measure for $p \in \mathscr{P}(\gamma)$,*

then f is integrable by γ-seminorm and

$$\lim_{n\to\infty}(\gamma)\text{-}\int_A f_A\,d\mu = (\gamma)\text{-}\int_A f\,d\mu \quad \textit{in } \gamma \textit{ for every } A \in \Sigma\,.$$

PROOF. By Vitali's convergence theorem, we see that for every $p \in \mathscr{P}(\gamma)$,

(1) f is measurable by p, and

(2) $\lim_{n\to\infty} \int_S p(f_n - f)\,d\mu = 0$.

The formula (2) implies that $((\gamma)\text{-}\int_A f_n\,d\mu)$ is Cauchy in E for every $A \in \Sigma$, and so $(\gamma)\text{-}\int_A f_n\,d\mu$ converges to some x_A in E. Making use of (2) again, we have

$$\langle x_A, x'\rangle = \lim_{n\to\infty}\left\langle (\gamma)\text{-}\int_A f_n\,d\mu, x'\right\rangle = \int_A \langle f, x'\rangle\,d\mu \text{ for each } x' \in E'_\gamma\,.$$

This shows that f is γ-Pettis integrable. Thus f is measurable by γ-seminorm and γ-Pettis integrable. Furthermore $p(f) \in L^1(\mu)$ for every $p \in \mathscr{P}(\gamma)$. Hence by Theorem 3.1 f is integrable by γ-seminorm. By (2) we conclude that $\lim_{n\to\infty}(\gamma)\text{-}\int_A f_n\,d\mu = (\gamma)\text{-}\int_A f\,d\mu$ in γ for every $A \in \Sigma$. q.e.d.

The following result is an immediate consequence of Theorem 3.7.

THEOREM 3.8 (*Lebesgue Dominated Convergence Theorem for Integrals by Seminorm*). *Let $f: S \to E$ be a function satisfying the following two conditions*:

(γ) *There exists a sequence $(f_n)_{n\in N}$ of E-valued functions on S which are integrable by γ-seminorm such that* $\lim_n p(f_n - f) = 0$ *in measure for every* $p \in \mathscr{P}(\gamma)$.

(δ) *For every $p \in \mathscr{P}(\gamma)$, there exists an integrable function $g_p: S \to [0, +\infty)$ such that $p(f_n) \leqq g_p$ μ-a.e. for every $n \in N$.*

Then f is integrable by γ-seminorm and

$$\lim_{n\to\infty}(\gamma)\text{-}\int_A f_n\,d\mu = (\gamma)\text{-}\int_A f\,d\mu \quad \textit{in } \gamma \textit{ for every } A \in \Sigma\,.$$

4. Integration in Souslin mixed topological spaces

In this section we advance an integration theory in mixed topological spaces $\gamma[\mathscr{B}, \tau]$ under the assumption that (E, τ) is a locally convex Souslin space. Let (S, Σ, μ) be a fixed complete nonnegative finite measure space. This is one of our main objectives.

The following lemma due to Thomas states a characteristic property of weakly measurable functions which take their values in locally convex quasi-complete Souslin spaces.

LEMMA 4.1. *Let (E, τ) be a locally convex quasi-complete Souslin space and $f: S \to E$ a τ-weakly measurable function. Then there exists a countable partition $S = \bigcup_{n=0}^{\infty} S_n$ of S into measurable subsets such that $\mu(S_0)$, $\mu(S_n) > 0$ and $f(S_n)$ is relatively compact for $n \in N$.*

Using this lemma, Thomas [23, Theorem 3] gave the following theorem.

THEOREM 4.2. *Let* (E, τ) *be a locally convex quasi-complete Souslin space. Let* $f: S \to E$ *be a* τ*-weakly measurable function such that* $\int_S p(f(s))\, d\mu < +\infty$ *for every* $p \in \mathscr{P}(\tau)$. *Then* f *is* τ*-Pettis integrable.*

If in Theorem 4.2 the function f is bounded, then we obtain stronger integrability.

PROPOSITION 4.3. *Let* (E, τ) *be a quasi-complete locally convex Souslin space. Then every bounded* τ*-weakly measurable function is* τ*-strongly integrable.*

PROOF. Let $f: S \to E$ be bounded and τ-weakly measurable. Then there exists a τ-closed bounded subset B such that $f(S) \subset B$. Therefore B is Souslin, and hence we see in the same way as in the proof of [2, Proposition 2.3] that we can take a sequence (f_n) of simple measurable functions such that $f_n(S) \subset B$ for $n \in N$ and $\lim_{n\to\infty} f_n(s) = f(s)$ for each $s \in S$. Thus the Lebesgue bounded convergence theorem implies that $\lim_n \int_S p(f_n - f)\, d\mu = 0$ for every $p \in \mathscr{P}(\tau)$.
q.e.d.

C. Blondia showed in [2] that weak measurability, measurability by seminorm and strong measurability are all equivalent in locally convex Souslin spaces. Using this fact, we have immediately the following proposition.

PROPOSITION 4.4. *Let* (E, τ) *be a locally convex Souslin space. Let* $(E, \mathscr{B}, \tau)$ *be a mixed space. For a function* $f: S \to E$ *the following conditions are equivalent*:

(1) f *is* τ*-weakly measurable*;
(2) f *is* γ*-weakly measurable*;
(3) $f^{-1}(B) \in \Sigma$ *for every* τ*-Borel subset* B *of* E;
(4) $f^{-1}(B) \in \Sigma$ *for every* γ*-Borel subset* B *of* E;
(5) $f^{-1}(C) \in \Sigma$ *for every* τ*-Souslin subset* C *of* E;
(6) $f^{-1}(C) \in \Sigma$ *for every* γ*-Souslin subset* C *of* E;
(7) f *is measurable by* τ*-seminorm*;
(8) f *is measurable by* γ*-seminorm*;
(9) f *is* τ*-strongly measurable*;
(10) f *is* γ*-strongly measurable.*

PROOF. Suppose that (E, τ) is Souslin. Then, by Proposition 1.4, (E, γ) is also Souslin. Hence by [2, Proposition 2.3], we have the implications $(1) \Leftrightarrow (3) \Leftrightarrow (5) \Leftrightarrow (7) \Leftrightarrow (9)$ and $(2) \Leftrightarrow (4) \Leftrightarrow (6) \Leftrightarrow (8) \Leftrightarrow (10)$. Thus it is sufficient to show that $(3) \Leftrightarrow (4)$. Since γ is finear than τ and a countable union of Souslin spaces is again Souslin, it follows that the Borel σ-fields with respect to τ and γ coincide.
q.e.d.

The following theorem shows that in the mixed spaces τ-Pettis integrability is equivalent to γ-Pettis integrability. It would be noted, however, that integrability by τ-seminorm is not always equivalent to integrability by γ-seminorm (see Remark 4.12).

THEOREM 4.5. *Let γ be the mixed topology associated with a mixed space $(E, \mathscr{B}, \tau)$. Then $f: S \to E$ is τ-Pettis integrable if and only if it is γ-Pettis integrable.*

PROOF. Since the sufficiency is obvious, we show the necessity. Let ν be the infefinite τ-Pettis integral of f. Let x' be any element of E'_γ. Since E'_γ with the strong dual topology is Fréchet and E'_τ is dense in E'_γ by Proposition 1.2 (ii), there exists a sequence (x'_n) of E'_τ such that $x'_n \to x'$ in E'_γ. Since $\langle f(s), x'_n \rangle \to \langle f(s), x' \rangle$ pointwise and $\langle \nu(A), x' \rangle = \lim_n \langle \nu(A), x'_n \rangle = \lim_n \int_A \langle f, x'_n \rangle \, d\mu$, it follows from the Vitali-Hahn-Saks theorem that $\langle f(s), x' \rangle$ is integrable, and

$$\langle \nu(A), x' \rangle = \lim_n \int_A \langle f, x'_n \rangle \, d\mu = \int_A \langle f, x' \rangle \, d\mu$$

for any $A \in \Sigma$. This shows that f is γ-Pettis integrable. q.e.d.

PROPOSITION 4.6. *Let $(E, \mathscr{B}, \tau)$ be a mixed space such that (E, τ) is a quasi-complete locally convex Souslin space. Then every $\mathscr{B}$-bounded τ-weakly measurable function is γ-strongly integrable.*

This is obvious from the property of γ that the $\mathscr{B}$-boundedness is equivalent to the γ-boundedness, Proposition 4.3 and Proposition 4.4.

COROLLARY 4.7. *Let (E, τ) be a quasi-complete locally convex Souslin space. Then every τ-weakly measurable function is locally τ-strongly integrable, i.e., for every $A \in \Sigma$ with $\mu(A) > 0$ there exists a $B \in \Sigma$ with $\mu(B) > 0$ such that $B \subset A$ and f is τ-strongly integrable on B. Therefore, if $(E, \mathscr{B}, \tau)$ is a mixed space, then every τ-weakly measurable function is locally γ-strongly integrable.*

PROOF. Let $f: S \to E$ be τ-weakly measurable. We see from Lemma 4.1 that there exists a partition $S = \bigcup_{n=0}^{\infty} S_n$ of S into measurable subsets such that $\mu(S_0) = 0$, $\mu(S_n) > 0$ and $f(S_n)$ is relatively compact for $n \in N$. Take any $A \in \Sigma$ with $\mu(A) > 0$. Then there exists $n \in N$ such that $\mu(A \cap S_n) > 0$. We put $B = A \cap S_n$. Then we see easily from Proposition 4.3 that f is τ-strongly integrable on B. As to the latter part of the assertion it is enough to note that τ-weak measurability and γ-weak measurability are equivalent. q.e.d.

Let E and F be locally convex spaces. A continuous linear map $u: E \to F$ is *nuclear* if it is of the form

$$x \to u(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x'_n \rangle y_n ,$$

where $\sum_{n=1}^{\infty}|\lambda_n| < +\infty$, (x_n') is an equicontinuous sequence in E', and (y_n) is a sequence contained in a closed bounded Banach ball B of F. A locally convex spaces E is *nuclear* if for every continuous seminorm p on E, the canonical map $E \to \hat{E}_p$ is a nuclear map. Here $\hat{E}_p$ stands for the completion of the quotient space $E/p^{-1}(0)$ with the norm $\|\hat{x}\|_p = p(x)$, where $\hat{x}$ denotes the coset containing x. It is known [8, p. 257] that if E is nuclear, every weakly integrable function $f: S \to E$ satisfies the condition that $p(f) \in L^1(\mu)$ for every continuous seminorm p on E.

E. Thomas gave the following theorem.

THEOREM 4.8 ([23, *Theorem* 7]). *Let (E, τ) be a Souslin space such that E is the topological dual of a quasi-complete barrelled nuclear space F and $E_\tau' = F$. If $f: S \to E$ is τ-weakly measurable, then there exist a τ-bounded Banach ball B in E and $S_0 \in \Sigma$ with $\mu(S \backslash S_0) = 0$ such that $f(s) \in E_B$ for all $s \in S_0$ and $f: S_0 \to E_B$ is Bochner integrable in the usual sense.*

The following result is obtained in the case of Souslin spaces as mentioned in Theorem 4.8. This contrasts with Proposition 4.3.

THEOREM 4.9. *Let (E, τ) be as in the above Theorem 4.8. Then τ-weakly integrable function $f: S \to E$ is τ-strongly integrable.*

PROOF. Using the previous theorem we can take a τ-bounded Banach ball B in E such that $f(s) \in E_B$ μ-a.e. (i.e., $s \in S_0 \in \Sigma$ with $\mu(S \backslash S_0) = 0$) and f: $S_0 \to E_B$ is Bochner integrable. Hence there is a sequence (f_n) of E_B-valued simple functions such that $\|f_n(s) - f(s)\|_B \to 0$ μ-a.e. and $\int_S \|f_n - f\|_B \, d\mu \to 0$. If $p \in \mathscr{P}(\tau)$ and $M = \sup\{p(x) : x \in B\}$, then $p(x) \leqq M\|x\|_B$ for all $x \in E$, whence

$$\int_S p(f_n - f)\, d\mu \leqq M \int_S \|f_n - f\|_B \, d\mu .$$

Thus we have $\lim_{n\to\infty} \int_S p(f_n - f)\, d\mu = 0$ for all $p \in \mathscr{P}(\tau)$. q.e.d.

Nuclear Fréchet spaces and the function spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{S}$ and $\mathscr{S}'$, which appear in distribution theory are all Souslin spaces satisfying the assumption of the above Theorem. See for instance [23].

THEOREM 4.10. *Let $(E, \mathscr{B}, \tau)$ be a mixed space such that (E, τ) is a quasi-complete locally convex Souslin space. If (E, γ) is nuclear, a function $f: S \to E$ is integrable by τ-seminorm if and only if it is integrable by γ-seminorm.*

PROOF. Since γ is finer than τ, the sufficiency is obvious. We show the necessity. To this end, let $f: S \to E$ be integrable by τ-seminorm. Then, in view of Proposition 4.4, f is measurable by γ-seminorm. As f is τ-Pettis integrable, f is γ-Pettis integrable by Theorem 4.5. By the nuclearity of (E, γ),

$p(f(s)) \in L^1(\mu)$ for each $p \in \mathscr{P}(\gamma)$ as mentioned before Theorem 4.8. Thus we see from Theorem 3.1 that f is integrable by γ-seminorm. q.e.d.

REMARK 4.11. If E is an infinite dimensional Saks space, then (E, γ) is never nuclear. In fact, suppose that (E, γ) is nuclear. Then (E, γ) has property (B) by Proposition 1.3, and so its strong dual E'_γ is also nuclear (see [19, 4.3.1]). This is a contradiction because E'_γ is in this case an infinite dimensional Banach space. In general, the class of complete nuclear spaces of the form $(E, \gamma[\mathscr{B}, \tau])$ coincides with the class of the strong duals of nuclear Fréchet spaces. Let (E, γ) be a complete nuclear space. Then (E, γ) is Montel, and so reflexive. Hence (E, γ) is the strong dual of the nuclear Fréchet space E'_γ by Proposition 1.3 and [19, 4.3.1]. Conversely, let F be a nuclear Fréchet space and denote its topological dual by E. Let $\mathscr{B}$ be the family of absolutely convex, equicontinuous subsets of E and let $\tau = \sigma(E, F)$. Then we see from [5, Corollary 4.2] and the Banach-Dieudonné theorem [15, p. 181] that $\gamma[\mathscr{B}, \tau]$ is equivalent to each of the following topology: the finest topology on E which coincides with τ on each equicontinuous subset of E, the topology of precompact convergence, $\beta(E, F)$ (= the strong topology on E). This shows that (E, γ) is the strong dual of the nuclear Fréchet space.

REMARK 4.12. If (E, γ) is not nuclear, integrability by τ-seminorm is not always equivalent to integrability by γ-seminorm even if (E, τ) is nuclear. Indeed, let (S, Σ, μ) be the Lebesgue measure space with $S = [0, 1]$ and let E be a separable Hilbert space. Let τ be the weak topology on E. Then it is obvious that (E, τ) is a quasi-complete locally convex Souslin space and that $(E, \|\cdot\|, \tau)$ is a Saks space. Let (e_n) be an orthonormal basis on E. Take any $(\lambda_n) \in c_0 \backslash l^2$ with $\lambda_n > 0$. Then there exists $(\xi_n) \in l^2$ with $\xi_n > 0$ $(n = 1, 2, \dots)$ such that $\sum_{n=1}^{\infty} \lambda_n \xi_n = +\infty$. We define $f(s) = 2^n \xi_n e_n$ on $(1/2^n, 1/2^{n-1})$, $n = 1, 2, \dots$, and $f(s) = 0$ elsewhere. Then it is easily verified that f is τ-Pettis integrable, and hence f is integrable by τ-seminorm since f is norm-measurable. Put $p_n(x) = |(e_n, x)|$, $n = 1, 2, \dots$ Let $p(x) = \sup_n \lambda_n p_n(x)$. Since $B_{\|\cdot\|}$ is τ-compact (i.e., weakly compact), Proposition 1.1 implies that p is γ-continuous, and $p(f(s)) = \lambda_n \xi_n 2^n$ on $(1/2^n, 1/2^{n-1})$, $n = 1, 2, \dots$ Thus we have $\int_S p(f(s))\, d\mu = \sum_{n=1}^{\infty} \lambda_n \xi_n = +\infty$, and so $p(f(s)) \notin L^1(\mu)$. Consequently we see from Theorem 3.1 and Theorem 4.5 that f is not integrable by γ-seminorm.

The integrability by γ-seminorm can be characterized in terms of bornology $\mathscr{B}$ and τ-Pettis integrability.

THEOREM 4.13. *Let $(E, \mathscr{B}, \tau)$ be a mixed space such that (E, τ) is a quasi-complete locally convex Souslin space. If $f : S \to E$ is τ-Pettis integrable, then f is integrable by γ-seminorm if and only if there exists a τ-closed $B \in \mathscr{B}$ such that*

$B \supset \nu(\Sigma)$ ($= \{\nu(A): A \in \Sigma\}$) and the indefinite τ-Pettis integral of f is of $\|\cdot\|_B$-bounded variation.

PROOF. Let $f: S \to E$ be τ-Pettis integrable. Then it follows from Proposition 4.4 and Theorem 4.5 that f is measurable by γ-seminorm by γ-Pettis integrable. Hence we see from Theorem 3.1 and Proposition 2.12 that f is integrable by γ-seminorm if and only if there exists a τ-closed $B \in \mathscr{B}$ with $B \supset \nu(\Sigma)$ and the indefinite τ-Pettis integral of f is of $\|\cdot\|_B$-bounded variation.

q.e.d.

Let (E, τ) be a locally convex Souslin space. First we note that if B is a closed ball in E, then $\|f(s)\|_B$ is measurable for any τ-weakly measurable function $f: S \to E$. See Proposition 4.4. The following definition is introduced by E. Thomas [23] and originated in the concept of totally summable sequences introduced by A. Pietsch. Let (E, τ) be a locally convex Souslin space. A τ-weakly measurable function $f: S \to E$ is said to be *τ-totally integrable* (*smmmable*) if there exists a τ-closed and τ-bounded ball B in E such that $\int_S \|f(s)\|_B \, d\mu < +\infty$. Let $\mathscr{B}$ be a bornology on E (satisfying the compatibility condition $(*)$ stated in Section 1). Let γ be the mixed topology $\gamma[\mathscr{B}, \tau]$. Then τ-weakly measurable function $f: S \to E$ is said to be *$\mathscr{B}$-integrable* if there exists a τ-closed set $B \in \mathscr{B}$ such that $\int_S \|f(s)\|_B \, d\mu < +\infty$. Therefore, if in particular $\mathscr{B}$ is the bornology defined by a norm $\|\cdot\|$ on E, i.e., $(E, \|\cdot\|, \tau)$ is a Saks space, then $\mathscr{B}$-integrability just implies $\int_S \|f(s)\| \, d\mu < +\infty$. It is easy to verify that every τ-totally integrable function $f: S \to E$ satisfies the following condition:

$$(**) \qquad \int_S p(f(s)) \, d\mu < +\infty \quad \text{for every } p \in \mathscr{P}(\tau).$$

Thus if (E, τ) is quasi-complete, then it follows from Theorem 4.2 that f is τ-Pettis integrable. According to Thomas [23], functions satisfying condition $(**)$ is said to be *τ-absolutely summable* (*integrable*).

PROPOSITION 4.14. *Let (E, τ) be a quasi-complete locally convex Souslin space. If f is τ-totally integrable, then it is integrable by τ-seminorm.*

PROOF. This is obvious from the above observation, Theorem 3.1 and Proposition 4.4.

We here recapitulate the types of integrability considered so far. Let (E, τ) be a quasi-complete locally convex Souslin space. Then for a function f: $S \to E$, each of the following conditions is more restrictive than the next: (1) f is τ-totally integrable; (2) f is integrable by τ-seminorm; (3) f is τ-absolutely integrable; (4) f is τ-Pettis integrable.

PROPOSITION 4.15. *Let $(E, \mathscr{B}, \tau)$ be a mixed space such that (E, τ) is a locally convex Souslin space. Let $f: S \to E$ be τ-weakly measurable. Then f is γ-totally integrable if and only if it is $\mathscr{B}$-integrable.*

PROOF. Let $f: S \to E$ be τ-weakly measurable. Then we see from Proposition 4.4 that f is γ-weakly measurable. Thus $\|f(s)\|_B$ is measurable for every γ-closed and γ-bounded ball B. Assume that f is γ-totally integrable. Then there exists a γ-closed and γ-bounded ball B in E such that $\int_S \|f(s)\|_B \, d\mu < +\infty$. Since B is γ-bounded, there exists a τ-closed $B' \in \mathscr{B}$ with $B \subset B'$. Hence $\|f(s)\|_{B'} \leqq \|f(s)\|_B$ for $s \in S$, and so $\int_S \|f(s)\|_{B'} \, d\mu < +\infty$. Thus f is $\mathscr{B}$-integrable. The converse is obvious since τ-closed set $\mathscr{B}$ is γ-closed and γ-bounded. q.e.d.

Finally, we give a result concerning Fubini's theorem. It is known that Fubini's theorem is not valid for Pettis integrable functions even though they take values in a separable Hilbert space. But it is seen that Fubini's theorem is valid for totally integrable functions.

Let (S, Σ, μ) be the completion of the product measure space of two complete nonnegative finite measure spaces (S_i, Σ_i, μ_i), $i = 1, 2$. Then by Theorem 4.15 and the result of Thomas [23, Theorem 8] together imply the following type of Fubini's theorem for $\mathscr{B}$-integral in mixed spaces.

THEOREM 4.16. *Let $(E, \mathscr{B}, \tau)$ be a mixed space such that (E, τ) is a locally convex Souslin space. Let $f: S \to E$ be τ-weakly measurable. If f is $\mathscr{B}$-integrable, then we have the following properties*:

(1) $s_2 \to f(s_1, s_2)$ *is $\mathscr{B}$-integrable with respect to μ_2 for almost all $s_1 \in S_1$.*

(2) $s_1 \to (\gamma)\text{-}\int_{S_2} f(s_1, s_2) \, d\mu_2(s_2)$ *is $\mathscr{B}$-integrable over $S_1 \backslash N_1$, where N_1 is the set of points excluded in* (1).

(3) $(\gamma)\text{-}\int_S f \, d\mu = (\gamma)\text{-}\int_{S_1} d\mu_1(s_1) (\gamma)\text{-}\int_{S_2} f(s_1, s_2) \, d\mu_2(s_2)$.

ACKNOWLEDGMENT. The author would like to express his thanks to Professor Shinnosuke Oharu for his valuable guidance all over this work.

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Added in Proof: After receiving the galley proofs of this paper, Professor Musial has called the attention of the paper published in Atti Sem. Mat. Fis. Univ. Modena, 35, 159–166(1987) which contains, among others, the results on Pettis integration given in Section 3. The author is grateful for his comments.