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# Scaled one-step methods with one interpolation point

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## 1. Introduction

Consider the initial value problem

(1.1) $$y' = f(x, y), \qquad y(x_0) = y_0,$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(t)$ be the solution of (1.1), let

(1.2) $$x_t = x_0 + th \qquad (h > 0,\ t > 0),$$

and denote by $y_t$ an approximation of $y(x_t)$, where $h$ is a stepsize. We are concerned with the case where the problem (1.1) is solved by one-step methods. Let

(1.3) $$y_1 = y_0 + h\textstyle\sum_{i=1}^{q} p_i k_i$$

be a one-step method of order $p$ for approximating $y(x_1)$, where

(1.4) $$k_1 = f(x_0, y_0),$$

(1.5) $$k_i = f(x_0 + a_i h,\ y_0 + h\textstyle\sum_{j=1}^{i-1} b_{ij} k_j) \qquad (i = 2, 3, \cdots, q),$$

(1.6) $$a_i = \textstyle\sum_{j=1}^{i-1} b_{ij}, \qquad a_i \neq 0 \qquad (i = 2, 3, \cdots, q),$$

$p_i$ $(i = 1, 2, \cdots, q)$, $a_i$ and $b_{ij}$ $(j = 1, 2, \cdots, i-1;\ i = 2, 3, \cdots, q)$ are constants. It is well known that the minimum of $q$ is 4, 6 for $p = 4$, 5 respectively.

Consider a method of the form

(1.7) $$y_t = y_0 + h\textstyle\sum_{i=1}^{q+r} p_{it} k_i$$

that provides an approximation of $y(x_t)$ with $r$ additional evaluations of $f$, where $p_{it}$ $(i = 1, 2, \cdots, q+r)$ are functions of $t$, $k_i$ and $b_{ij}$ $(j = 1, 2, \cdots, i-1;\ i = 2, 3, \cdots, q+r)$ satisfy (1.5) and (1.6) with $q$ replaced by $q+r$, but $a_i$ and $b_{ij}$ $(j = 1, 2, \cdots, i-1;\ i = q+1, q+2, \cdots, q+r)$ may be functions of $t$. On the basis of Fehlberg's (4)5 method with $q = 6$, Horn [1] has shown that for $r = 1$ there exists a method (1.7) which is of order 4 for any number of values of $t$ and that for $r = 2$ there exists a method (1.7) of order 5 for any specified value of $t (t \neq 1)$. We require that the methods (1.6) and (1.7) are of the same order $p$. In our previous paper [2] it has been shown that for $q = 2, 3, 4, 6$ and $r = 0, 1, 2, 3$ there exists a method (1.7) which is of order 2, 3, 4, 5 respectively for any number of values of $t$.

Let

(1.8) $$e = h\sum_{i=1}^{q} s_i k_i + h s_{q+1} \bar{k},$$

where $\bar{k} = f(x_1, y_1)$. Then in this paper it is shown that for $q=4$, 6 and for any specified value of $t(t \neq 1)$ there exist a method (1.7) with $r=1$ for which $p=4$, 5 respectively and a method (1.8) such that $e = O(h^p)$. The method $y_1 + e$ is of order $p-1$, so that the quantity $e$ can be used to control the stepsize. Finally numerical examples are presented.

## 2. Preliminaries

Let $m = q + r$ and

(2.1) $$c_i = \sum_{j=2}^{i-1} a_j b_{ij},\ d_i = \sum_{j=2}^{i-1} a_j^2 b_{ij},\ e_i = \sum_{j=2}^{i-1} a_j^3 b_{ij} \quad (i = 3, 4, \dots),$$

(2.2) $$l_i = \sum_{j=3}^{i-1} c_j b_{ij},\ m_i = \sum_{j=3}^{i-1} d_j b_{ij},\ q_i = \sum_{j=3}^{i-1} a_j c_j b_{ij} \quad (i = 4, 5, \dots).$$

Let $D$ be the differential operator defined by

(2.3) $$D = \frac{\partial}{\partial x} + k_1 \frac{\partial}{\partial y}$$

and put

(2.4) $$D^j f(x_0, y_0) = T^j, \quad D^j f_y(x_0, y_0) = S^j \quad (j = 1, 2, \dots),$$
$$(Df)^2(x_0, y_0) = P, \quad (Df_y)^2(x_0, y_0) = Q, \quad Df_{yy}(x_0, y_0) = R,$$
$$f_y(x_0, y_0) = f_y, \quad f_{yy}(x_0, y_0) = f_{yy}.$$

Then $y_t$ can be expanded into power series in $h$ as follows:

(2.5) $$y_t = y_0 + hA_1 k_1 + h^2 A_2 T + (h^3/2!)\,(A_3 T^2 + 2A_4 f_y T) + (h^4/3!)\,(B_1 T^3 + 6B_2 TS + 3B_3 f_y T^2 + 6B_4 f_y^2 T) + (h^5/4!)\,(C_1 T^4 + 12C_2 TS^2 + 12C_3 T^2 S + 12C_4 f_{yy} P + 4C_5 f_y T^3 + 12C_6 f_y^2 T^2 + 24C_7 f_y^3 T + 24C_8 f_y TS) + O(h^6),$$

where

(2.6) $$A_1 = \sum_{i=1}^{m} p_{it},\ A_2 = \sum_{i=2}^{m} a_i p_{it},$$

(2.7) $$A_3 = \sum_{i=2}^{m} a_i^2 p_{it},\ B_1 = \sum_{i=2}^{m} a_i^3 p_{it},\ C_1 = \sum_{i=2}^{m} a_i^4 p_{it},$$

(2.8) $$A_4 = \sum_{i=3}^{m} c_i p_{it},\ B_1 = \sum_{i=3}^{m} a_i c_i p_{it}, B_3 = \sum_{i=3}^{m} d_i p_{it},\ C_2 = \sum_{i=3}^{m} a_i^2 c_i p_{it},$$

$$C_3 = \sum_{i=3}^{m} a_i d_i p_{it},\ C_4 = \sum_{i=3}^{m} c_i^2 p_{it},\ C_5 = \sum_{i=3}^{m} e_i p_{it},$$

(2.9) $$B_4 = \sum_{i=4}^{m} l_i p_{it},\ C_6 = \sum_{i=4}^{m} m_i p_{it},\ C_7 = \sum_{i=5}^{m} (\sum_{j=4}^{i-1} l_j b_{ij}) p_{it},$$

$$C_8 = \sum_{i=4}^{m} (a_i l_i + g_i) p_{it}.$$

Put

(2.10) $$A_{1t} = A_1 - t,\ A_{2t} = A_2 - t^2/2,\ A_{3t} = A_3 - t^3/3,\ A_{4t} = A_4 - t^3/6,$$

(2.11) $$B_{it} = B_i - t^4/(4u_i)\ (i = 1, 2, 3, 4),\ C_{jt} = C_j - t^5/(5v_j)\ (j = 1, 2, \dots, 8),$$

where

(2.12) $$u_1 = 1,\ u_2 = 2,\ u_3 = 3,\ u_4 = 6,\ v_i = i\ (i = 1, 2, 3, 4),$$

$$v_5 = 4,\ v_6 = 12,\ v_7 = 24,\ v_8 = 24/7.$$

Then we have

(2.13) $$y_t - y(x_t) = hA_{1t}k_1 + h^2 A_{2t} T + (h^3/2)\ (A_{3t} T^2 + 2A_{4t} f_y T) + \cdots.$$

Similarly we have

(2.14) $$e = h\tilde{A}_1 k_1 + h^2 \tilde{A}_2 T + (h^3/2)\ (\tilde{A}_3 T^2 + 2\tilde{A}_4 f_y T) + \cdots.$$

Put

(2.15) $$L_{ij} = a_i \prod_{k=2}^{j} (a_i - a_k),\ M_{ij} = a_i \prod_{k=3}^{j} (a_i - a_k)\ (i > j).$$

If we impose the condition

(2.16) $$p_{2t} = 0,\ c_i = a_i^2/2,\ d_i = a_i^3/3\ (i = 3, 4, \dots, m),$$

then we have

(2.17) $$2A_{4t} = A_{3t},\ 2B_{2t} = 3B_{3t} = B_{1t},\ 2C_{2t} = 3C_{3t} = 4C_{4t} = C_{1t},$$

(2.18) $$3a_2 = 2a_3,$$

(2.19) $$a_3^2 b_{i3} + 3\sum_{j=4}^{i-1} a_j\ (a_j - a_2)\ b_{ij} = a_i^2\ (a_i - a_3)\ (i = 4, 5, \dots, m).$$

Put

(2.20) $$X_1 = a_2 + a_3,\ Y_1 = a_2 a_3,\ X = a_3 + a_4,\ Y = a_3 a_4,\ U = a_5 + X,$$

$$V = a_5 X + Y,\ W = a_5 Y,$$

(2.21) $$P_{ik} = \sum_{j=k+1}^{i-1} M_{jk} b_{ij}\ (i \geqq k+2),\ Q_{ik} = \sum_{j=k+2}^{i-1} P_{jk} b_{ij}\ (i \geqq k+3),$$

(2.22) $$P_{i3} = \sum_{j=4}^{i-1} M_{ij} E_j\ (i \geqq 5),\ P_{i4} = \sum_{j=5}^{i-1} M_{ij} F_j\ (i \geqq 6),$$

$$P_{i5} = \sum_{j=6}^{i-1} M_{ij} G_j\ (i \geqq 7).$$

## 3. Construction of the methods

We shall show the following

THEOREM. *For $q=4$, 6 and any $t>0$ $(t\neq 1)$ there exist a method* (1.7) *with $r=1$ for which $p=4$, 5 respectively and a formula* (1.8) *such that $e=O(h^p)$.*

### 3.1. Case $q=4$

The condition $A_{it}=B_{it}=0$ $(i=1, 2, 3, 4)$ yields

(3.1) $\sum_{i=1}^{5} p_{it}=t,\ 2\sum_{i=2}^{5} a_i p_{it}=t^2,\ 6\sum_{i=3}^{5} c_i p_{it}=t^3,$

$24\sum_{i=4}^{5} l_i p_{it}=t^4,$

(3.2) $\sum_{i=3}^{5} L_{i2} p_{it}=r_1(t),\ \sum_{i=4}^{5} L_{i3} p_{it}=r_2(t),\ \sum_{i=4}^{5} (a_i-a_3)c_i p_{it}=r_3(t),$

$\sum_{i=4}^{5} (\sum_{j=3}^{i-1} L_{j2} b_{ij}) p_{it}=r_4(t),$

where

(3.3) $6r_1(t)=t^2(2t-3a_2),\ 12r_2(t)=t^2(3t^2-4X_1 t+6Y_1),$

$24r_3(t)=t^3(3t-4a_3),\ 12r_4(t)=t^3(t-2a_2).$

The choice $t=1$ and $p_{51}=0$ leads to

(3.4) $c_3 b_{43}\neq 0,\ a_4=1,\ a_3\neq 1,\ L_{32}=2(1-2a_3)c_3,$

$(1-a_3)c_4=(3-4a_3)c_3 b_{43},\ L_{43}=2(3-4X_1+6Y_1)c_3 b_{43}.$

Using (3.4) and (3.1), we have from (3.2)

(3.5) $24c_3 b_{43} b_{54} p_{5t}=t^3(t-1),$

(3.6) $6K_1 p_{5t}=a_2 t^2(t-1)\ (3-t),\ 6K_2 p_{5t}=a_3 t^3(t-1),$

$3K_3 p_{5t}=t^3(t-1)\ (X_1-2Y_1),$

where

(3.7) $K_1=L_{52}-2(1-2a_2)c_5-4a_2 l_5,\ K_2=(a_5-a_3)c_5-(3-4a_3)l_5,$

$K_3=a_5 L_{52}-2a_3(1-2a_2)c_5-2(3-4X_1+8Y_1)l_5.$

Elimination of $p_{5t}$ from (3.6) yields

(3.8) $a_3 K_3=2(X_1-2Y_1)K_2,\ t a_3 K_1=(3-t)a_2 K_2,\ K_2=4a_3 c_3 b_{43} b_{54}.$

Put

(3.9) $d=a_5(3-4X_1+8Y_1)-2Y_1-t(a_5+2-4X_1+6Y_1).$

Then from (3.8) we have

(3.10) $6dc_3b_{43}b_{54} = -tL_{54},\ 6dl_5 = L_{53}[(3-t)a_5-2t],$

$2dc_5 = L_{52}[a_5(3-t-4a_3)-2t(1-2a_3)].$

Hence for $a_2$, $a_3$ and $a_5$ that satisfy (3.4) and the condition

(3.11) $L_{54} \neq 0,\ d \neq 0$ for all $t \geqq 0$,

the quantities $l_5$, $c_5$, $b_{54}$ and $p_{5j}$ are determined as functions of $t$; $p_{it}$ $(i=1, 2, 3, 4)$ are obtained from (3.1); $b_{5j}$ $(j=1, 2, 3, 4)$ are determined from $l_5$, $c_5$ and (1.6).

The choice $\tilde{A}_i=0$ $(i=1, 2, 3, 4)$ yields

(3.12) $\sum_{i=1}^{5} s_i=0,\ 2\sum_{i=3}^{4} c_i s_i + s_5 = 0,\ 2L_{43}s_4 + (3-4X_1+6Y_1)s_5 = 0,$

(3.13) $e = (h^4/4!)\ [B_1^* T^3 + 3B_2^* TS + B_3^* f_y T^2 + B_4^* f_y^2 T] + O(h^5),$

where

(3.14) $B_1^* = 2(1-2a_2)\ (2a_3-1)s_5, B_2^* = 2(2a_3-1)s_5,$

$B_3^* = 2(3a_2-1)s_5,\ B_4^* = -2s_5.$

Example 1. For the choice $a_2=1/3$, $a_3=2/3$, $a_5=7/12$ and $s_5=1/6$ we have

(3.15) $b_{21} = 1/3,\ b_{31} = -1/3,\ b_{32} = 1,\ b_{41} = -b_{42} = b_{43} = 1,$

(3.16) $b_{51} = (444t-409)b(t)/3,\ b_{52} = 5(42-29t)b(t),\ b_{53} = 7(14t-9)b(t),$

$b_{54} = -5tb(t),$

(3.17) $8p_{1t} = -9t^4 + 24t^3 - 22t^2 + 8t + p(t)/7, 8p_{2t} = 27t^4 - 60t^3 + 36t^2 - p(t),$

$8p_{3t} = -27t^4 + 48t^3 - 18t^2 - 3p(t),\ 8p_{4t} = 9t^4 - 12t^3 + 4t^2 + p(t)/5,$

$35p_{5t} = 16p(t),$

(3.18) $y_1 = y_0 + h(k_1+3k_2+3k_3+4k_4)/8,$

$e = h(-k_1+3k_2-3k_3-3k_4+4\bar{k})/24,$

where

(3.19) $128(1+9t)b(t) = 7,\ p(t) = t^2(1-t)\ (1+9t).$

Example 2. For the choice $a_2=2/5$, $a_3=3/5$, $a_5=14/25$ and $s_5=1/6$ we have

(3.20) $b_{21} = 2/5, b_{31} = -3/20, b_{32}=3/4, b_{41}=19/44, b_{42} = -15/44, b_{43} = 10/11,$

(3.21) $b_{51} = 14(2471t-2460)/61875,\ b_{52} = 14(1071-631t)/12375,$

$b_{53} = 98(23t-12)/12375,\ b_{54} = -154t/5625,$

(3.22) $72p_{1t} = -75t^4 + 200t^3 - 186t^2 + 72t + 33p(t)/7,\ 72p_{2t} = 375t^4 - 800t^3$

$$+450t^2 - 165p(t)/2,\ 72p_{3t} = -375t^4 + 700t^3 - 300t^2 - 330p(t),$$
$$72p_{4t} = 75t^4 - 100t^3 + 36t^2 + 6p(t),\ 112p_{5t} = 625p(t),$$

(3.23) $$y_1 = y_0 + h(11k_1 + 25k_2 + 25k_3 + 11k_4)/72,$$
$$e = h(-k_1 + 5k_2 - 5k_3 - 11k_4 + 12\tilde{k})/72,$$

(3.24) $$B_1^* = 1/75,\ B_2^* = B_3^* = 1/15,\ B_4^* = -1/3,$$

where

$$p(t) = t^2(1-t).$$

### 3.2. Case $q=6$

We impose the condition (2.16) and assume that $a_i$ ($i=3, 4, 5, 6, 7$) are all distinct. The condition $A_{it}=B_{it}=0$ ($i=1, 2, 3, 4$) and $C_{jt}=0$ ($j=1, 2, \cdots, 8$) yields

(3.25) $$\textstyle\sum_{i=1}^{7} p_{it} = t,\ 2\sum_{i=3}^{7} a_i p_{it} = t^2,\ \sum_{i=4}^{7} M_{i3} p_{it} = r_1(t),$$
$$\textstyle\sum_{i=5}^{7} M_{i4} p_{it} = r_2(t),\ \sum_{i=6}^{7} M_{i5} p_{it} = r_3(t),$$

(3.26) $$M_{76}E_6 p_{7t} = r_4(t),\ M_{76}F_6 p_{7t} = r_5(t),\ (E_5 P_{75} + E_6 P_{76}) p_{7t} = r_6(t),$$
$$[E_5 + (a_7 - a_5)E_6] M_{76} p_{7t} = r_9(t),$$

where

(3.27) $$6r_1(t) = t^2(2t - 3a_3),\ 12r_2(t) = t^2(3t^2 - 4Xt + 6Y),$$
$$60r_3(t) = t^2(12t^3 - 15Ut^2 + 20Vt - 30W),$$
$$r_4(t) = t^3(t - 2a_3)/12 - E_4 r_2(t) - E_5 r_3(t),\ r_5(t) = r_8(t) - F_5 r_3(t),$$
$$r_6(t) = t^4(2t - 5a_3)/120 - E_4 r_8(t),\ 60r_8(t) = t^3(3t^2 - 5Xt + 10Y),$$
$$r_7(t) = t^3[8t^2 - 5(3a_3 + 2a_5)t + 20a_3a_5]/120 - [E_4 + (a_6 - a_5)E_5] r_3(t).$$

The choice $t=1$ and $p_{71}=0$ leads to the condition

(3.28) $$r_8(1) = F_5 r_3(1),\ 2 - 5a_3 = 120E_4 r_8(1),\ 1 - 2a_3 = 12E_4 r_2(1) - 12E_5 r_3(1),$$

(3.29) $$(a_6 - 1)\ [2a_4(5a_3^2 - 4a_3 + 1) - a_3] = 0.$$

Hence we choose $a_6 = 1$ and $a_i$ ($i=3, 4, 5$) so that $r_3(1) \neq 0$ and $r_8(1) \neq 0$. Then $E_4$, $E_5$ and $F_5$ are determined from (3.28).

Using (3.28), we have from (3.26)

(3.30) $$E_5 M_{76} p_{7t} = p(t) q_1(t),\ G_6 E_5 M_{76} p_{7t} = p(t) q_2(t),\ F_6 M_{76} p_{7t} = p(t) q_3(t),$$
$$E_6 M_{76} p_{7t} = p(t) q_4(t),$$

where

(3.31) $p(t)=t^2(t-1),\ q_i(t)=P_i t^2+Q_i t+R_i \quad (i=1,2,3,4),$

$15P_1=1-3E_4+3(a_7-a_6)E_5,\ 24Q_1=24P_1-3a_3-2a_7+6E_4(X+a_7)$
$-6(a_7-a_6)E_5U,$

$6R_1=6Q_1+a_3a_7-2E_4(a_7X+Y)+2(a_7-a_6)E_5V,\ P_2=0,$

$60Q_2=1-3E_4,\ 24R_2=24Q_2-a_3+2E_4X,\ 20P_3=1-4F_5,$

$12Q_3=12P_3-X+3F_5U,\ 5P_4=-E_5,\ 12Q_4=12P_4+1-3E_4+3E_5U,$

$6R_4=6Q_4-a_3+2E_4X-2E_5V,\ 6R_3=6Q_3+Y-2F_5V.$

Hence if we choose $a_3$, $a_4$, $a_5$ and $a_7$ so that

(3.32) $r_3(1)\neq 0,\ r_8(1)\neq 0,\ E_5\neq 0,\ q_1(t)\neq 0 \quad$ for all $\quad t\geqq 0,$

then $p_{7t}$, $G_6$, $F_6$ and $E_6$ are determined from (3.30); $p_{it}$ $(i=1,3,4,5,6)$ are obtained from (3.25); $b_{ij}$ $(j=4,5,\dots,i-1;\ i=5,6,7)$ are determined from (2.21) and (2.22); $b_{i3}$ $(i=4,5,\dots,7)$ are obtained from (2.19); $b_{j2}$ $(j=3,4,\dots,7)$ are determined from (2.16); $b_{k1}$ $(k=2,3,\dots,7)$ are obtained from (1.6).

Choosing $\tilde{A}_i=\tilde{B}_i=0$ $(i=1,2,3,4)$ and $s_2=0$, we have

(3.33) $6M_{65}E_5s_6=(6M_{64}E_4+3a_3-2)s_7,\ \sum_{i=5}^{6}M_{i4}s_i+M_{64}s_7=0,$

$\sum_{i=4}^{6}M_{i3}s_i+(1-a_3)s_7=0,\ \sum_{i=3}^{6}a_is_i+s_7=0,\ s_1+\sum_{i=3}^{7}s_i=0,$

(3.34) $\tilde{C}_1=M_{65}(s_6+s_7),\ \tilde{C}_5=3\tilde{C}_6=F_5M_{65}s_6+(3-4X+6Y)s_7/12,$

$2\tilde{C}_7=(F_5M_{65}-2M_{54}E_4b_{65})s_6+(1-2a_4+3a_3a_4)s_7/12,$

$24\tilde{C}_8=12[q-2E_4-(a_6-a_5)E_5]M_{65}s_6+[7-12X(1-a_3)-$
$12a_5(1-a_3)(1-a_4)-2(a_4-2a_3-2a_5)(2-3a_3)]s_7,$

(3.35) $e=(h^5/5!)\ [C_1^*(T^4+6TS^2+4T^2S+3f_{yy}P)+C_5^*(f_yT^3+f_y^2T^2)$
$+C_7^*f_y^3T+C_8^*f_yTS]+O(h^6),$

where

(3.36) $C_1^*=5\tilde{C}_1,\ C_5^*=20\tilde{C}_5,\ C_7^*=120\tilde{C}_7,\ C_8^*=120\tilde{C}_8.$

EXAMPLE 3. For the choice $a_3=1/4$, $a_4=1/2$, $a_5=3/4$, $a_6=1$, $a_7=19/44$ and $s_7=1/6$ we have

(3.37) $a_2=b_{21}=1/6,\ b_{31}=1/16,\ b_{32}=3/16,\ b_{41}=1/4,\ b_{42}=-3/4,$

$b_{43}=1,\ b_{51}=3/16,\ b_{52}=b_{53}=0,\ b_{54}=9/16,\ b_{61}=-4/7,\ b_{62}=3/7,$

$b_{63}=-b_{64}=12/7,\ b_{65}=8/7.$

(3.38) $2b_{71}=(231-623t)b(t)+76893/117128,\ 4b_{72}=10395tb(t)$
$-917301/58564,$

$b_{73}=(228t^2-3059t-255)b(t)+76456/14641,$

$4b_{74}=-3(1088t^2-1253t+3)b(t)-37449/14641,$

$b_{75}=(300t^2-119t-57)b(t)+304/14641,$

$4b_{76}=49(3-4t)b(t),$

(3.39) $90p_{1t}=192t^5-600t^4+700t^3-375t^2+90t-24p(t)/19,\ p_{2t}=0,$

$45p_{3t}=8t^2(-48t^3+135t^2-130t+45)+6p(t),$

$15p_{4t}=192t^5-480t^4+380t^3-90t^2+8p(t),$

$45p_{5t}=8t^2(-48t^3+105t^2-70t+15)-24p(t)/7,$

$90p_{6t}=192t^5-360t^4+220t^3-45t^2+24p(t)/25,$

$p_{7t}=-29282p(t)/49875,$

(3.40) $y_1=y_0+h(7k_1+32k_3+12k_4+32k_5+7k_6)/90,$

$e=h(-4k_1+16k_3-24k_4+16k_5-49k_6+45\bar{k})/270,$

(3.41) $C_1^*=-1/144,\ C_5^*=-13/72,\ C_7^*=43/48,\ C_8^*=2/3,$

where

(3.42) $14641(9+16t^2)b(t)=475,\ p(t)=t^2(t-1)\ (9+16t^2).$

## 4. Numerical examples

The following six problems are solved by the methods in Examples 1, 2 and 3 with $h=1/2$.

Problem 1. $y'=y,\ y(0)=1.$
Problem 2. $y'=2xy,\ y(0)=1.$
Problem 3. $y'=-y^2,\ y(0)=1.$
Problem 4. $y'=1-y^2,\ y(0)=0.$
Problem 5. $y'=-5y,\ y(0)=1.$
Problem 6. $y'=y-2x/y,\ y(0)=1.$

The errors $e_t=y_t-y(x_t)$ $(t=1/2,\ 1)$ are listed in Table 1.

Table 1.

| Prob \ Ex | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|
| | $e_{1/2}$ | $e_1$ | $e_{1/2}$ | $e_1$ | $e_{1/2}$ | $e_1$ |
| 1 | −8.06E−4 | −2.84E−4 | −8.92E−5 | −2.84E−4 | 5.08E−5 | 1.06E−6 |
| 2 | −3.09E−4 | 6.97E−4 | 1.46E−4 | 3.49E−4 | 8.57E−6 | 4.88E−5 |
| 3 | −5.21E−3 | −1.63E−3 | −1.09E−3 | −5.80E−4 | −3.53E−4 | 1.70E−5 |
| 4 | −1.23E−4 | 1.51E−4 | 1.47E−5 | −3.01E−5 | 8.07E−6 | −1.52E−5 |
| 5 | −1.72E−1 | 5.66E−1 | 2.75E−1 | 5.66E−1 | −7.17E−1 | 1.34E−1 |
| 6 | 2.86E−4 | 2.83E−4 | 2.68E−4 | 7.88E−4 | −3.80E−5 | 2.05E−5 |

## References

[1] M. K. Horn, *Fourth- and fifth-order, scaled Runge-Kutta algorithms for treating dense output,* SIAM J. Numer. Anal., **20** (1983), 558–568.

[2] H. Shintani, *On scaled one-step methods,* Hiroshima Math. J., **18** (1988), 113–126.

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