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Homotopy commutativity of the loop space of a finite CW-complex

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0. Introduction

As a generalization of a topological group in the homotopy theory, an H-space (or a Hopf space) is defined to be a topological space Y with a base point $*$ admitting a continuous multiplication $\mu: Y \times Y \to Y$ such that $*$ acts as a two sided homotopy unit, that is, the restrictions $\mu | Y \times *$ and $\mu | * \times Y$ are both homotopic (preserving $*$) to the identity map $\mathrm{id}_Y: Y \to Y$; and an H-space $Y = (Y, \mu)$ is said to be homotopy associative when the two maps $\mu(\mu \times \mathrm{id}_Y)$ and $\mu(\mathrm{id}_Y \times \mu)$ of $Y \times Y \times Y$ to Y are homotopic. The loop space ΩX of a based space X admitting the usual loop multiplication is another important example; and in the homotopy theory, ΩX can be regarded as a topological group G when $X = B_G$ is the classifying space of G.

Moreover, as a generalization of a topological abelian group, an H-space $Y = (Y, \mu)$ is said to be homotopy commutative when $\mu: Y \times Y \to Y$ is homotopic to μT for the homeomorphism T on $Y \times Y$ commuting coordinates. A compact connected Lie group G is homotopy commutative if and only if G is abelian, that is, G is a torus, the product of some copies of the circle group S^1, by Araki-James-Thomas [2]. Moreover Hubbuck[13] proved that if a connected finite CW-complex Y is a homotopy commutative H-space, then Y has the homotopy type of a torus.

In this paper, we are concerned with the homotopy commutativity of the loop space ΩX of a connected, simply connected finite CW-complex X. It is easy to see that

(i) If X itself is an H-space, then ΩX is homotopy commutative.

But the converse is not true for the complex projective 3-space $CP(3)$. In fact, Stasheff [21; Th.1.18] proved that $\Omega CP(3)$ is homotopy commutative; but $CP(3)$ is not an H-space which is seen by Borel's theorem on the cohomology ring of an H-space.

We note also that X is an H-space if and only if ΩX is strongly homotopy commutative in the sense of Sugawara [23].

Now the purpose of this paper is to prove the following

THEOREM 1. *Let X be the suspension $X = \Sigma A$ of a connected finite CW-*

complex A. Then the loop space ΩX is homotopy commutative only if X is an H-space, that is, only if X is contractible or is homotopy equivalent to the n-sphere S^n for $n = 3, 7$.

THEOREM 2. *Let X be the mapping cone $C(\alpha) = S^k \cup_\alpha e^n$ of $\alpha \in \pi_{n-1}(S^k)$ $(k, n \geq 2)$. Then $\Omega C(\alpha)$ is homotopy commutative if and only if $C(\alpha)$ is contractible, that is, $k = n - 1$ and $\pm\alpha = [\mathrm{id}_{S^k}] = \iota_k \in \pi_k(S^k)$.*

Let E_γ denote the k-sphere bundle over S^n with characteristic class $\gamma \in \pi_{n-1}(O_{k+1})$ $(k, n \geq 2)$, which is the CW-complex of the form

$$E_\gamma = S^k \cup_\alpha e^n \cup e^{n+k} \qquad \text{for} \qquad \alpha = q_*(\gamma) \in \pi_{n-1}(S^k) \qquad ([15]),$$

where O_k is the orthogonal group of transformations on the real k-space and $q: O_{k+1} \to O_{k+1}/O_k = S^k$ is the projection. Also, in addition to the generator $\iota_n \in \pi_n(S^n) \cong Z$, consider those $\eta_i \in \pi_{i+1}(S^i) \cong Z$ $(i = 2)$, $\cong Z_2$ $(i \geq 3)$ and $\omega \in \pi_6(S^3) \cong Z_{12}$.

THEOREM 3. *ΩE_γ of the bundle E_γ is not homotopy commutative, except for the following cases* (1) $\sim$ (4):

(1) *E_γ is an H-space, that is, it is homotopy equivalent to $S^3 \times S^3$, $S^3 \times S^7$, $S^7 \times S^7$ $(\alpha = 0)$, S^7 $(\alpha = \pm\iota_3)$, $SU(3)$ $(\alpha = \eta_3)$, or $\alpha = \pm a\omega$ for $a = 1, 3, 4, 5$.*

(2) *$\alpha = \pm\eta_2$, that is, E_γ is homotopy equivalent to $CP(3)$.*

(3) *$(k, n) = (3, 4)$, $(3, odd)$ and $(7, odd)$, and E_γ is not an H-space.*

(4) *k and n are odd, $k + 2 \leq n \leq 2k - 1$ $(k \neq 3, 7)$, the order of α is even and the Whitehead product $[\iota_k, \iota_k]$ is in the image of $\alpha_*: \pi_{2k-1}(S^{n-1}) \to \pi_{2k-1}(S^k)$.*

In Theorem 3, the case (1) is determined by Zabrodsky [25]. As is stated above, ΩE_γ is homotopy commutative in the first two cases; but the author cannot determine whether so is or not in the last two cases.

In case when X is not simply connected, we note only the following

PROPOSITION 4. *When $X = S^n/Z_m$ is the real projective space $(n \geq 2, m = 2)$ or the lens space $(n: odd \geq 3, m \geq 3)$, ΩX is homotopy commutative if and only if $n = 3$ or 7; and X is an H-space if and only if $n = 3, 7$ and $m = 2$.*

Our method to prove the above theorems is based on the following results which may be well known:

(ii) (cf. Arkowitz [3].) If ΩX is homotopy commutative, then all Whitehead products on X vanish.

(iii) (cf. [4], [11], [18].) For any set P of primes, the loop space ΩX_P of the P-localization X_P of X is homotopy commutative if so is ΩX.

We also use Adams' theorem which states that S^n is an H-space if and only if $n = 1$, 3 or 7. For the properties of the homotopy groups of spheres and the calculation of several Whitehead products, we use Toda's book [24] and so on.

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1. Preliminaries

In this paper, all topological spaces will be assumed to have a base point $*$ and to have the homotopy type of connected CW-complexes, and all maps and homotopies to preserve $*$. We denote by $\pi(X, Y)$ the homotopy set of all homotopy classes $[f]$ of maps $f: X \to Y$ (preserving $*$), and f and $[f]$ are denoted frequently by the same letter.

A space $X = (X, \mu_X)$ is an *H-space* with multiplication $\mu = \mu_X: X \times X \to X$ if $\mu | X \times * \simeq \mathrm{id} \simeq \mu | * \times X$, and is *homotopy associative* (resp. *commutative*) if

$$\mu(\mu \times \mathrm{id}) \simeq \mu(\mathrm{id} \times \mu): X \times X \times X \to X \qquad (\text{resp}, \mu \simeq \mu T: X \times X \to X).$$

(Here $\simeq$ means 'homotopic (preserving $*$)', $\mathrm{id} = \mathrm{id}_X: X \to X$ is the identity map, and $T: X \times X \to X \times X$, $T(x, y) = (y, x)$, is a commuting map.) A map $f: X = (X, \mu_X) \to Y = (Y, \mu_Y)$ between H-spaces is an *H-map* if $f\mu_X \simeq \mu_Y(f \times f)$.

The *loop space* ΩX of a connected space X is the space of all loops $\omega: (I, \partial I) \to (X, *)$ $(I = [0, 1], \partial I = \{0, 1\})$ with the compact open topology, whose base point is the constant map $*$. By the loop multiplication

$$\mu = \mu_{\Omega X}: \Omega X \times \Omega X \to \Omega X, \qquad \mu(\omega_1, \omega_2) = \omega_1 \cdot \omega_2,$$

given by $(\omega_1 \cdot \omega_2)(t) = \omega_1(2t)$ if $t \leq 1/2$, $= \omega_2(2t - 1)$ if $t \geq 1/2$, ΩX is a homotopy associative H-space; and ΩX is homotopy commutative if X is an H-space. Also $\tau: \Omega X \to \Omega X$, $\tau(\omega) = \omega^{-1}$, is given by $\omega^{-1}(t) = \omega(1 - t)$, which satisfies $\mu(\mathrm{id} \times \tau) \simeq * \simeq \mu(\tau \times \mathrm{id})$.

We note that ΩX has the homotopy type of a CW-complex by Milnor's theorem. Also, ΩX is connected if and only if X is simply connected. Hereafter, we are concerned with ΩX by assuming that X is simply connected unless otherwise stated.

For $f, g: A \to \Omega X$, we have $f \cdot g: A \to \Omega X$ given by $(f \cdot g)(a) = f(a) \cdot g(a)$; and the homotopy set $\pi(A, \Omega X)$ forms a group by $[f] \cdot [g] = [f \cdot g]$ so that we have the natural isomorphism

$$\Omega_0: \pi(\Sigma A, X) \cong \pi(A, \Omega X) \qquad (\Sigma A = A \times I/(A \times \partial I \cup * \times I), \text{ the } \textit{suspension})$$

by sending $f: \Sigma A \to X$ to its *adjoint* $\Omega_0 f = f': A \to \Omega X$, $f'(a)(t) = f(a, t)$ for $a \in A$ and $t \in I$.

Now consider the *commutator map*

$$\varphi : \Omega X \times \Omega X \to \Omega X\,, \qquad \varphi(\omega_1, \omega_2) = (\omega_1 \cdot \omega_2)\cdot(\omega_1^{-1} \cdot \omega_2^{-1})\,.$$

Then $\varphi|\Omega X \vee \Omega X$ is null homotopic, and there exists a map

$$\bar{\varphi} : \Omega X \wedge \Omega X \to \Omega X \qquad \text{with} \qquad \bar{\varphi}\,\mathrm{pr} \simeq \varphi\,.$$

(Here $A \vee B = A \times * \cup * \times B$ is the *wedge*, $A \wedge B = A \times B/A \vee B$ is the *smash product*, and $\mathrm{pr} : A \times B \to A \wedge B$ is the natural projection.)

DEFINITION 1.1. (1) The *Samelson product* of $f_i' : A_i \to \Omega X$ $(i = 1, 2)$ is given by

$$\langle f_1', f_2' \rangle = \bar{\varphi}(f_1' \wedge f_2') : A_1 \wedge A_2 \to \Omega X \wedge \Omega X \to \Omega X\,.$$

(2) The *Whitehead product* of $f_i : \Sigma A_i \to X$ $(i = 1, 2)$ is given by

$$[f_1, f_2] = \Omega_0^{-1}\langle \Omega_0 f_1, \Omega_0 f_2 \rangle : \Sigma(A_1 \wedge A_2) \to X\,,$$

by using the adjoint operator Ω_0.

(3) The homotopy classes of these products give us the *Samelson product* $\langle [f_1'], [f_2'] \rangle$ and the *Whitehead product* $[[f_1], [f_2]]$ of homotopy classes.

For f_i in (2), consider $\bar{f_i} = f_i\,\mathrm{pr} : (CA_i, A_i) \to (X, *)$ and define the map

$$h : A_1 * A_2 = CA_1 \times A_2 \cup A_1 \times CA_2 \to X$$

by $h|CA_1 \times A_2 = \bar{f_1}\,\mathrm{pr}_1$ and $h|A_1 \times CA_2 = \bar{f_2}\,\mathrm{pr}_2$. (Here $CA = A \times I/A \times 1 \cup * \times I$ $(A = A \times 0)$ is the *cone*, $\mathrm{pr} : CA \to \Sigma A = CA/A$ is the projection, $A_1 * A_2$ is the *reduced join* and $\mathrm{pr}_i : X_1 \times X_2 \to X_i$ is the projection to the i-th factor.) Then Arkowitz[3; Th.2.4] proved that

$$\nu[f_1, f_2] \simeq h : A_1 * A_2 \to X$$

for the projection $\nu : A_1 * A_2 \to \Sigma(A_1 \wedge A_2)$, where ν is a homotopy equivalence; and for a homotopy inverse ν^{-1} of ν, $\nu^{-1}h$ is known to be the usual definition of the Whitehead product $[f_1, f_2]$.

To study the homotopy commutativity of ΩX, we use the following

PROPOSITION 1.2. (i) *The loop space ΩX is homotopy commutative if and only if the commutator map $\varphi : \Omega X \times \Omega X \to \Omega X$ is null homotopic.*

(ii) (Arkowitz[3; Prop.5.1].) *Let A_i be a finite CW-complex. Then the Whitehead product $[f_1, f_2]$ vanishes for $f_i : \Sigma A_i \to X$ if and only if there exists a map $f : \Sigma A_1 \times \Sigma A_2 \to X$ of type (f_1, f_2), that is, $f|\Sigma A_1 \times * \simeq f_1$ and $f|* \times \Sigma A_2 \simeq f_2$.*

(iv) (Stasheff[21; Th.1.10].) *Let* $d = \Omega_0^{-1}\,\mathrm{id}_{\Omega X} : \Sigma\Omega X \to X$ *be the adjoint of* $\mathrm{id}_{\Omega X}$. *Then* ΩX *is homotopy commutative if and only if there exists a map* $\psi : \Sigma\Omega X \times \Sigma\Omega X \to X$ *of type* (d, d).

On the other hand, for a set P of primes, we consider the P-localizations

$$l = l_X : X \to X_P \qquad \text{and} \qquad f_P : X_P \to Y_P$$

of X and $f : X \to Y$ (cf., e.g., [4], [11] and [18]), satisfying the following

LEMMA 1.3. (i) *For each* i, *there exists a natural isomorphism* $\pi_i(X_P) \cong \pi_i(X)_P$ *such that the diagram*

$$\begin{tikzcd} \pi_i(X) \arrow[r, phantom, "="] \arrow[d, "l_*"'] & \pi_i(X) \arrow[d, "l"] \\ \pi_i(X_P) \arrow[r, phantom, "\cong"] & \pi_i(X)_P \end{tikzcd}$$

commutes where $l : G \to G_P$ *is the P-localization of a group G.*

(ii) *We have the homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "l_X"'] & Y \arrow[r, "g"] \arrow[d, "l_Y"'] & Z \arrow[d, "l_Z"'] \\ X_P \arrow[r, "f_P"] & Y_P \arrow[r, "g_P"] & Z_P \end{tikzcd}$$

for a given upper sequence; and if the upper sequence is a fibration or a cofibration, then so is the lower one up to homotopy equivalence.

LEMMA 1.4. *If* ΩX *is homotopy commutative, then so is* ΩX_P.

PROOF. We see the following by [4] and [11]: If Y is an H-space, then so is Y_P and $l : Y \to Y_P$ is an H-map; and if Y is homotopy commutative, then so is Y_P. Moreover, the induced map $\Omega l : \Omega X \to \Omega X_p$ of $l : X \to X_P$ is also the P-localization so that there exists a homotopy equivalence

$$r : (\Omega X)_P \to \Omega X_P \qquad \text{with} \qquad r l_{\Omega X} \simeq \Omega l$$

for $l_{\Omega X} : \Omega X \to (\Omega X)_P$ and r is an H-map since so is Ωl. These show the lemma.

REMARK 1.5. For a simply connected space X and a positive integer n, consider the space X_n obtained from X by attaching $(i+1)$-cell to kill the homotopy groups $\pi_i(X)$ for $i \geq n$. If ΩX is homotopy commutative, then so is ΩX_n.

2. Proof of Theorem 1 and a note on the non-simply connected case

We prove Theorem 1 in the introduction which says in particular that ΩS^n of the n-sphere S^n ($n \geq 2$) is homotopy commutative only if $n = 3, 7$.

PROOF OF THEOREM 1. Assume that ΩX of $X = \Sigma A$ is homotopy commutative. Then the Whitehead product $[\mathrm{id}_X, \mathrm{id}_X] \in \pi(\Sigma(A \wedge A), X)$ is defined and vanishes by Proposition 1.2 (ii). Hence there is a map $\mu: X \times X \to X$ of type $(\mathrm{id}_X, \mathrm{id}_X)$ by Proposition 1.2 (iii), which means that X is an H-space by definition. Therefore $X = \Sigma A$ is S^3 or S^7 by West[26].

In the rest of this section, we note on the case that X is not simply connected. In this case, we consider the universal covering

$$p: \tilde{X} \to X = \tilde{X}/\pi, \qquad \pi = p^{-1}(*) = \pi_1(X).$$

Here, π is the covering transformation group identified with $p^{-1}(*)$ and also with the fundamental group $\pi_1(X)$, by identifying $\alpha \in \pi$ with $\alpha = \alpha(*) \in p^{-1}(*)$ ($1(*) = *$ for the unit 1 of π) and with $\alpha = [pl_\alpha] \in \pi_1(X)$ by a fixed path $l_\alpha: (I; 0, 1) \to (\tilde{X}; *, \alpha)$ ($l_1 = *$).

LEMMA 2.1. *If ΩX is homotopy commutative, then so is $\Omega\tilde{X}$ and π is abelian.*

PROOF. Consider the space $L\tilde{X}$ of all paths $l: I \to \tilde{X}$ and its subspace $L(\tilde{X}; A, B) = \{l \in L\tilde{X} \mid l(0) \in A, l(1) \in B\}$. Then, p induces the homeomorphism

$$p_\alpha: L(\tilde{X}; *, \alpha) \approx (\Omega X)_\alpha = \{\omega \in \Omega X \mid [\omega] = \alpha\}, \qquad p_\alpha(l) = pl,$$

by the unique lifting property. Here $(\Omega X)_\alpha$ is the path component of ΩX containing pl_α; hence so is $(\Omega X)_1 \ni *$, and $(\Omega X)_1$ is an H-space by the loop multiplication. Also, $p_1: \Omega\tilde{X} = L(\tilde{X}; *, *) \approx (\Omega X)_1$ is an H-map. Thus, if ΩX is homotopy commutative, then so are $(\Omega X)_1$ and $\Omega\tilde{X}$.

LEMMA 2.2. (i) *We have the homotopy equivalence*

$$\varphi: \Omega\tilde{X} \times \pi \to \Omega X$$

given by $\varphi(\tilde{\omega}, \alpha) = p\tilde{\omega} \cdot pl_\alpha$ for $\tilde{\omega} \in \Omega\tilde{X}$, $\alpha \in \pi$.

(ii) *Assume that $\tilde{X}$ is an H-space with multiplication $\mu: \tilde{X} \times \tilde{X} \to \tilde{X}$ such that $\mu(\alpha, x) = \alpha(x)$ for $\alpha \in \pi$. Then φ is an H-map; hence the converse of Lemma 2.1 is also valid.*

PROOF. In the path space $L\tilde{X}$, we have l^{-1} and the path multiplication $l \cdot l'$ when $l(1) = l'(0)$, as usual.

(i) A homotopy inverse of φ is obtained by sending $\omega \in \Omega X$ to $(\tilde{\omega} \cdot l_\alpha^{-1}, \alpha) \in \Omega \tilde{X} \times \pi$, where $\alpha = [\omega]$, $\tilde{\omega} = p_\alpha^{-1}(\omega)$, and p_α is the homeomorphism in the above proof.

(ii) We have to show that the two maps of $\Omega\tilde{X} \times \Omega\tilde{X} \times \pi \times \pi$ to ΩX sending $(\tilde{\omega}_1, \tilde{\omega}_2, \alpha_1, \alpha_2)$ to $(p\tilde{\omega}_1 \cdot pl_{\alpha_1}) \cdot (p\tilde{\omega}_2 \cdot pl_{\alpha_2})$ and $p(\tilde{\omega}_1 \cdot \tilde{\omega}_2) \cdot pl_{\alpha_1\alpha_2}$, respectively, are homotopic. Here, $pl_{\alpha_1\alpha_2}$ and $pl_{\alpha_1} \cdot pl_{\alpha_2}$ are in the same path component $L(\tilde{X}; *, \alpha_1\alpha_2)$. Therefore, by the homotopy associativity of the path multiplication, it is sufficient to show that

$$\varphi \simeq \varphi' : \Omega\tilde{X} \times \pi \to \Omega X , \qquad \text{where} \qquad \varphi'(\tilde{\omega}, \alpha) = pl_\alpha \cdot p\tilde{\omega} .$$

Now, μ in the assumption gives us the map $\mu : L\tilde{X} \times L\tilde{X} \to L\tilde{X}$ given by $\mu(l, l')(t) = \mu(l(t), l'(t))$. Then, $\mu(l_\alpha, \tilde{\omega})$ can be deformed continuously to $\mu(l_\alpha \cdot *_\alpha, * \cdot \tilde{\omega}) = \mu(l_\alpha, *) \cdot \mu(*_\alpha, \tilde{\omega})$ ($*_\alpha$ is the constant path to α) and also to $\mu(* \cdot l_\alpha, \tilde{\omega} \cdot *) = \mu(*, \tilde{\omega}) \cdot \mu(l_\alpha, *)$, and so are these to $l_\alpha \cdot \alpha\tilde{\omega}$ and $\tilde{\omega} \cdot l_\alpha$, respectively, because $\mu | \tilde{X} \times * \simeq \mathrm{id}$ and $\mu(\alpha, x) = \alpha(x)$ by assumption. Since $p(l_\alpha \cdot \alpha\tilde{\omega}) = pl_\alpha \cdot p\tilde{\omega} = \varphi'(\tilde{\omega}, \alpha)$ and $p(\tilde{\omega} \cdot l_\alpha) = \varphi(\tilde{\omega}, \alpha)$, these show that $\varphi \simeq \varphi'$.

By these lemmas, we can prove Proposition 4 in the introduction.

Proof of Proposition 4. In this case, we have the universal covering $S^n \to S^n/Z_m = X$. If ΩX is homotopy commutative, then so is ΩS^n by Lemma 2.1; hence $n = 3$ or 7 by Theorem 1. Conversely, if $n = 3$ or 7, then the multiplication μ of quaternions or Cayley numbers on S^n satisfies the assumption of Lemma 2.2 (ii), which shows that ΩX is homotopy commutative. S^n/Z_m ($n = 3, 7$) is an H-space when $m = 2$ by the multiplication induced from μ on S^n of above, and is not an H-space when $m \geq 3$ by Browder [8] and [9].

3. The case when X has two cells

For the homotopy group $\pi_m(S^n)$ ($n \geq 2$), we use the following results (see Adams[1], James[14], Serre[20] and Toda[24]):

(3.1) (i) $\pi_m(S^n) = 0$ if $m < n$.

(ii) $\pi_n(S^n) \cong Z$ generated by $\iota_n = [\mathrm{id}]$.

(iii) $\pi_m(S^n)$ ($m > n$) is finite except for the case that $m = 2n - 1$ and n is even.

(iv) If n is even, then $\pi_{2n-1}(S^n) \cong Z \oplus F_n$. Here,

$$F_n = \mathrm{Im}\,[\Sigma : \pi_{2n-2}(S^{n-1}) \to \pi_{2n-1}(S^n)] = \{\alpha \,|\, H(\alpha) = 0\}$$

(Σ is the *suspension homomorphism* and $H(\alpha)$ is the *Hopf invariant* of α) is finite and $F_2 = 0$. Also the infinite cyclic part Z is generated by α_n such that

$$H(\alpha_n) = \pm 1 \quad \text{for} \quad n = 2, 4, 8, \qquad \text{and} \qquad H(\alpha_n) = \pm 2 \quad \text{otherwise} ;$$

and $H(\alpha) = \pm 1$ when α is the *Hopf class* η_2, ν_4 or σ_8, and $H([\iota_n, \iota_n]) = \pm 2$ for the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$.

(v) For odd n, $\Sigma: \pi_{2n-2}(S^{n-1}) \to \pi_{2n-1}(S^n)$ is epimorphic, and $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ is 0 if $n = 3, 7$, and is of order 2 if $n \neq 3, 7$. Moreover $[\iota_n, \iota_n]$ $(n > 2)$ is not contained in $2\pi_{2n-1}(S^n)$ unless $n + 1$ is a power of 2.

Berstein-Ganea [6] introduced the numerical invariant of homotopy type, nil ΩX, for any space X; and in particular, nil $\Omega X \leq 1$ means that ΩX is homotopy commutative. Hereafter we use this notation frequently for the simplicity.

In this section, we prove Theorem 2 in the introduction for the mapping cone

$$C(\alpha) = S^k \cup_\alpha e^n \qquad \text{of} \qquad \alpha \in \pi_{n-1}(S^k) \qquad (k, n \geq 2).$$

If $n - 1 < k$ or $\alpha = 0$, then $C(\alpha) \simeq S^k \vee S^n$. If $n - 1 = k$, then $\alpha = s\iota_k = \Sigma(s\iota_{k-1})$ and $C(\alpha) \simeq \Sigma C(s\iota_{k-1})$. Thus, in these cases, $C(\alpha)$ is the suspension type, and Theorem 2 follows from Theorem 1 by noticing that $C(s\iota_k)$ is contractible if and only if $s = \pm 1$.

Therefore, in the rest of this section, we assume that

$$n - 1 > k \geq 2 \qquad \text{and} \qquad \alpha \neq 0.$$

LEMMA 3.2. *If $\alpha \in \pi_{n-1}(S^k)$ is of finite order, then* nil $\Omega C(\alpha) > 1$.

PROOF. Consider the $\varnothing$-localizations $X_\varnothing$ of X and $\alpha_\varnothing: X_\varnothing \to Y_\varnothing$ of $\alpha: X \to Y$, which are the localizations with respect to the empty set $\varnothing$. Then, Lemma 1.3 shows that $\alpha_\varnothing: S^{n-1}_\varnothing \to S^k_\varnothing$ is null homotopic by assumption, and that $C(\alpha)_\varnothing \simeq C(\alpha_\varnothing)$. Therefore,

$$C(\alpha)_\varnothing \simeq C(\alpha_\varnothing) \simeq S^k_\varnothing \vee \Sigma S^{n-1}_\varnothing \simeq S^k_\varnothing \vee S^n_\varnothing \simeq (S^k \vee S^n)_\varnothing .$$

Consider the inclusions $i: S^k \to S^k \vee S^n$, $j: S^n \to S^k \vee S^n$. Then, $[i, j] \in \pi_{k+n-1}(S^k \vee S^n)$ is of infinite order, because $\partial: \pi_{k+n}(S^k \times S^n, S^k \vee S^n)$ $(\cong Z) \to \pi_{k+n-1}(S^k \vee S^n)$ is monomorphic and $[i, j]$ is the ∂-image of a generator. Therefore,

$$[li, lj] = l[i, j] \in \pi_{k+n-1}((S^k \vee S^n)_\varnothing)$$

is non-trivial for the $\varnothing$-localization $l: X \to X_\varnothing$. Thus $\Omega(S^k \vee S^n)_\varnothing$ is not homotopy commutative by Proposition 1.2 (ii), and so is $\Omega C(\alpha)_\varnothing$, which implies the lemma by Lemma 1.4.

For a space X, an even integer $n \geq 2$ and a map $h: S^n \to X$, consider the induced homomorphism $h^*: H^*(X) \to H^*(S^n)$ of the integral cohomology groups. Then:

PROPOSITION 3.3. *Assume that a cohomology class $u \in H^*(X)$ is mapped by h^* to a generator of $H^*(S^n) \cong Z$ and that*

$$[h, h] = 0 \qquad in \ \pi_{2n-1}(X).$$

Then $u^2 \in H^{2n}(X)$ is of infinite order. Moreover, if

$$u^2 = tv \qquad for\ some\ \ v \in H^{2n}(X) \qquad and \qquad t \in Z,$$

then $t = \pm 1$ or ± 2.

PROOF. Consider the Whitehead product $\alpha = [\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$. Then, $h_*(\alpha) = [h, h] = 0$ in $\pi_{2n-1}(S^n)$ by assumption. Therefore, there exists a map

$$\bar{h}: C = C(\alpha) = S^n \cup_\alpha e^{2n} \to X \qquad \text{with} \qquad \bar{h}i = h$$

for the inclusion $i: S^n \to C$. Consider the induced homomorphism

$$\bar{h}^*: H^*(X) \to H^*(C).$$

Here $i^*: H^n(C) \cong H^n(S^n)$, $H^{2n}(C) \cong Z$ and generators $e_j \in H^j(C)$ for $j = n,\ 2n$ satisfy

$$e_n{}^2 = \pm H(\alpha)e_{2n} = \pm 2e_{2n} \qquad \text{in} \ \ H^{2n}(C) \cong Z$$

by definition of the Hopf invariant. Therefore, $\bar{h}^*(u) = \pm e_n$ by assumption, and

$$\bar{h}^*(u^2) = (\bar{h}^*(u))^2 = e_n{}^2 = \pm 2e_{2n}.$$

Thus u^2 is of infinite order, since so is $\pm 2e_{2n}$. If $u^2 = tv$ for $v \in H^{2n}(X)$ and $t \in Z$, then $\bar{h}^*(v) = se_{2n}$ for some integer s and

$$\pm 2e_{2n} = \bar{h}^*(u^2) = \bar{h}^*(tv) = tse_{2n};$$

hence $ts = \pm 2$ and $|t| = 1$ or 2.

COROLLARY 3.4. *Assume that k is even and the Hopf invariant $H(\alpha)$ of $\alpha \in \pi_{2k-1}(S^k)$ is not equal to ± 1 and ± 2. Then $[i, i] \neq 0$ in $\pi_{2k-1}(C(\alpha))$ for the inclusion $i: S^k \to C(\alpha)$.*

PROOF. Assume that $[i, i] = 0$. Then the inclusion $i: S^k \to C(\alpha)$ satisfies the assumption of Proposition 3.3. On the other hand, $H^*(C(\alpha))$ has a Z-basis $\{1, e_k, e_{2k}\}$ ($\deg e_j = j$) with a relation $e_k{}^2 = H(\alpha)e_{2k}$, where $H(\alpha) \neq \pm 1,\ \pm 2$, which contradicts the conclusion of Proposition 3.3.

LEMMA 3.5. *Assume that $H(\alpha) = \pm 1$ for $\alpha \in \pi_{2k-1}(S^k)$ ($k = 2, 4, 8$). Then*

$$[p, p] \neq 0 \qquad in \ \ \pi_9(C(\alpha)) \qquad if \qquad k = 2,$$

where $p: S^5 \to C(\alpha) = C(\pm\eta_2) = CP(2)$ *is the projection; and*

$$[i\eta_k, i] \neq 0 \quad \text{in} \quad \pi_{2k}(C(\alpha)) \quad \text{if} \quad k = 4, 8,$$

where $i: S^k \to C(\alpha)$ *is the inclusion and* $\eta_k \in \pi_{k+1}(S^k)$.

PROOF. By (3.1) (iv), we have the following two cases:
(a) $k = 2$ and $\alpha = \pm\eta_2$.
(b) $k = 4, 8$ and $\alpha = \pm h + \Sigma\beta$ ($h = \nu_4, \sigma_8$) for some $\beta \in \pi_{2k-2}(S^{k-1})$.

Case (a): Then $C(\alpha)$ is homotopy equivalent to the complex projective plane $CP(2)$. From the homotopy exact sequence associated with the fibration $S^1 \to S^5 \xrightarrow{p} CP(2)$, we see that

$$p_*: \pi_j(S^5) \cong \pi_j(CP(2)) \qquad \text{for } j \geq 3,$$

since $\pi_j(S^1) = 0$ for $j \geq 2$. Thus we have

$$[p_*\iota_5, p_*\iota_5] = p_*[\iota_5, \iota_5] \neq 0 \qquad \text{in } \pi_9(CP(2)),$$

because $[\iota_5, \iota_5] \neq 0$ in $\pi_9(S^5)$ by (3.1) (v).

Case (b): From the theorem of Blakers-Massey, we obtain the exact sequence

$$\pi_{2k}(S^{2k-1}) \xrightarrow{\alpha_*} \pi_{2k}(S^k) \xrightarrow{i_*} \pi_{2k}(C(\alpha)),$$

where $i: S^n \to C(\alpha)$ is the inclusion.

Let η_{2k-1} be a generator of $\pi_{2k}(S^{2k-1}) \cong Z_2$. Also let h' be a generator of the cyclic group $\pi_{2k-2}(S^{k-1})$ ($k = 4, 8$), i.e., $h' = \nu'$ ($k = 4$) or σ' ($k = 8$) in [24]. Then, $\alpha = \pm h + \Sigma\beta = \pm h + b\Sigma h'$ for some integer b, and

$$\alpha_*(\eta_{2k-1}) = (\pm h + b\Sigma h')\eta_{2k-1} = \pm h\eta_{2k-1} + b(\Sigma h')\eta_{2k-1}$$

and $h\eta_{2k-1} \neq 0$ by [24; Prop.5.8 and 7.1]. On the other hand,

$$[\eta_k, \iota_k] = (\Sigma h')\eta_{2k-1} \qquad \text{by [24; (5.11) and p.63]}.$$

These show that $[\eta_k, \iota_k] \notin \operatorname{Im} \alpha_* = \operatorname{Ker} i_*$ in the above exact sequence.

Thus

$$[i\eta_k, i] = i_*[\eta_k, \iota_k] \neq 0 \qquad \text{in } \pi_{2k}(C(\alpha)).$$

By Corollary 3.4 and Lemma 3.5, nil $\Omega C(\alpha) > 1$ in these cases according to Proposition 1.2 (ii).

To consider the case that $H(\alpha) = \pm 2$, we prepare the following

PROPOSITION 3.6. *Assume that the cohomology $H^*(X; k)$ with coefficient in a field k for $* < 3n$ has a k-basis*

$$\{1, e_n, e_{2n}\} \quad (\deg e_j = j) \qquad \textit{with} \qquad e_n{}^2 = 0\,,$$

and $e_n e_{2n} = 0$ in $H^{3n}(X; k)$, for some $n \geq 2$. Then, nil $\Omega X > 1$.

PROOF. In this proof, we omit the coefficient field k for the simplicity. Consider the projection $p: L = L(X; *, X) \to X$, $p(l) = l(1)$, the adjoint d: $\Sigma\Omega X \to X$ of $\mathrm{id}_{\Omega X}$ and the map $d': C\Omega X \to L$, $d'(\omega, t)(s) = \omega(ts)$. Then, $pd' = d\,\mathrm{pr}$ and we have the commutative diagram

$$\begin{array}{ccccc}
H^j(X) & \xrightarrow{p^*} & H^j(L, \Omega X) & \underset{\cong}{\xleftarrow{\delta^*}} & H^{j-1}(\Omega X) \\
\Big\downarrow{\scriptstyle d^*} & & \Big\downarrow{\scriptstyle d'^*} & & \Big\| \\
H^j(\Sigma\Omega X) & \underset{\cong}{\xrightarrow{\mathrm{pr}^*}} & H^j(C\Omega X, \Omega X) & \underset{\cong}{\xleftarrow{\delta^*}} & H^{j-1}(\Omega X)\,.
\end{array}$$

By assumption and by studying the cohomology spectral sequence for the fibration $\Omega X \to L \to X$ (cf. [20]), we see that $H^*(\Omega X)$ for $* \leq 3n - 3$ has k-basis $\{1, \sigma(e_n), e', \sigma(e_{2n}), \sigma(e_n)e'\}$ ($\deg e' = 2n - 2$), where $\sigma = (\delta^*)^{-1}p^*$ is the suspension homomorphism. Thus, $H^*(\Sigma\Omega X)$ for $* \leq 3n - 2$ has k-basis

$$\{1, a_n, b_{2n-1}, a_{2n}, b_{3n-2}\} \qquad (\deg c_j = j)\,,$$

where the suspension isomorphism $\sigma^* = (\delta^*)^{-1}\,\mathrm{pr}^*$ maps a_j to $\sigma(e_j)$, b_{2n-1} to e', and b_{3n-2} to $\sigma(e_n)e'$. Hence, the above commutative diagram shows that

$$d^*(e_j) = a_j \qquad \text{for} \qquad j = n \text{ and } 2n\,.$$

Suppose that ΩX is homotopy commutative. Then, by the result of Stasheff stated in Proposition 1.2 (iv), there exists a map

$$\psi: \Sigma\Omega X \times \Sigma\Omega X \to X$$

such that $\psi|\Sigma\Omega X \times * \simeq d \simeq \psi|* \times \Sigma\Omega X$. Consider the homomorphism

$$\psi^*: H^*(X) \to H^*(\Sigma\Omega X) \otimes H^*(\Sigma\Omega X)$$

induced by ψ. Then, by the above results on the cohomologies, we see that

$$\psi^*(e_n) = d^*(e_n) \otimes 1 + 1 \otimes d^*(e_n) = a_n \otimes 1 + 1 \otimes a_n\,,$$

$$\begin{aligned}
\psi^*(e_{2n}) &= d^*(e_{2n}) \otimes 1 + 1 \otimes d^*(e_{2n}) + g d^*(e_n) \otimes d^*(e_n) \\
&= a_{2n} \otimes 1 + 1 \otimes a_{2n} + g a_n \otimes a_n
\end{aligned}$$

for some $g \in k$, and so

$$0 = \psi^*(e_n e_{2n}) = \psi^*(e_n)\psi^*(e_{2n}) = a_n \otimes a_{2n} + a_{2n} \otimes a_n \neq 0\,,$$

which is a contradiction. Thus nil $\Omega X > 1$.

COROLLARY 3.7. *If* $H(\alpha) = \pm 2$ *for* $\alpha \in \pi_{2k-1}(S^k)$ $(k : \textit{even})$, *then* nil $\Omega C(\alpha) > 1$.

PROOF. $H^*(C(\alpha); Z_2)$ has a Z_2-basis $\{1; e_k, e_{2k}\}$ $(\deg e_j = j)$ and $e_k^2 = H(\alpha)e_{2k} = 0$ by assumption. Thus the result is a special case of Proposition 3.6.

Thus Theorem 2 is proved completely.

4. The case when X is a sphere bundle over a sphere

By Steenrod[22; 18.5], the k-sphere bundles over the n-sphere S^n with group O_{k+1} are classified, up to bundle equivalence, by equivalence classes of elements of $\pi_{n-1}(O_{k+1})$ under the operations of $\pi_0(O_{k+1})$. Hereafter, we denote by E_γ the k-sphere bundle over S^n which corresponds to the equivalence class of

$$\gamma \in \pi_{n-1}(O_{k+1}) \qquad (k, n \geq 2)\,,$$

which is called the characteristic class of E_γ, and by

$$p : E_\gamma \to S^n \qquad \text{and} \qquad i : S^k = p^{-1}(*) \to E_\gamma$$

the projection and the inclusion, respectively. Then, we have the following exact sequence

$$\cdots \longrightarrow \pi_m(S^k) \xrightarrow{i_*} \pi_m(E_\gamma) \xrightarrow{p_*} \pi_m(S^n) \xrightarrow{\Delta} \pi_{m-1}(S^k) \longrightarrow \cdots;$$

and there holds the equality

$$\alpha = \Delta(\iota_n) = q_*(\gamma) \in \pi_{n-1}(S^k) \tag{4.1}$$

for the homomorphism $q_* : \pi_{n-1}(O_{k+1}) \to \pi_{n-1}(S^k)$ induced by the natural projection $q : O_{k+1} \to O_{k+1}/O_k = S^k$. Also, for the boundary homomorphism Δ, we have the formula

$$\Delta(\Sigma\beta) = \Delta(\iota_n)\beta = \alpha\beta \qquad \text{for any } \beta \in \pi_{m-1}(S^{n-1})\,. \tag{4.2}$$

In particular, (4.1) shows that E_γ admits a cross section if and only if $\alpha = 0$.

In the first place, we consider the case that

$$\alpha = 0\,, \qquad \text{e.g.,} \qquad n \leq k\,, \qquad \text{or} \qquad n = k+1 \text{ and } k \text{ is even}\,.$$

In fact, $\pi_{n-1}(S^k) = 0$ if $n \leq k$, and $q_* = 0 : \pi_k(O_{k+1}) \to \pi_k(S^k)$ if k is even by [22; 23.7].

PROPOSITION 4.3. *Assume that a fibration (E, p, B) with fiber $F = p^{-1}(*)$ admits a cross section $s: B \to E$, and B and F are simply connected. Then*:

(i) *$\mu(\Omega i \times \Omega s): \Omega F \times \Omega B \to \Omega E$ is a homotopy equivalence, where $i: F \to E$ is the inclusion and μ is the loop multiplication on ΩE.*

(ii) *If ΩE is homotopy commutative, then so are ΩB and ΩF.*

PROOF. (i) We see that $\mu(\Omega i \times \Omega s)$ induces the isomorphisms of the homotopy groups, which implies (i) by J. H. C. Whitehead's theorem.

(ii) If ΩE is homotopy commutative, then we can see that $\mu(\Omega i \times \Omega s)$ is an H-map. By (i), $\Omega F \times \Omega B$ is also homotopy commutative and so are ΩB and ΩF.

COLLORARY 4.4. *For the bundle E_γ with $\alpha = q_*(\gamma) = 0 \in \pi_{n-1}(S^k)$, ΩE_γ is not homotopy commutative unless $\{k, n\} \subset \{3, 7\}$.*

PROOF. This follows from Proposition 4.3 and Theorem 1.

From now on, we consider E_γ for $\gamma \in \pi_{n-1}(O_{k+1})$ such that

$$n > k \geq 2 \qquad \text{and} \qquad q_*(\gamma) = \Delta(\iota_n) = \alpha \neq 0 \qquad \text{in } \pi_{n-1}(S^k)\,.$$

By James-Whitehead[15], the bundle E_γ admits a CW-structure

$$E_\gamma = S^k \cup_\alpha e^n \cup e^{n+k} = C(\alpha) \cup_\beta e^{n+k} = C(\beta)\,.$$

Here $\beta: S^{n+k-1} \to C(\alpha)$ is the attaching map of the top cell e^{n+k} so that

$$\beta = \bar{\beta}|S^{n+k-1} \qquad \text{for} \qquad \bar{\beta}: (V^{n+k}, S^{n+k-1}) \to (E_\gamma, C(\alpha))\,,$$

where $\bar{\beta}$ is the characteristic map for e^{n+k}. Also,

$$\alpha = \bar{\alpha}|S^{n-1} \qquad \text{for} \qquad \bar{\alpha}: (V^n, S^{n-1}) \to (C(\alpha), S^{n-1})$$

where $\bar{\alpha}$ is the one for e^n, and there holds the following

PROPOSITION 4.5 ([15]). (i) *$S^k = p^{-1}(*)$ and $p|C(\alpha): (C(\alpha), S^k) \to (S^n, *)$ is a relative homeomorphism for the projection $p: E_\gamma \to S^n$.*

(ii) $$j_*(\beta) = [\bar{\alpha}, \iota_k] \qquad \text{for} \qquad j_*: \pi_{n+k-1}(C(\alpha)) \to \pi_{n+k-1}(C(\alpha), S^k)\,,$$

where j is the inclusion and $[\bar{\alpha}, \iota_k]$ is the relative Whitehead product.

LEMMA 4.6. *If $k < n - 1$, then $[\bar{\alpha}, \iota_k]$ and β are of infinite order.*

PROOF. Consider the homotopy exact sequence

$$\cdots \to \pi_m(E_\gamma, S^k) \to \pi_m(E_\gamma, C(\alpha)) \xrightarrow{\partial'} \pi_{m-1}(C(\alpha), S^k) \to \pi_{m-1}(E_\gamma, S^k) \to \cdots$$

of the triple $(E_\gamma, C(\alpha), S^k)$, where

$$\partial' = j_*\partial : \pi_m(E_\gamma, C(\alpha)) \to \pi_{m-1}(C(\alpha)) \to \pi_{m-1}(C(\alpha), S^k)$$

and $\pi_{n+k}(E_\gamma, C(\alpha)) \cong Z$ generated by $\bar{\beta}$. Then, $\pi_{n+k}(E_\gamma, S^k)$ is finite, because $p_* : \pi_m(E_\gamma, S^k) \cong \pi_m(S^n)$ and $n < n+k < 2n-1$ by assumption. Thus ∂' for $m = n+k$ is monomorphic; hence $\partial'(\bar{\beta}) = j_*(\beta) = [\bar{\alpha}, \iota_k]$ is of infinite order, and so is also β.

LEMMA 4.7. *Assume that n or k is even, and that $\alpha = q_*(\gamma) \in \pi_{n-1}(S^k)$ is of finite order. Then,*

$$[\rho, \rho] \neq 0 \quad \text{in } \pi_{2n-1}(E_\gamma) \quad \text{for some } \rho \in \pi_n(E_\gamma) \text{ if } n \text{ is even},$$

and

$$[i, i] \neq 0 \quad \text{in } \pi_{2k-1}(E_\gamma) \quad \text{if } k \text{ is even}.$$

PROOF. Consider the exact sequence

$$\cdots \longrightarrow \pi_m(S^k) \xrightarrow{i_*} \pi_m(E_\gamma) \xrightarrow{p_*} \pi_m(S^n) \xrightarrow{\Delta} \pi_{m-1}(S^k) \longrightarrow \cdots.$$

Assume that n is even and $s\alpha = 0$ in $\pi_{n-1}(S^k)$ for $s \neq 0$. Then, from (4.1) and the exactness,

$$\Delta(s\iota_n) = s\Delta(\iota_n) = s\alpha = 0 \qquad \text{and so} \qquad s\iota_n = p_*(\rho)$$

for some $\rho \in \pi_n(E_\gamma)$. Thus

$$p_*([\rho, \rho]) = [p_*(\rho), p_*(\rho)] = s^2[\iota_n, \iota_n] \neq 0 \qquad \text{in } \pi_{2n-1}(S^n)$$

by (3.1) (iv). Thus $[\rho, \rho] \neq 0$ in $\pi_{2n-1}(E_\gamma)$.

Assume that k is even. Then $[\iota_k, \iota_k] \in \pi_{2k-1}(S^k)$ is of infinite order by (3.1) (iv). On the other hand, the image of

$$\Delta : \pi_{2k}(S^n) \to \pi_{2k-1}(S^k)$$

is finite because $\pi_{2k}(S^n)$ is finite if $n \neq 2k$ by (3.1), and Im Δ is generated by $\Delta(\iota_n) = \alpha$ if $n = 2k$. Therefore $[\iota_k, \iota_k]$ is not contained in Im Δ. Hence, by the above exact sequence, we have

$$[i, i] = i_*[\iota_k, \iota_k] \neq 0 \qquad \text{in } \pi_{2k-1}(E_\gamma).$$

Thus we have nil $\Omega E_\gamma > 1$ in case of Lemma 4.7 by Proposition 1.2 (ii).

LEMMA 4.8. *Assume that $n = k+1 \neq 4$ and $\alpha = q_*(\gamma) \neq 0$ in $\pi_k(S^k)$. Then* nil $\Omega E_\gamma > 1$.

PROOF. k is odd by assumption, as is noticed in front of Proposition 4.3.

Put $\alpha = s\iota_k$ ($s \neq 0$), and consider the homotopy exact sequence

$$\cdots \longrightarrow \pi_m(C(\alpha)) \xrightarrow{j_*} \pi_m(C(\alpha), S^k) \xrightarrow{\partial} \pi_{m-1}(S^k) \longrightarrow \pi_{m-1}(C(\alpha), S^k) \longrightarrow \cdots$$

of the pair $(C(\alpha), S^k)$ for $C(\alpha) = S^k \cup_\alpha e^{k+1}$. Then, by Proposition 4.5 (ii) and [7] on the relation of the relative Whitehead product and the absolute one, we see that

$$0 = \partial j_*(\beta) = \partial([\bar{\alpha}, \iota_k]) = -[\partial\bar{\alpha}, \iota_k] = -[\alpha, \iota_k] = -s[\iota_k, \iota_k]$$

in $\pi_{2k-1}(S^k)$. Thus s is even if $k \neq 3, 7$ by (3.1) (v).

(a) The case that s is even: By Blakers-Massey[7], $\pi_{n+k-1}(C(\alpha), S^k)$ is the direct sum

$$\operatorname{Im}[\bar{\alpha}_* : \pi_{n+k-1}(V^n, S^{n-1}) \to \pi_{n+k-1}(C(\alpha), S^k)] \oplus Z \qquad (k = n-1)$$

where Z is generated by $[\bar{\alpha}, \iota_k]$. Consider $\partial : \pi_{2k}(V^{k+1}, S^k) \cong \pi_{2k-1}(S^k)$. Then, for any $\rho \in \pi_{2k}(V^{k+1}, S^k)$, we have

$$\partial(\bar{\alpha}_*(\rho)) = (\partial\bar{\alpha})_*(\partial\rho) = \alpha_*(\partial\rho) = (s\iota_k)_*(\partial\rho) = s\partial\rho$$

since $\partial\rho \in \pi_{2k-1}(S^k) = \Sigma\pi_{2k-2}(S^{k-1})$ by (3.1)(v). Thus,

$$\operatorname{Im}[\partial : \pi_{2k}(C(\alpha), S^k) \to \pi_{2k-1}(S^k)] = s\pi_{2k-1}(S^k)$$

by the above direct sum decomposition, since $\partial([\bar{\alpha}, \iota_k]) = 0$ as is shown in the above. Since s is even, this and (3.1) (v) show that

$$[\iota_k, \iota_k] \notin \operatorname{Im}\partial \qquad \text{if } k+1 \text{ is not a power of } 2\,.$$

On the other hand, we have the commutative diagram

$$\begin{tikzcd}
\pi_{2k}(S^{k+1}) \arrow[r, "\Delta"] & \pi_{2k-1}(S^k) \arrow[r, "i_*"] & \pi_{2k-1}(E_\gamma) \\
\pi_{2k}(E_\gamma, S^k) \arrow[u, "p_*", "\cong"'] & \pi_{2k}(C(\alpha), S^k)\,, \arrow[l, "t_*"'] \arrow[u, "\partial"] &
\end{tikzcd}$$

where t is the inclusion map and t_* is epimorphic since $E_\gamma = C(\alpha) \cup e^{2k+1}$. Therefore $\operatorname{Im}\Delta = \operatorname{Im}\partial \not\ni [\iota_k, \iota_k]$ and

$$[i, i] = i_*[\iota_k, \iota_k] \neq 0 \qquad \text{in } \pi_{2k-1}(E_\gamma)$$

if $k+1$ is not a power of 2.

Now, consider the case that $k+1 = n$ is a power of 2. This proof can be applicable in the case $k = 3, 7$ and $\alpha = s\iota_k$ (s is even).

We consider the exact sequence

$$\cdots \longrightarrow \pi_m(S^k) \xrightarrow{i_*} \pi_m(E_\gamma) \xrightarrow{p_*} \pi_m(S^{k+1}) \xrightarrow{\Delta} \pi_{m-1}(S^k) \longrightarrow \cdots.$$

Then, for the generator $\eta_{k+1} = \Sigma^2\eta_{k-1} \in \pi_{k+2}(S^{k+1}) \cong Z_2$, we see that

$$\begin{aligned} \Delta(\eta_{k+1}) &= \Delta(\iota_{k+1})\eta_k && \text{by (4.2)} \\ &= \alpha\eta_k = s\eta_k && \text{by (4.1)} \\ &= 0 \end{aligned}$$

since s is even. Thus there exists an element $\rho \in \pi_{k+2}(E_\gamma)$ such that $\eta_{k+1} = p_*(\rho)$. Therefore, by Hilton[10],

$$p_*([\rho, \rho]) = [\eta_{k+1}, \eta_{k+1}] \neq 0 \qquad \text{in } \pi_{2k+3}(S^{k+1})\,;$$

hence $[\rho, \rho] \neq 0$ in $\pi_{2k+3}(E_\gamma)$ and nil $\Omega E_\gamma > 1$.

(b) The case that $k = n - 1 = 7$, $\alpha = s\iota_7$ and s is odd: We consider the set P of primes p with $(p, s) = 1$. Then $2 \in P$ and the P-localization $\alpha_P : S^7_P \to S^7_P$ is a homotopy equivalence. Thus $C(\alpha_P)$ has the homotopy type of a point $*$. Therefore $\beta_P \simeq * : S^{14}_P \to C(\alpha_P) \simeq C(\alpha)_P \simeq *$ and

$$(E_\gamma)_P \simeq C(\beta_P) \simeq S^{15}_P\,.$$

For the P-localization $l : S^m \to S^m_P$ of S^m, we note that

$$[l\iota_m, l\iota_m] = l[\iota_m, \iota_m] \neq 0 \qquad \text{in } \pi_{2m-1}(S^m) \text{ if } m \neq 3, 7 \text{ and } 2 \in P\,.$$

In fact, this is shown by Lemma 1.3, since $2 \in P$ and the order of $[\iota_m, \iota_m]$ is 2 or infinite by (3.1).

Therefore $[l\iota_{15}, l\iota_{15}] \neq 0$ in $\pi_{29}(S^{15}_P)$ in the above case, and ΩS^{15}_P is not homotopy commutative by Proposition 1.2 (ii), and so is $\Omega(E_\gamma)_P$, which implies the lemma by Lemma 1.4.

LEMMA 4.9. *Assume that* $n = 2k \geq 8$, k *is even and* $\alpha = q_*(\gamma) \in \pi_{2k-1}(S^k)$ *satisfies* $H(\alpha) \neq 0$. *Then* $[i, i] \neq 0$ *in* $\pi_{2k-1}(E_\gamma)$.

PROOF. Consider the case $H(\alpha) \neq \pm 1, \pm 2$. Then,

$$[i', i'] \neq 0 \qquad \text{in } \pi_{2k-1}(C(\alpha))$$

for the inclusion $i' : S^k \to C(\alpha)$, by Corollary 3.4. Also,

$$\pi_m(C(\alpha)) \cong \pi_m(E_\gamma) \qquad \text{for } m \leq 3k - 2$$

by the homomorphism induced by the inclusion $C(\alpha) \to E_\gamma = C(\alpha) \cup e^{3k}$. Therefore $[i, i] \neq 0$ in $\pi_{2k-1}(E_\gamma)$.

Now, we show that the assumption implies $H(\alpha) \neq \pm 1, \pm 2$.

By Barratt-Mahowald[5] and Krishnarao[17], we see that $\pi_{2k-1}(O_{k+1})$ for even $k \geq 10$ is the direct sum of a finite group and an infinite cyclic group generated by θ and

$$q_*(\theta) = \lambda[\iota_k, \iota_k] + \theta' \qquad \text{for } \lambda = \varepsilon(k)((k-1)!)/8 \geq 2$$

for the homomorphism $q_* : \pi_{2k-1}(O_{k+1}) \to \pi_{2k-1}(S^k)$ induced by the projection q, where $\theta' \in F_k$ in (3.1) (iv) and $\varepsilon(k) = 1$ or 2 according as $k/2$ is even or odd.

Consider the case $k = 4, 6, 8$. By [16] (cf. the table of $\pi_m(O_k)$ and $\pi_m(S^k)$ in [27; II, pp.1415–7]), we can see that $\pi_{2k-1}(O_{k+1})$ is the direct sum of a finite group and an infinite cyclic group generated by θ which satisfies

$$q_*(\theta) = \begin{cases} 6[\iota_4, \iota_4] + \theta' & \text{for } k = 4 \\ 2[\iota_6, \iota_6] & \text{for } k = 6 \\ (7!/4)[\iota_8, \iota_8] + \theta' & \text{for } k = 8 \end{cases}$$

where $\theta' \in F_k$. Therefore,

$$H(\alpha) \neq \pm 1, \pm 2 \qquad \text{if} \qquad H(\alpha) \neq 0 \text{ and } k \text{ is even} \geq 4\,.$$

since $\alpha = q_*(\gamma)$; and the lemma is proved.

REMARK 4.10. Let X be a CW-complex obtained from $C(\alpha) = S^k \cup_\alpha e^n$ by attaching r-cells with $r \geq m$ for some m. Assume that $[\xi, \zeta] \neq 0$ in $C(\alpha)$ for $\xi \in \pi_a(C(\alpha))$ and $\zeta \in \pi_b(C(\alpha))$ and $a + b < m$. Then $[j\xi, j\zeta] \neq 0$ in X for the inclusion $j : C(\alpha) \to X$, because $j_* : \pi_s(C(\alpha)) \cong \pi_s(X)$ for $s < m$, and we see that $\operatorname{nil} \Omega X > 1$.

LEMMA 4.11. *Let E_γ be the bundle with $\alpha = q_*(\gamma) \in \pi_{n-1}(S^k)$ ($k < n-1$).*
(i) *If $n > 2k$ and $k \neq 3, 7$, then $[i, i] \neq 0$ in $\pi_{2k-1}(E_\gamma)$.*
(ii) *If the order of α is odd and $(k, n) \neq (3, 7)$, then* $\operatorname{nil} \Omega E_\gamma > 1$.

PROOF. (i) Consider the homotopy exact sequence

$$\cdots \longrightarrow \pi_{2k}(S^n) \xrightarrow{\Delta} \pi_{2k-1}(S^k) \xrightarrow{i_*} \pi_{2k-1}(E_\gamma) \longrightarrow \cdots.$$

Then i_* is a monomorphism since $\pi_{2k}(S^n) = 0$. Thus $i_*[\iota_k, \iota_k] = [i, i] \neq 0$ in $\pi_{2k-1}(E_\gamma)$.

(i) We consider the 2-localization $(E_\gamma)_2$ of E_γ. Then the 2-localization α_2 of α is null homotopic. Thus the fibration $S^k_2 \to (E_\gamma)_2 \to S^n_2$ has a cross section. Therefore we have $\operatorname{nil} \Omega E_\gamma > 1$ by Proposition 4.3 and 1.2 (ii), because $[l\iota_m, l\iota_m] \neq 0$ in $\pi_{2m-1}(S^m_2)$ for $m \neq 3, 7$ as noted in the case (b) of the proof of Lemma 4.8.

LEMMA 4.12. *Assume that $\alpha = q_*(\gamma) \in \pi_{n-1}(S^k)$ ($k < n-1$) satisfies the following condition* (1) *or* (2):

(1) $[\iota_k, \iota_k]$ *is not contained in the image of* $\alpha_*: \pi_{2k-1}(S^{n-1}) \to \pi_{2k-1}(S^k)$ *and* $k \neq 3, 7$,

(2) $\alpha = 2\alpha'$ *for some* $\alpha' \in \pi_{n-1}(S^k)$, $k+1$ *is not a power of* 2 *and* $k \geq 4$.

Then $[i, i] \neq 0$ *in* $\pi_{2k-1}(E_\gamma)$.

PROOF. In the exact sequence

$$\cdots \longrightarrow \pi_{2k}(S^n) \xrightarrow{\Delta} \pi_{2k-1}(S^k) \xrightarrow{i_*} \pi_{2k-1}(E_\gamma) \longrightarrow \cdots,$$

we see by (4.2) and the suspension theorem that

$$\alpha_* = \Delta\Sigma : \pi_{2k-1}(S^{n-1}) \cong \pi_{2k}(S^n) \to \pi_{2k-1}(S^k)$$

since $k < n-1$.

Case (1): In this case, we have

$$\Delta(\pi_{2k}(S^n)) = \alpha_*(\pi_{2k-1}(S^{n-1})) \not\ni [\iota_k, \iota_k].$$

Therefore $[i, i] = i_*[\iota_k, \iota_k] \neq 0$ in $\pi_{2k-1}(E_\gamma)$, by (3.1) (v) and the exactness.

Case (2): In this case, we have

$$\Delta(\pi_{2k}(S^n)) = \alpha_*(\pi_{2k-1}(S^{n-1})) \subset 2\pi_{2k-1}(S^k)$$

since $\alpha = 2\alpha'$ and $\Sigma : \pi_{2k-2}(S^{n-2}) \to \pi_{2k-1}(S^{n-1})$ is epimorphic. On the other hand, $[\iota_k, \iota_k] \notin 2\pi_{2k-1}(S^k)$, by (3.1) (v), since $k+1$ is not a power of 2. Thus we obtain $\Delta(\pi_{2k}(S^n)) \not\ni [\iota_k, \iota_k]$. Therefore $[i, i] \neq 0$ in $\pi_{2k-1}(E_\gamma)$.

REMARK 4.13. $[\iota_k, \iota_k]$ does not lie in the image of $\alpha_*: \pi_{2k-1}(S^{n-1}) \to \pi_{2k-1}(S^k)$ for the following $\alpha \in \pi_{n-1}(S^k)$:

η_k for $k \equiv 1 \bmod 4$, η_{11}, η_{15}; ν_k, ν_k^3, μ_k, $\eta_k\varepsilon_{k+1}$ for $k = 11$, 13 and 15; ζ_k for $k = 13$ and 15, (the notation are the ones in [24]).

LEMMA 4.14. *Assume that* $n \equiv 0, 1 \bmod 4$ *and* $n \neq 5$, *and that* $\alpha = q_*(\gamma) \in \pi_{n-1}(S^k)$ *satisfies*

$$\alpha\eta_{n-1} = 0 \qquad \text{for } \eta_{n-1} \in \pi_n(S^{n-1}) \cong Z_2, \qquad \text{e.g., } \alpha \in 2\pi_{2n-1}(S^k).$$

Then $[\tilde{\eta}_n, \tilde{\eta}_n] \neq 0$ *in* $\pi_{2n+1}(E_\gamma)$ *for any* $\tilde{\eta}_n \in \pi_{n+1}(E_\gamma)$.

PROOF. Consider the exact sequence

$$\cdots \longrightarrow \pi_{n+1}(S^k) \xrightarrow{i_*} \pi_{n+1}(E_\gamma) \xrightarrow{p_*} \pi_{n+1}(S^n) \xrightarrow{\Delta} \pi_n(S^k) \longrightarrow \cdots.$$

Then,

$$\begin{aligned} \Delta(\eta_n) = \Delta(\Sigma\eta_{n-1}) = \Delta(\iota_n)\eta_{n-1} \qquad & \text{by (4.2)} \\ = \alpha\eta_{n-1} = 0 \qquad & \text{by (4.1)}. \end{aligned}$$

Thus, there exists an element $\tilde{\eta}_n \in \pi_{n+1}(E_\gamma)$ such that $p_*(\tilde{\eta}_n) = \eta_n$.

Therefore, by Hilton [10],

$$p_*([\tilde{\eta}_n, \tilde{\eta}_n]) = [\eta_n, \eta_n] \neq 0 \qquad \text{in } \pi_{2n+1}(S^n);$$

hence $[\tilde{\eta}_n, \tilde{\eta}_n] \neq 0$ in $\pi_{2n+1}(E_\gamma)$.

LEMMA 4.15. $\mathrm{nil}\, \Omega X > 1$ *for any 2-sphere bundle over* S^4 *such that* $\alpha = q_*(\gamma) \neq \eta_2 \in \pi_3(S^2)$.

PROOF. Let E_m denote the bundle E_γ with $\alpha = m\eta_2$. Then E_m has a CW-structure

$$E_m \simeq S^2 \cup_{m\eta_2} e^4 \cup e^6,$$

where $\eta_2 : S^3 \to S^2$ is the Hopf map.

From the homotopy exact sequence associated with the bundle E_m and (4.1), we have

$$\pi_2(E_m) \cong Z \qquad \text{generated by } i_* \iota_2 = i,$$

$$\pi_3(E_m) \cong Z_m \qquad \text{generated by } i_* \eta_2 = i\eta_2 \qquad (=0 \text{ if } m = \pm 1),$$

where $i : S^2 \to E_m$ is the inclusion.

Let $m \neq \pm 1, \pm 2$. Then

$$[i_* \iota_2, i_* \iota_2] = i_*[\iota_2, \iota_2] = i_*(2\eta_2) = 2i_*\eta_2 \neq 0 \qquad \text{in } \pi_3(E_m).$$

When $m = \pm 2$, $[\tilde{\eta}_4, \tilde{\eta}_4] \neq 0$ in $\pi_9(E_m)$ by Lemma 4.14.

When $m = \pm 1$, E_m is homotopy equivalent to the complex projective space $CP(3)$. By Stasheff [21; Th.1.18], $\Omega CP(3)$ is homotopy commutative.

Therefore ΩE_m is homotopy commutative if and only if $m = \pm 1$.

Now, Theorem 3 in the introduction is proved by Corollary 4.4 and Lemmas 4.7–15.

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