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On the products $\beta_s\beta_t$ in the stable homotopy groups of spheres

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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§1. Introduction

For a prime $p \geqq 5$, H. Toda [7] introduced the β-family $\{\beta_s | s \geqq 1\}$ and showed the relation $uv\beta_s\beta_t = st\beta_u\beta_v$ $(s + t = u + v)$ in the p-component of the stable homotopy groups $\pi_*(S)$ of spheres. An easy consequence of this relation is $\beta_s\beta_t = 0$ if $p | st$, since the order of β_s is p. In this paper we find the following

THEOREM 1.1. *Let s and t be positive integers with $p \nmid st$. Then,*

$$\beta_s\beta_t \neq 0 \ \ in \ \ \pi_*(S) \qquad \text{if } s + t \in I,$$

where $I = \{kp^i - (p^{i-1} - 1)/(p - 1) | i \geqq 1, \ p \nmid k + 1\}$.

Consider the Adams-Novikov spectral sequence converging to $\pi_*(S)$, in which Miller, Revenel, and Wilson [1] defined the β-elements β_s $(s \geqq 1)$ surviving to β_s in $\pi_*(S)$. This sequence has sparsity in its E_2-term enough not to kill the product $\beta_s\beta_t$. Therefore the above theorem follows from the non-triviality in the following

THEOREM 1.2. *Let s and t be positive integers with $p \nmid st$. Then, in the E_2-term of the Adams-Novikov spectral sequence,*

$$\beta_s\beta_t \neq 0 \ \ if \ \ s + t \in I.$$

Furthermore suppose that $s + t \geqq p^2 + p + 2$. Then we have

$$\beta_s\beta_t = 0 \ \ if \ \ p \nmid (s + t)(s + t + 1), \ \ or \ \ if \ \ s + t + 1 = kp \ \ and \ \ p \nmid k(k + 1).$$

Notice that $p | n(n + 1)$ if $n \in I$. We also note that the relation "$\beta_s\beta_t = 0$ if $p | st$" is also valid in the E_2-term ([2], [6; Cor. 2.8]), and that $\beta_s\beta_t = 0$ if and only if $p | st$ in both $\pi_*(S)$ and the E_2-term for the case when $p = 5$ and $s + t \leqq p^2 - p + 1$ ([3; Chap. 7]).

This theorem does not determine whether or not $\beta_s\beta_t$ $(p \nmid st)$ is trivial in the E_2-term for the following cases:

a) $\quad p^2 | s + t + p$, b) $s + t = kp^3 - p^2 - 1$, and c) $s + t = kp^2 - p - 1 \notin I$.

In §2, we recall the Brown-Peterson spectrum BP at p and the E_2-term

H^*BP_* of the above spectral sequence; and give the elements $B_s\beta_t$ in $H^2N_0^2$ mapped to $\beta_s\beta_t$ in H^4BP_* by the Greek letter map G: $H^2N_0^2 \to H^4BP_*$. Then the triviality in Theorem 1. 2 is proved in Theorem 2. 9 by noticing $uv\beta_s\beta_t = st\beta_u\beta_v (s+t=u+v)$ in H^4BP_* and by showing $B_u\beta_v=0$ for $u=1, 2$. Furthermore G is an isomorphism (see Lemma 3.3); and the non-triviality in Theorem 1. 2 is proved for the case $p|s+t$ in §3 by mapping $B_1\beta_v$ to $H^3M_1^1$ whose structure is given in [4], and for the case $p|s+t+1$ in §4 after determining the structure of $H^1M_0^2$ at the corresponding degree.

§2. Triviality in the E_2-term

Throughout the paper p denotes a prime $\geqq 5$. Let BP be the Brown-Peterson spectrum at the prime p. Then the coefficient ring BP_* and the BP_*-homology BP_*BP are the polynomial rings

$$A = BP_* = Z_{(p)}[v_1, v_2, \cdots] \text{ and } \Gamma = BP_*BP = BP_*[t_1, t_2, \cdots],$$

where $|v_i| = e(i) = |t_i|$. Here we use the notation

$$(2.1) \qquad |x| = (\deg x)/(2p-2) \text{ and } e(i) = (p^i-1)/(p-1).$$

The pair $(A, \Gamma) = (BP_*, BP_*BP)$ is the Hopf algebroid (cf. [3]), and we use here the following formulae for the right unit $\eta: A \to \Gamma$ and the coproduct Δ: $\Gamma \to \Gamma \otimes_A \Gamma$.

$$(2.2) \qquad \eta v_1 = v_1 + pt_1, \quad \eta v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod(p).$$

$$\eta v_3 \equiv v_3 + v_2 t_1^{p^2} - v_2^p t_1 + v_1 t_2^p + v_1^2 V \mod(p, v_1^p) \text{ and}$$

$$\eta v_4 \equiv v_4 + v_3 t_1^{p^3} + v_2 t_2^{p^2} - t_1 \eta v_3^p - t_2 v_2^{p^2} \mod(p, v_1); \text{ and}$$

$$\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1 \text{ and } \Delta t_2 = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 + v_1 T.$$

Here $pv_1 V = v_1^p t_1^{p^2} - v_1^{p^2} t_1^p + v_2^p - \eta v_2^p$ and $pT = t_1^p \otimes 1 + 1 \otimes t_1^p - \Delta t_1^p$. For a Γ-comodule M with coaction ψ, we define the homology H^*M as the homology of the codar complex

$$\Omega^k M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (k \text{ copies of } \Gamma)$$

with the differential d_k: $\Omega^k M \to \Omega^{k+1} M$ given inductively by

$$(2.3) \qquad d_0 m = \psi m - m, \; d_1 x = x \otimes 1 - \Delta x + 1 \otimes x,$$

$$d_1 m \otimes x = d_0 m \otimes x + m \otimes d_1 x \text{ and}$$

$$d_{k+1} m \otimes x \otimes y = d_1 m \otimes x \otimes y - m \otimes x \otimes d_k y$$

for $m \in M$, $x \in \Gamma$ and $y \in \Omega^k M$ $(k \geqq 1)$.

Consider the Γ-comodules N_n^i and M_n^i with the coaction η induced from the right unit η of Γ defined inductively by the equalities

$N_n^0 = A/(p, v_1, \cdots, v_{n-1})(N_0^0 = A)$, $M_n^i = v_{n+i}^{-1}N_n^i$ and the exact sequence

$$0 \longrightarrow N_n^i \xrightarrow[\subset]{\lambda} M_n^i \longrightarrow N_n^{i+1} \longrightarrow 0.$$

Then we have the long exact sequence

$$\text{(2.4)} \qquad \cdots \longrightarrow H^k N_n^i \xrightarrow{\lambda} H^k M_n^i \longrightarrow H^k N_n^{i+1} \xrightarrow{\delta_k} H^{k+1} N_n^i \longrightarrow \cdots$$

for each i, k and n, and the Greek letter map

$$G = \delta_{k+i-1} \cdots \delta_{k+1}\delta_k\colon\ H^k N_0^i \longrightarrow H^{k+i}N_0^0 = H^{k+i}A,$$

whose range is the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. As usual, we write an element of M_n^i as a summation of fractions x/v for $x \in v_{n+i}^{-1}N_n^0$ and $v = a_n a_{n+1} \cdots a_{n+i-1}$ ($a_j = v_j^{e_j}$ for $n \leqq j < n+i$ and $e_j > 0$) with a convention that $x/v = 0$ if $a_j | x$ for some j. Hereafter, $v_0 = p$. With a calculation by (2. 2–3), we find the element

$$B_s = v_2^s/pv_1 \ \text{ in } \ H^0 N_0^2,$$

and define

$$\beta_s = GB_s \in H^2 A,$$

which is the β-elements given in [1].

From here on we study the product $\beta_s\beta_t$ in the E_2-term H^4A.

Lemma 2.5 [2; Lemma 4.4]. *The element β_s is represented by*

$$\beta_s \equiv \binom{s}{2}K_{s-2} + sT_{s-1} \ \operatorname{mod}(p, v_1) \ \textit{in } \Omega^2 A,$$

where $K_i = v_2^i(2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p})$ and $T_i = v_2^i T$.

Lemma 2.6 [2; Remark after Prop. 6. 1]. *In H^4A, we have*

$$uv\beta_s\beta_t = st\beta_u\beta_v \ \textit{ if } s + t = u + v.$$

We notice that H^*M is an H^*A-module for a comodule M, and that each map of (2.4) is an H^*A-module map. Thus we have

$$\beta_s\beta_t = GB_s\beta_t \in H^4A \ \text{ for } B_s\beta_t \in H^2N_0^2.$$

Lemma 2.7. *Let u be a positive integer. Then in $H^2N_0^2$,*

$$\text{(2.7.1)} \qquad uK_{u-1}/pv_1 = -2T_u/pv_1, \ \textit{and}$$

$$\text{(2.7.2)} \qquad B_1\beta_u = T_u/pv_1 \ \textit{ if } p \nmid u \ \textit{ and } B_2\beta_{u-1} = K_{u-1}/pv_1 \ \textit{ if } p | u.$$

Proof. By (2.2–3), we compute

(2.7.3) $$d_1H/p^2v_1^p = (uK_{u-1} + 2T_u)/pv_1,$$

where $$H = v_1^{p-1}v_2^u t_1^p - pv_1^{p-2}v_2^u t_2 - puv_1^{p-1}v_2^{u-1}t_1^p t_2.$$

Thus we have (2.7.1). (2.7.2) follows from (2.7.1) since $B_1\beta_u = (\binom{u}{2}K_{u-1} + uT_u)/pv_1$ and $B_2\beta_{up-1} = (K_{up-1} - T_{up})/pv_1$ by Lemma 2.5. q.e.d.

LEMMA 2.8. *Let u be an integer such that $u \geqq p^2 + p + 1$. Then,*

$$T_u/pv_1 = K_{u-1}/pv_1 = 0 \text{ in } H^2N_0^2$$

if $p \not| (u+1)(u+2)$, or if $u + 2 = kp$ and $p \not| k(k+1)$.

PROOF. Put $D = v_2^{u-1}t_1^p t_2 + v_2^{u-p^2-p-1}(v_2 t_2^p \eta v_3^p + v_2^p t_1^p \eta v_4 - t_1^p \eta v_3^{p+1})$, $E = v_3 t_1^{p^3} + v_2 t_2^{p^2} - t_1 \eta v_3^p - v_2^{p^2} t_2$ and $F = v_1 v_2^u \zeta^p t_1^p + (u+1)^{-1}v_2^{u+1}\zeta^p$ for the element $\zeta = v_2^{-1}t_2 - v_2^{-1}t_1^{p+1} + v_2^{-p}t_2^p - v_2^{-p-1}t_1^p \eta v_3$. Then D/pv_1, E/pv_1^2 and F/pv_1^2 belong to $\Omega^1N_0^2$ by the assumption on u. By (2.3), we have $(d_1 x\eta v) = d_1 x(1 \otimes \eta v) - x \otimes d_0 v$ for $x \in \Omega^1 M$ and $v \in \Omega^0 M$ (M is a Γ-comodule). Using this and the equalities in (2.2–3), we compute

(2.8.1) $$d_1D/pv_1 = (v_2^u\zeta \otimes t_1^p - K_{u-1} - v_2^{u-p^2}g^p)/pv_1,\ d_1E/pv_1^2 = (g^p + v_2^{p^2}T)/pv_1,$$
$$d_1\zeta^p/pv_1^2 = 0 \text{ and } d_1F/pv_1^2 = -v_2^u\zeta^p \otimes t_1^p/pv_1,$$

where $g = t_1 \otimes t_2^p + t_2 \otimes t_1^{p^2}$. We also have the element $U = v_3^{p+1} - v_2^p v_4 \in A$ such that $d_1U/pv_1 = v_2^{p^2+p+1}(\zeta - \zeta^p)/pv_1$ (cf. [1; (3.20)]). Now consider the element

$$C_u = (v_1D + v_2^{u-p^2}E + F - v_1v_2^{u-p^2-p-1}Ut_1^p + uv_1v_2^{u-p^2-1}t_1^p\eta v_4)$$

of $\Omega^2N_0^2$, and we have

(2.8.2) $$d_1C_u/pv_1^2 = T_u/pv_1 - K_{u-1}/pv_1 \in \Omega^2N_0^2.$$

Thus $T_u/pv_1 = K_{u-1}/pv_1$ in $H^2N_0^2$ if $p \not| (u+1)$, which is trivial if $p \not| (u+2)$ by (2.7.1).

For the case $u = kp - 2$, we further consider the elements

$$X_1 = v_2^{kp-2p}(t_1^{p^2}\eta v_3 - 2^{-1}v_2 t_1^{2p^2} - v_1 t_1^{p^2} t_2^p) \text{ and}$$
$$X_2 = v_2^{kp-2p}((t_1^{p^2+p} - t_2^p)\eta v_2^{p-1} - 2^{-1}v_1v_2^{p-3}v_3t_1^{2p} + \zeta^p\eta v_2^{2p-1} + v_1v_2^{-p^2+p-3}Ut_1^p)$$

and recall [6; Lemma 2.6] that the element V satisfies

$$V \equiv -v_2^{p-1}t_1^p + 2^{-1}v_1v_2^{p-2}t_1^{2p} \bmod (p, v_1^2) \text{ and}$$
$$d_1v_2^{sp}V/pv_1^{p+2} = T_s^p/pv_1^3 + sv_2^{sp-p}t_1^{p^2} \otimes V/pv_1^2.$$

Then we obtain

$$d_1X_1/pv_1^4 = v_2^{kp-p}t_1^{p^2} \otimes t_1/pv_1^4 + 2^{-1}K_{k-2}^p/pv_1^3 - v_2^{kp-2p}t_1^{p^2} \otimes V/pv_1^2 \text{ and}$$
$$d_1X_2/pv_1^2 = v_2^{kp-2p}t_1^{p^2} \otimes V/pv_1^2 + 2^{-1}K_{kp-3}/pv_1$$

in the same way as we compute above. Therefore,

$$d_1Y/pv_1^{p+4} = -2^{-1}\binom{k+1}{2}K_{kp-3}/pv_1$$

for $Y = v_2^{kp}t_1 - kv_1^pX_1 - 2^{-1}kv_1C_{k-1}^p + 2^{-1}kv_1^2v_2^{kp-p}V - \binom{k+1}{2}v_1^{p+2}X_2$, which implies $K_{u-1}/pv_1 = 0$ in $H^2N_0^2$ if $p \not| k(k+1)$. Thus the lemma is also valid for this case. q.e.d.

THEOREM 2.9. *In the E_2-term H^4A, we have*

$\beta_s\beta_t = 0$ *if* $p|st$, *if* $p \not| (s+t)(s+t+1)$ *and* $s+t \geqq p^2+p+2$,
or if $s+t+1 = kp$, $p \not| k(k+1)$ *and* $s+t \geqq p^2+p+2$.

PROOF. Lemma 2.6 implies the triviality for the case $p|st$, and the equalities $u\beta_s\beta_t = st\beta_1\beta_u$ and $2(u-1)\beta_s\beta_t = st\beta_2\beta_{u-1}$ for $u = s+t-1$. Let $u \geqq p^2+p+1$ and suppose that $p \not| u(u+1)(u+2)$, or that $u+2 = kp$ and $p \not| k(k+1)$. Then $B_1\beta_u = 0$ in $H^2N_0^2$ by Lemmas 2.7–8, and so $\beta_s\beta_t = stu^{-1}\beta_1\beta_u = 0$. In case $p|u$, the triviality similarly follows from the equality $B_2\beta_{u-1} = 0$ shown by Lemmas 2.7–8. q.e.d.

§3. Non-triviality for the case $p | s+t$

In §§3–4, we study the element

$\beta_s\beta_t$ in H^4A for $s, t \geqq 1$ with $p|(u+1)(u+2)$, where $u = s+t-1$.

In this section we assume that $p|u+1$ and prove the non-triviality of $\beta_s\beta_t$ by showing that $\delta\lambda B_1\beta_u \neq 0$ in $H^3M_1^1$. Here λ: $H^2N_0^2 \to H^2M_0^2$ is the localization map in §2, and δ: $H^2M_0^2 \to H^3M_1^1$ is the boundary homomorphism associated to the short exact sequence

(3.1) $$0 \longrightarrow M_1^1 \xrightarrow{f} M_0^2 \xrightarrow{p} M_0^2 \to 0 \qquad (fx = x/p).$$

LEMMA 3.2. $\lambda B_1\beta_u = -v_2^ut_1^p \otimes \zeta/pv_1$ *in* $H^2M_0^2$.

PROOF. Note that (2.8.1) is also valid in $\Omega^2M_0^2$ for the case $u < p^2 + p + 1$. Then we obtain

(3.2.1) $$d_1Z/pv_1^2 = (-v_2^ut_1^p \otimes \zeta - K_{u-1} + T_u)/pv_1,$$

for $Z = v_1D + v_2^{u-p^2}E + uv_1v_2^{u-p^2-1}t_1^p\eta v_4 + v_1v_2^u\zeta t_1^p$. Now apply Lemma 2. 7 to get the lemma. q.e.d.

Since $H^kM_0^0 = 0 = H^kM_0^1$ for $k \geqq 2$ by [1; Th. 3.16, Th. 4.2], the exact sequences (2.4) for $(k, n, i) = (3, 0, 0)$, $(2, 0, 1)$ imply the following

LEMMA 3.3. *The Greek letter map* G: $H^2N_0^2 \to H^4A$ *is an isomorphism.*

PROPOSITION 3.4. *In the* E_2*-term* H^4A, *we have the non-triviality*

$$\beta_s\beta_t \neq 0 \text{ if } p \not| st,\ p|s+t \text{ and } p^2 \not| s+t+p.$$

PROOF. Note first that $\zeta/v_1 = \zeta^{p^i}/v_1$ $(i \geqq 0)$ in $H^1M_1^1$ (cf. [1; Lemma 3. 19]) and the following:

(3.4.1)[5; Lemma 2.6] *There exists an element* ζ' *of* $v_2^{-1}\Gamma/(p^2, v_1^p)$ *such that*

$$d_1\zeta'/p^2v_1^p = 0 \text{ and } \zeta'/pv_1 = \zeta^{p^2}/pv_1 \text{ in } \Omega^*M_0^2.$$

We also have the relations $v_2^ut_1^p \otimes \zeta \otimes \zeta/v_1 = 0$ and $T_u/v_1 = -v_2^{u+1}g_1/v_1$ in $H^*M_1^1$. In fact, these are given by $2^{-1}d_2v_2^ut_1^p \otimes \zeta^2/v_1$ and $(d_1v_2^{u-p^2}E + uv_1v_2^{u-p^2-1}t_1^p\eta v_4)/v_1^2$. Then by the definition of δ, (2.7.3), (3.2.1) and Lemma 3.2,

$$\begin{aligned}\delta\lambda B_1\beta_u &= f^{-1}(d_2(-H + pv_1^{p-2}Z) \otimes \zeta'/p^2v_1^p)\\ &= v_2^{u+1}g_1 \otimes \zeta/v_1,\end{aligned}$$

which equals the generator $x_1^aG_1 \otimes \zeta^{(2)}/v_1$ $(ap = u+1)$ of $H^3M_1^1$ if $p \not| a+1$ by [4;Th.4.4]. Thus we see that $B_1\beta_u \neq 0$ and so is $\beta_1\beta_u$ by Lemma 3.3. Hence we have the proposition by Lemma 2.6. q.e.d.

§4. Non-triviality for the case $p|s+t+1$

The integer u also denotes $s+t-1 \geqq 1$ here, and is supposed to be $p|u+2$. Consider the long exact sequence

$$(4.1) \qquad 0 \longrightarrow H^0M_1^1 \xrightarrow{f_0} H^0M_0^2 \xrightarrow{p} H^0M_0^2 \xrightarrow{\delta_0'} H^1M_1^1$$
$$\xrightarrow{f_1} H^1M_0^2 \xrightarrow{p} H^1M_0^2 \xrightarrow{\delta_1'} H^2M_1^1 \xrightarrow{f_2} H^2M_0^2$$

assiciated to the short exact sequence (3.1). Note that this exact sequence is homogeneous. We first determine $X = H^1M_0^2$ at the degree

$$kq = \{(ap-1)(p+1)-2\}q \quad (q = 2p-2)$$

for $u = ap-2$ by the following

LEMMA 4.2. *Let* B *be a direct sum of submodules* $L\langle g\rangle$ $(g \in Y(j) \subset X = H^1M_0^2)$, *where* $Y(j)$ *is a homogeneous subset of* X *with the degree* j *and* $L\langle g\rangle$ *denotes the* $\boldsymbol{Z}$*-module generated by* g *which is isomorphic to* $\boldsymbol{Z}/n$ *if the order of* g

is n. Then $B = X$ at the degree j if B contains $\operatorname{Im} f_1$ *and the set* $\{\delta'_1 g \mid g \in Y(j)\}$ *is linearly independent.*

This is proved in a same manner to [6; Lemma 3.9] by using [1; Remark 3.11]. We also need

LEMMA 4.3. *The $\boldsymbol{Z}/p[v_1]$-module $H^n M_1^1$ at the degree kq is*
0) [1; Th. 5.3] *the direct sum of* $L'\langle x_i^s/v_1^j\rangle$ *for* $(i, s, j) \in \Lambda(k)$, *if* $n = 0$,
1) [6; Th. 3.10] *the direct sum of* $L'\langle x_i^s\zeta/v_1^j\rangle$ *for* $(i, s, j) \in \Lambda(k)$, *and*

$L'\langle y_m/v_1^j\rangle$ *for* $m = sp^i$ *with* $(i, s, j) \in \Lambda_0(k)$, *if* $n = 1$, *and*

2) [4; Th. 4.4] *the direct sum of* $L'\langle x_i^s G_i/v_1^j\rangle$ *for* $(i, s, j) \in \Lambda(k - |G_i|)$

with $p \nmid s + 1$, *and* $L'\langle y_m \otimes \zeta/v_1^j\rangle$ *for* $m = sp^i$ *with* $(i, s, j) \in \Lambda_0(k)$

$(|G_i| = -(p+1)e(i-1) - 1)$, *if* $n = 2$.

Here $L'\langle x/v_1^j\rangle$ *denotes the submodule generated by the element* x/v_1^j *which is isomorphic to* $\boldsymbol{Z}/p[v_1]/(v_1^j)$, $\Lambda(l)$ *and* $\Lambda_0(l)$ *are the sets of triples of integers*

$$\Lambda(l) = \{(i, s, j) \mid i \geqq 0,\ j \leqq a_i \text{ and } sp^i(p+1) - j = l \text{ for } s \text{ with } p \nmid s\}, \text{ and}$$
$$\Lambda_0(l) = \{(i, s, j) \mid i \geqq 0,\ j \leqq A(sp^i) \text{ and } sp^i(p+1) + 1 - j = l$$
$$\text{for } s \text{ with } p \nmid s(s+1) \text{ or } p^2 \mid s+1\},$$

for the integers a_i and $A(m)$ defined by

$$a_0 = 1,\ a_i = p^i + p^{i-1} - 1; \text{ and } A(sp^i) = 2 + \varepsilon(s)p^i(p^2-1) + (p+1)e(i)$$

$(p \nmid s$, *and* $\varepsilon(s) = 1$ *if* $p^2 \mid s+1$, *and* $= 0$ *otherwise), and the generators satisfy*

(4.3.1) $$y_m/v_1^3 = v_2^m(t_1 - 2^{-1}v_1\zeta^p)/v_1^3 \text{ if } p \mid m, \text{ and}$$

(4.3.2) $$x_i^s G_i/v_1 = v_2^m T/v_1 = T_m/v_1 \ (m = sp^i - e(i-1) - 1) \text{ if } i \geqq 2.$$

Here we note that $i \geqq 2$ if $(i, s, j) \in \Lambda(k) \cup \Lambda_0(k)$ except for the case $i = 1$ and $p \mid s+1$ and so we see that $p \nmid j$ if $(i, s, j) \in \Lambda(k) \cup \Lambda_0(k)$ since $k \equiv -3 \bmod p$.

PROPOSITION 4.4. *The $\boldsymbol{Z}$-module $H^1 M_0^2$ at degree kq is the direct sum of* $L\langle g\rangle$ *for* $g \in Y(kq)$, *where*

$$Y(kq) = \{ym/pv_1^j \mid m = sp^i \text{ and } (i, s, j) \in \Lambda_0(k)\}.$$

PROOF. Let B denote the direct sum of $L\langle g\rangle$ for $g \in Y(kq)$. If $\Lambda(k) = \phi$, then $B \supset \operatorname{Im} f_1$ by Lemma 4.3. For the element $x = x_i^s/v_1^j \in H^0 M_1^1$ with $(i, s, j) \in \Lambda(k)$, we obtain

$$\delta'_0(f_0 x) = -jy_m/v_1^{j+1} - 2^{-1}jx\zeta + \cdots$$

by [1; prop. 6.9] and (4.3.1), where f_0 is the map in (4.1) and $\cdots$ denotes an element killed by v_1^{j-1}. Therefore $x\zeta$ is dependent on the elements of $Y(kq)$ in $\operatorname{Im} f_1$ and so we have $B \supset \operatorname{Im} f_1$.

By the definition of δ'_1, we have

(4.4.1) $$\delta'_1(y) = -jt_1 \otimes y/v_1 + f_1^{-1}(d_1 y_m^{\sim})/p^2 v_1^j$$

for $y = y_m/pv_1^j \in Y(kq)$ and $y_m^{\sim}/p^2v_1^j \in \Omega^1 M_0^2$ with $y = y_m^{\sim}/pv_1^j$, whose last term is killed by v_1^{j-1} since $y_m^{\sim}/p^2v_1^j = v_2^m(t_1 - 2^{-1}v_1\zeta')/p^2v_1^j + \cdots$ and $(d_1 y_m^{\sim})/p^2v_1^j = mv_1v_2^{m-1}t_1^p \otimes (t_1 - 2^{-1}v_1\zeta')/p^2v_1^j + \cdots$ by (3.4.1) and (4.3.1). The element $v_2^m t_1 \otimes t_1/v_1^{j+1}$ turns out to be a part of $\cdots$ by considering $d_1 v_2^m t_1^2/v_1^{j+1}$. Then the first term of (4.4.1) turns into

$$-jt_1 \otimes y/v_1 = 2^{-1}jy \otimes \zeta + \cdots$$

by (4.3.1). Therefore we see that the set $\{\delta'_1(y) | y \in Y(kq)\}$ is linearly independent by Lemma 4.3, since $p \nmid j$. Hence the proposition follows from Lemma 4.2. q.e.d.

By observing the proof of this proposition, we also have

PROPOSITION 4.5. Im δ'_1 *at the degree* k *is the submodule of* $H^2M_1^1$ *generated by* $2t_1 \otimes y/v_1 = -y \otimes \zeta + \cdots$ *for* $y \in Y(kq)$.

COROLLARY 4.6. *In the* E_2*-term* H^4A, *we have the non-triviality*

$$\beta_s\beta_t \neq 0 \text{ if } p \nmid st,\ p | s+t+1 \text{ and } s+t \in I.$$

PROOF. By virtue of the exact sequence (4.1), we see that $T_u/pv_1 \neq 0$ for $u + 1 \in I$ by (4.3.2), Lemma 4.3 and Proposition 4.5. Now the corollary follows from Lemmas 2.6–7 and 3.3. q.e.d.

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