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On renormings of nonreflexive Banach spaces with preduals

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Introduction

Let X be a Banach space. Let X^* and X^{**} denote the first and second dual spaces of X. We denote by π the canonical map of X into X^{**}. In what follows, X will be identified with $\pi(X)$. It is well known that if a Banach space X is isometric to a dual Banach space then there exists a projection P with norm 1 from its second dual X^{**} onto X. Davis and Johnson showed in [3] that every nonreflexive Banach space can be equivalently renormed in such a way that the renormed space is not isometric to a dual space. Dulst and Singer [4] then proved that this conclusion can be improved in the following form: Every nonreflexive Banach space X admits an equivalent norm $\|\cdot\|$ such that for each projection P: $X^{**} \to X$, $\|P\| > 1$ with respect to the norm. After this, Godun gave in [5] a more general result that each nonreflexive Banach space X admits an equivalent norm $\|\cdot\|$ such that for each projection P: $X^{**} \to X$, $\|P\| \geq 2$ with respect to the norm. These results are all related to the existence of an equivalent norm which admits no preduals. On the contrary, we are concerned with equivalent norms which admit preduals. In this paper we consider the class of such equivalent norms and demonstrate that given a nonreflexive Banach space X "most" of the equivalent norms on X do not admit preduals in the above sense.

Let $(\mathfrak{N}, |\cdot|)$ be a nonreflexive Banach space with norm $|\cdot|$. We denote by $\mathfrak{N}(X)$ the class of all the equivalent norms on X and by $\mathfrak{N}_p(X)$ the class of all equivalent norms on X which admit preduals. The purpose of this paper is to show in terms of metric space theory that $\mathfrak{N}_p(X)$ is a meager subset of the space $\mathfrak{N}(X)$. To accomplish this we introduce a metric ρ: $\mathfrak{N}(X) \times \mathfrak{N}(X) \to [0, \infty)$ on $\mathfrak{N}(X)$ defined by

$$\begin{aligned}&\rho(|\cdot|_1, |\cdot|_2)\\ &\quad = \log\{\inf\{\lambda > 0\colon |x|_1 \leq \lambda |x|_2 \leq \lambda^2 |x|_1 \quad \text{for all } x \in X\}\}.\end{aligned}$$

Then ρ defines a complete metric on $\mathfrak{N}(X)$, and it is shown (see Theorem 2 below) that $\mathfrak{N}_p(X)$ is nowhere dense in $\mathfrak{N}(X)$ with respect to the metric topology ρ. This means that $\mathfrak{N}_p(X)$ is meager in $\mathfrak{N}(X)$ and in this sense "most" of the equivalent norms do not admit preduals.

1. Preliminaries

In what follows, a continuous projection is simply called a projection. Let $\mathscr{P}(X)$ be the set of all projections from X^{**} onto X. It is obvious that $\mathscr{P}(X) \neq \phi$ if X itself is a dual space. However it should be mentioned that $\mathscr{P}(X) \neq \phi$ does not always mean that X is a dual space. For instance, it is known that the space $L^1[0, 1]$ is not isomorphic to any dual Banach space. But it is proved that there is a projection with norm 1 from its second dual $(L^1[0, 1])^{**}$ onto $L^1[0, 1]$, and hence $\mathscr{P}(L^1[0, 1]) \neq \phi$. We next define an extended real-valued function Φ on $\mathfrak{N}(X)$ by

$$\Phi(|\cdot|) = \begin{cases} \inf\{|P|: P \in \mathscr{P}(X)\} & \text{if } \mathscr{P}(X) \neq \phi, \\ \infty & \text{if } \mathscr{P}(X) = \phi. \end{cases} \tag{1}$$

It is not difficult to show that either $\Phi(|\cdot|) = \infty$ for all $|\cdot| \in \mathfrak{N}(X)$ or $1 \leq \Phi(|\cdot|) < \infty$ for all $|\cdot| \in \mathfrak{N}(X)$. If $(X, |\cdot|)$ is a dual Banach space, then $\Phi(|\cdot|) = 1$. Godun's result states that there exists $|\cdot| \in \mathfrak{N}(X)$ with $\Phi(|\cdot|) \geq 2$ for any nonreflexive Banach space X. Moreover, Φ is continuous on $(\mathfrak{N}(X), \rho)$ if $\mathscr{P}(X) \neq \phi$. In fact, it is seen that $|P|_1 \leq \lambda^2 |P|_2 \leq \lambda^4 |P|_1$ for any projection $P \in \mathscr{P}(X)$ provided that $|X|_1 \leq \lambda |X|_2 \leq \lambda^2 |X|_1$ for all $x \in X$. Hence we have $\Phi(|\cdot|_1) \leq \lambda^2 \Phi(|\cdot|_2) \leq \lambda^4 \Phi(|\cdot|_1)$ if $|x|_1 \leq \lambda |x|_2 \leq \lambda^2 |x|_1$ for all $x \in X$. This means that Φ is continuous on $(\mathfrak{N}(X), \rho)$. We here recall (see[1]) the notion of characteristic of a subspace V of a dual Banach space X^*. The *characteristic* $r(V)$ of V is defined as the maximum of nonnegative numbers r such that the unit ball $B_V = \{f \in V: |f| \leq 1\}$ of V is $\sigma(X^*, X)$-dense in the r-ball $rB_{X^*} = \{f \in X^*: |f| \leq r\}$ of X^*. Clearly, $0 \leq r(V) \leq 1$ and $r(V)$ is characterized as

$$\begin{aligned} r(V) &= \{\sup\{|x|: x \in \bar{B}_X^{\sigma(X,V)}\}\}^{-1} \\ &= \inf\{|\pi(x) - F|: x \in X,\ |x| = 1,\ F \in V^{\perp}\}, \end{aligned} \tag{2}$$

where $\bar{B}_X^{\sigma(X,V)}$ denotes the closure with respect to the weak topology $\sigma(X, V)$ of the unit ball $B_X = \{x \in X: |x| \leq 1\}$ and $V^{\perp} = \{F \in X^{**}: F(f) = 0, f \in V\}$. The following lemma is immediate.

LEMMA 1. *Let $(X, |\cdot|)$ be a nonreflexive Banach space with $\mathscr{P}(X) \neq \varnothing$. Let $P \in \mathscr{P}(X)$. Then we have*

$$|P|^{-1} = \inf\{r(\ker F): P(F) = 0,\ F \in X^{**}\}$$

and

$$|P| = \sup\{\sup\{|x|: x \in \bar{B}_X^{\sigma(X,\ker F)}\}: P(F) = 0,\ F \in X^{**}\}.$$

PROOF. Since the first equality was already proved in [4], we prove the second identity. To this end, we use the characterization (2) for $r(V)$. For any $F \in X^{**}$ with $P(F) = 0$, we take $V = \ker F \subset X^*$. Then $V^{\perp} = \langle F \rangle$, where $\langle F \rangle$

stands for the one dimensional linear space spanned by F. Therefore

$$\begin{aligned} r(\ker F) &= \{\sup\{|x|: x \in \bar{B}_X^{\sigma(X,\ker F)}\}\}^{-1} \\ &= \inf\{|\pi(x) - x^{**}|: x \in X,\ |x| = 1,\ x^{**} \in \langle F \rangle\} \\ &= 1/|P|_{\pi(X)\oplus\langle F\rangle}|. \end{aligned}$$

and $\sup\{|x|: x \in \bar{B}_X^{\sigma(X,\ker F)}\} = |P|_{\pi(X)\oplus\langle F\rangle}|$. From this it follows that

$$\sup\{\sup\{|x|: x \in \bar{B}_X^{\sigma(X,\ker F)}\}: P(F) = 0,\ F \in X^{**}\} = |P|.$$

Q.E.D.

2. Lemmas

We begin by introducing two numbers. For a pair of positive numbers λ_1, λ_2 and a pair of bounded subsets B_1, B_2 of X, we define

$$M_{i,j} = 1 + (\lambda_j/\lambda_i)\inf\{r > 0: B_j \subset rB_i\},$$

where $(i, j) = (1, 2)$ or $(2, 1)$. It is seen that $M_{1,2}$ and $M_{2,1}$ make sense as finite numbers provided that both B_1 and B_2 contain the origin 0 as an interior point.

LEMMA 2. *Let B_1 and B_2 be bounded convex subsets of X both of which contain 0 as an interior point. Let $\lambda_1 > 0$, $\lambda_2 > 0$, and set $B_3 = \lambda_1 B_1 + \lambda_2 B_2$. Let V be a subspace of X^* and suppose that there exists $\varepsilon \geq 0$ satisfying*

$$\bar{B}_3^{\sigma(X,V)} \subset (1 + \varepsilon)\bar{B}_3. \tag{3}$$

Then we have the following two inclusions;

$$\bar{B}_1^{\sigma(X,V)} \subset (1 + M_{1,2}\varepsilon)\bar{B}_1 \tag{4}$$

$$\bar{B}_2^{\sigma(X,V)} \subset (1 + M_{2,1}\varepsilon)\bar{B}_2. \tag{5}$$

PROOF. By condition (3) we have

$$\lambda_1\bar{B}_1^{\sigma(X,V)} + \lambda_2 B_2 \subset \overline{\lambda_1 B_1 + \lambda_2 B_2}^{\sigma(X,V)} \subset (1 + \varepsilon)(\overline{\lambda_1 B_1 + \lambda_2 B_2}).$$

We now define

$$M(A, x^*) = \sup\nolimits_{x\in A} x^*(x), \qquad \text{for } A \subset X,\ x^* \in X^*.$$

Then we have

$$\begin{aligned} M(\lambda_1\bar{B}_1^{\sigma(X,V)} + \lambda_2 B_2,\ x^*) &\leq (1 + \varepsilon)M(\overline{\lambda_1 B_1 + \lambda_2 B_2}, x^*) \\ &= (1 + \varepsilon)\lambda_1 M(B_1, x^*) + (1 + \varepsilon)\lambda_2 M(B_2, x^*). \end{aligned}$$

This implies

$$\lambda_1 M(\bar{B}_1^{\sigma(X,V)}, x^*) + \lambda_2 M(B_2, x^*)$$
$$\leq (1+\varepsilon)\{\lambda_1 M(B_1, x^*) + \lambda_2 M(B_2, x^*)\},$$

and hence $\lambda_1 M(\bar{B}_1^{\sigma(X,V)}, x^*) \leq (1+\varepsilon)\lambda_1 M(B_1, x^*) + \varepsilon\lambda_2 M(B_2, x^*)$. Therefore we get

$$M(\bar{B}_1^{\sigma(X,V)}, x^*) \leq (1+\varepsilon)M(B_1, x^*) + (\varepsilon\lambda_2/\lambda_1)M(B_2, x^*)$$
$$\leq (1+\varepsilon)M(B_1, x^*) + (\lambda_2/\lambda_1)M(B_1, x^*) \inf\{r > 0: B_2 \subset rB_1\}$$
$$= M(B_1, x^*)(1 + M_{1,2}\varepsilon).$$

Since the above relation holds for every x^* of X^*, we have

$$\bar{B}_1^{\sigma(X,V)} \subset (1 + M_{1,2}\varepsilon)\bar{B}_1.$$

Thus we obtain the first inclusion (4). The second inclusion (5) can be verified in a similar way. Q.E.D.

Let X be a Banach space with norm $|\cdot|$ and V be a closed subspace of X^*. Then V becomes a Banach space, and it is seen that V is a predual of X if and only if the following two conditions are satisfied:

[A] For any continuous linear function f on V, there exists a unique element x of X such that $f(x^*) = x^*(x)$ for all $x^* \in V$.

[B] For any $x \in X$, $|x| = \sup\{x^*(x): x^* \in V, |x^*| = 1\}$.

For a Banach space X Property [A] is invariant under renormings of X, while Property [B] depends in general upon renormings of X.

LEMMA 3. *Let $|\cdot|_1, |\cdot|_2 \in \mathfrak{N}(X)$ and $\lambda_1 > 0$ and $\lambda_2 > 0$, and let V be a closed subspace of X^*. Let $B_1 = \{x \in X; |x|_1 \leq 1\}$, $B_2 = \{x \in X; |x|_2 \leq 1\}$ and $B_3 = \lambda_1 B_1 + \lambda_2 B_2$. Then the Minkowski functional $|\cdot|_3$ of B_3 gives an equivalent norm on X with the following three properties*:

(I) *If V is a predual with the norm $|\cdot|_3$, then it is a predual with any one of the norms $|\cdot|_1$ and $|\cdot|_2$.*

(II) *Let P be a continuous projection from X^{**} to X. If $|P|_3 = 1$, then $|P|_1 = |P|_2 = 1$.*

(III) *If $\Phi(|\cdot|_3) = 1$, then $\Phi(|\cdot|_1) = \Phi(|\cdot|_2) = 1$, where Φ is the functional defined by* (1).

PROOF. We first prove (I). By the assumption of (I), V satisfies conditions [A] and [B] for the norm $|\cdot|_3$ and $\bar{B}_3$ is $\sigma(X, V)$-closed. Therefore, letting $\varepsilon = 0$ in Lemma 3, we infer that B_1 is also $\sigma(X, V)$-closed. For any $x \in X$ with $|x|_1 = 1$ and any $\varepsilon > 0$, $(1+\varepsilon)x$ is not the element of B_1. By the separation theorem one finds an element x^* of V such that $\sup\{x^*(y): y \in B_1\} < (1+\varepsilon)x^*(x)$. Then we have $|x^*|_1 \leq (1+\varepsilon)x^*(x)$, or $1/(1+\varepsilon) < (1/|x^*|_1)x^*(x)$. This shows that $\sup\{y^*(x): y^* \in V, |y^*|_1 \leq 1\} \geq 1/(1+\varepsilon)$. Letting $\varepsilon \downarrow 0$, we have

$$\sup\{y^*(x): y^* \in V, |y^*|_1 \leq 1\} = 1.$$

This shows that V satisfies condition [B] for the norm $|\cdot|_1$, and that V is a predual of X with the norm $|\cdot|_1$. Similarly, V is a predual of X with the norm $|\cdot|_2$.

(II) and (III): Suppose that $P \in \mathscr{P}(X)$ with $|P|_3 \leq 1 + \varepsilon$. We wish to show that

$$|P|_1 \leq 1 + M_{1,2}\varepsilon \text{ and } |P|_2 \leq 1 + M_{2,1}\varepsilon \tag{6}$$

where $M_{1,2}$ and $M_{2,1}$ are the constants defined at the beginning of §2. To this end, fix any $x^{**} \in \ker P \subset X^{**}$ and let $V = \{x^* \in X^*: x^{**}(x^*) = 0\}$. Applying Lemma 1 to X equipped with the norm $|\cdot|_3$, we have

$$\sup\{|x|_3 : x \in \bar{B}_3^{\sigma(X,V)}\} \leq |P|_3 \leq 1 + \varepsilon.$$

This shows that $\bar{B}_3^{\sigma(X,V)} \subset (1 + \varepsilon)\bar{B}_3$. Thus Lemma 2 implies that

$$\bar{B}_1^{\sigma(X,V)} \subset (1 + M_{1,2}\varepsilon)B_1.$$

From this it follows that

$$\sup\{|x|_1 : x \in \bar{B}_1^{\sigma(X,V)}\} \leq 1 + M_{1,2}\varepsilon.$$

We then apply Lemma 1 again and conclude that $|P|_1 \leq 1 + M_{1,2}\varepsilon$. Similarly we obtain $|P|_2 \leq 1 + M_{2,1}\varepsilon$.

To prove assertion (II), it is sufficient to put $\varepsilon = 0$ in (6). Then $|P|_1 \leqq 1$ and $|P|_2 \leq 1$.

Finally, suppose that $\Phi(|\cdot|_3) = 1$. Then for any $\varepsilon > 0$, there exists a projection P such that $|P|_3 \leq 1 + \varepsilon$. Then we infer from (6) that $|P|_1 \leq 1 + M_{1,2}\varepsilon$, and so $\Phi(|\cdot|_1) \leq 1 + M_{1,2}\varepsilon$. Letting $\varepsilon \downarrow 0$ yields $\Phi(|\cdot|_1) = 1$. By the same reasoning we have $\Phi(|\cdot|_2) = 1$. Thus we obtain the third assertion (III).

Q.E.D.

3. Theorems

We are now in a position to state our main theorems and give their proofs.

THEOREM 1. *Let $(X, |\cdot|_0)$ be a nonreflexive Banach space. Then the set $\tilde{\mathfrak{N}}(X) = \{|\cdot| \in \mathfrak{N}(X): \Phi(|\cdot|) > 1\}$ is open dense in $\mathfrak{N}(X)$ with respect to the metric ρ.*

PROOF. We only give the proof for the case in which $\Phi < \infty$. Since Φ is continuous on $(\mathfrak{N}(X), \rho)$, $\tilde{\mathfrak{N}}(X)$ is open. Next by Godun's theorem [5] there exists a norm $\|\cdot\| \in \mathfrak{N}(X)$ such that

$$\Phi(\|\cdot\|) \geq 2 \ (> 1)$$

Now take any element $|\cdot| \in \mathfrak{N}(X)$ and any $\lambda > 0$. Let $B = \{x \in X: |x| \leq 1\}$ and $C = \{x \in X: \|x\| \leq 1\}$. We then consider the Minkowski functional $|||\cdot|||_\lambda$ of $B + \lambda C$. Then $\Phi(|||\cdot|||_\lambda) > 1$ by Lemma 3 (III). Moreover, it is seen that $\lim_{\lambda \downarrow 0} \rho(|\cdot|, |||\cdot|||_\lambda) = 0$. This means that $\tilde{\mathfrak{N}}(X)$ is dense in $(\mathfrak{N}(X), \rho)$. Q.E.D.

THEOREM 2. Let $(X, |\cdot|_0)$ be a nonreflexive Banach space. Then the set $\mathfrak{N}_p(X) = \{|\cdot| \in \mathfrak{N}(X): X$ with the norm $|\cdot|$ has a predual$\}$ is a nowhere dense subset of the metric space $(\mathfrak{N}(X), \rho)$.

PROOF. Let $\tilde{\mathfrak{N}}(X) = \{|\cdot| \in \mathfrak{N}(X): \Phi(|\cdot|) > 1\}$. Since $\Phi(|\cdot|) = 1$ for all $|\cdot| \in \mathfrak{N}_p(X)$, it follows that $\tilde{\mathfrak{N}}(X) \cap \mathfrak{N}_p(X) = \phi$. Theorem 1 then implies that $\mathfrak{N}_p(X)$ is nowhere dense. Q.E.D.

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