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Derivation of a porous medium equation from many Markovian particles and the propagation of chaos

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§ 0. Introduction

We consider the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \triangle (u^{\alpha}), \qquad (t > 0,\ x \in \boldsymbol{R}^d), \tag{1}$$

for a given real number $\alpha > 1$, where $\triangle$ is the d-dimensional Laplacian. This equation was introduced by Muskat as an (empirical) equation of the density u of a gas flowing through a homogeneous porous medium and is called a *porous medium equation* ([1]). Analogously to Kac's approach to a Boltzmann equation [10] we introduce a Markov system of many particles as a simple model of the gas. The porous medium equation (1) is derived from the equation for the empirical density of the number of particles. We prove that a macroscopic limit of the empirical density is a solution of (1). We also prove Kac-McKean's propagation of chaos for the system as follows.

Let $S_h = \{(hz_1, \cdots, hz_d): z_1, \cdots, z_d \in \boldsymbol{Z}\}$ be a d-dimensional lattice of the width $h > 0$, and $\tau > 0$ be a unit time. We define a system of N-particles on S_h with the following stochastic interaction. For each integer $n \geq 0$, let

$$X_n^{N,1}, \cdots, X_n^{N,N} \in S_h$$

denote the positions of N-particles at time $n\tau$. If the number of particles at a position $x (\in S_h)$ is $m (\geq 1)$, then each particle at x jumps to one of the nearest neighbor lattice points $x \pm (0, \cdots, 0, \underset{(j)}{h}, 0, \cdots, 0)$ $(j = 1, \cdots, d)$ with probability $\{m/N\}^{\alpha-1}/2d$ and stops on x with probability $1 - \{m/N\}^{\alpha-1}$ independently of the other particles. Thus all N-particles can move at the same time (for detail, see (M. 1), (M. 2) and Remark (3) in § 1).

We consider a macroscopic behaviour of this model. Let $\delta(x, y)$ be Kronecker's δ-function (i.e. $\delta(x, y) = 0$ for $x \neq y$ and $\delta(x, x) = 1$) and define by

$$\bar{X}_n^N(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(X_n^{N,i}, x), \qquad x \in S_h,$$

the *empirical measure* of the number of particles (on S_h) at time $n\tau$. Suppose that, for each lattice point $x=(x_1,\cdots,x_d)\in S_h$, $U_h(x)=[x_1, x_1+h)\times\cdots\times[x_d, x_d+h)$ is a unit cell in a porous medium and each particle stays in one unit cell during each time interval $[n\tau, (n+1)\tau)$. Then define by

(2) $$\bar{X}^N_{\tau,h}(t, y)=h^{-d}\cdot\bar{X}^N_{[t/\tau]}(x), \qquad (\text{if } y\in U_h(x) \text{ for } x\in S_h)$$

the *empirical density* of the number of particles (on $\boldsymbol{R}^d$) at time $t\geq 0$. Here we assume that N, τ and h satisfy the following relation

(3) $$c/\log(\log N)<\tau=\frac{1}{d}h^{d(\alpha-1)+2}$$

for a fixed constant $c>0$. We denote by

$$(N, \tau, h)\underset{(3)}{\longrightarrow}(\infty, 0, 0)$$

the limit of N, τ and h satisfying (3) as N tends to infinity and τ, h tend to zero. Under some initial conditions we will show that

$$\int_0^T dt\int_{\boldsymbol{R}^d}|\bar{X}^N_{\tau,h}(t, x)-u(t, x)|^2\,dx\longrightarrow 0$$

holds in probability as $(N, \tau, h)\underset{(3)}{\longrightarrow}(\infty, 0, 0)$ for each $T>0$, where $u=u(t, x)$ is a unique weak solution of a Cauchy problem for (1) (see Theorem 1 in §1).

Taking the limit in the same manner, we will show a *propagation of chaos* for the sysem of the N-particles. Namely if the initial positions of the N-particles are chaotic (= independently and identically distributed), then the processes $\{X^{N,i}_{[t/\tau]}: t\geq 0\}$ $(i=1,\cdots, m)$ become chaotic as $(N, \tau, h)\underset{(3)}{\longrightarrow}(\infty, 0, 0)$ for each integer $m\geq 1$. Further each process $\{X^{N,i}_{[t/\tau]}: t\geq 0\}$ converges in law to a d-dimensional diffusion process $(\{X(t)=(X_1(t),\cdots, X_d(t))\}, P)$ satisfying

(4) $$P(X(t)\in dx)=u(t, x)\,dx, \qquad (t\geq 0,\ x\in\boldsymbol{R}^d)$$

and

(5) $$X_j(t)=X_j(0)+\int_0^t u(s, X(s))^{(\alpha-1)/2}dB_j(s), \qquad (j=1,\cdots, d),$$

where $\{(B_1(t),\cdots, B_d(t))\}$ is a d-dimensional Brownian motion and $u=u(t, x)$ is the same unique weak solution of (1) (see Theorem 2 in §1).

The problems about N-particles of this kind were investigated originally by Kac [10]. Extending Kac's master equation approach to a Boltzmann equation, McKean [12] introduced an interacting random system of N-particles and proved the propagation of chaos by using Itô's calculus of stochastic differential equations. For some system of N-particles with an interaction (depending on the empirical measure), the propagation of chaos can be proved by the convergence of the empirical measure (see e.g. [15], [16], [17], [18]).

Our first result (Theorem 1) states a convergence of the empirical density (2) toward the unique weak solution u of (1). To prove this, we will show that the empirical density $\bar{X}^N_{\tau,h}$ converges to a deterministic version $u_{\tau,h}$ as $N \to \infty$ (see §3), and $u_{\tau,h}$ converges to u as τ, $h \to 0$ (see §2). To prove the propagation of chaos as $(N, \tau, h) \underset{(3)}{\longrightarrow} (\infty, 0, 0)$, in §4 we will estimate the rate of convergence for the propagation of chaos as $N \to \infty$ with fixed τ, $h > 0$. In §5 we will complete the proof of Theorem 2 by applying the random walk approach to a Brownian motion (cf. [8], [9]). In §6 we note two remarks. One is a note for the long time behaviour of the d-dimensional diffusion process $\{X(t)\}$ satisfying (4) and (5). That is the convergence of $X_k = \{k^{-\beta}X(kt): t \geq 0\}$ to a self-similar diffusion process X_∞ with the exponent $\beta = (d(\alpha-1)+2)^{-1}$. Another is a note for the order $h = O(\tau^\beta)$ in (3), which is concerned with a self-similarity of a sequence of Markov measures.

§1. Formulation and results

Let us consider the following parabolic Cauchy problem

$$(1.1) \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}(\varphi(u)u), \qquad (t > 0,\ x \in \boldsymbol{R}^d),$$

$$u(0, x) = u_0(x), \qquad (x \in \boldsymbol{R}^d),$$

where φ is a given function satisfying the following conditions:

$$(1.2a) \qquad \varphi \in C([0, \infty) \longrightarrow [0, \infty)) \cap C^1((0, \infty) \longrightarrow (0, \infty)) \text{ and}$$

$$\varphi'(x) \geq 0 \text{ for } x > 0,$$

(1.2b) there exists a constant $\rho \in (0, 1]$ such that

$$\sup_{0<x<1} \varphi'(x)\cdot x^{1-\rho} < \infty,$$

and

$$(1.2c) \qquad \sup_{x>0} \frac{\varphi'(x)\cdot x}{\varphi(x)} < \infty.$$

We assume the following conditions for the initial function u_0:
(1.3) u_0 is a bounded probability density function on $\boldsymbol{R}^d$ satisfying

$$\int_{\boldsymbol{R}^d} |x|^2 u_0(x)\,dx + \mathrm{V}(u_0) < \infty,$$

where

$$\mathrm{V}(f) = \sup_{0<h<1} \textstyle\sum_{x\in S_h} \sup_{y,y'\in I_h(x)} |f(y) - f(y')| h^{d-1},$$

$$S_h = \{(hz_1, \cdots, hz_d) : z_1, \cdots, z_d \in \boldsymbol{Z}\},\ h > 0, \text{ and}$$

$$I_h(x) = [x_1, x_1 + h] \times \cdots \times [x_d, x_d + h] \qquad \text{for } x = (x_1, \cdots, x_d) \in S_h.$$

We consider a *Markov system of N-particles* $\{\boldsymbol{X}_n^N = (X_n^{N,1}, \cdots, X_n^{N,N}) : n \geq 0\}$ whose transition rule is given as follows. For each $h > 0$ and $N \in \boldsymbol{N}$, let

$$\Omega_{N,h} = \{\omega = (\omega_0^N, \omega_1^N, \cdots) : \omega_n^N = (\omega_n^{N,1}, \cdots, \omega_n^{N,N}) \in (S_h)^N\}$$

be a path space and $\boldsymbol{X}_n^N = (X_n^{N,1}, \cdots, X_n^{N,N})$ be a function on $\Omega_{N,h}$ defined by $\boldsymbol{X}_n^N(\omega) = \omega_n^N$ and $X_n^{N,i}(\omega) = \omega_n^{N,i}$. For each $\tau > 0$, let $P_{N,\tau,h}$ be a Markov measure on $\Omega_{N,h}$ characterized by

(M. 1) (*independency of individual transitions*)

$$P_{N,\tau,h}(\boldsymbol{X}_{n+1}^N = \boldsymbol{y} \mid \boldsymbol{X}_n^N = \boldsymbol{x}) = \textstyle\prod_{i=1}^N P_{N,\tau,h}(X_{n+1}^{N,i} = y_i \mid \boldsymbol{X}_n^N = \boldsymbol{x})$$

for $\boldsymbol{y} = (y_1, \cdots, y_N)$, $\boldsymbol{x} \in (S_h)^N$ and

(M.2) (*transition rule of each particle*)

$$P_{N,\tau,h}(X_{n+1}^{N,i} = x_i \pm h_j \mid \boldsymbol{X}_n^N = \boldsymbol{x}) = \frac{1}{2d}(\varphi(h^{-d}\cdot\bar{\boldsymbol{x}}(x_i))\cdot d\tau h^{-2} \wedge 1), \quad (j = 1, \ldots, d),$$

$$P_{N,\tau,h}(X_{n+1}^{N,i} = x_i \mid \boldsymbol{X}_n^N = \boldsymbol{x}) = 1 - (\varphi(h^{-d}\cdot\bar{\boldsymbol{x}}(x_i))\cdot d\tau h^{-2} \wedge 1)$$

for all $i = 1, \cdots, N$, $n = 0, 1, \cdots$, and $\boldsymbol{x} = (x_1, \cdots, x_N) \in (S_h)^N$ where $h_j = (0, \cdots, 0, \underset{(j)}{h}, 0, \cdots, 0)$ and $\bar{\boldsymbol{x}}(y) = \frac{1}{\mathrm{N}} \sum_{k=1}^N \delta(x_k, y)$.

We note that all N-particles can move simultaneously. We are concerned with the empirical measure

$$\bar{\boldsymbol{X}}_n^N(x) = \frac{1}{N} \textstyle\sum_{i=1}^N \delta(X_n^{N,i}, x), \qquad n = 0, 1, \cdots, \quad x \in S_h,$$

and the empirical density

$$\bar{\boldsymbol{X}}_{\tau,h}^N(t, x) = h^{-d}\cdot \bar{\boldsymbol{X}}_{[t/\tau]}^N(([x_1/h]h, \cdots, [x_d/h]h)), \tag{1.4}$$

$$t \geq 0, \qquad x = (x_1, \cdots, x_d) \in \boldsymbol{R}^d,$$

where we take $\tau > 0$ as the unit time of this system.

For φ satisfying (1.2a) and u_0 satisfying (1.3), we choose the unit time τ, the width h of the lattice and the total number N of the particles such as

$$C_1/\log(\log N) \leq \tau \leq C_2 h^2 \tag{1.5}$$

for fixed constants C_1, $C_2 > 0$, where $C_2 < 1/db(\|u_0\|_\infty)$ and $b(u) = \varphi'(u)\cdot u + \varphi(u)$. We denote by

$$(N, \tau, h) \underset{(1.5)}{\longrightarrow} (\infty, 0, 0)$$

the limit of N, τ and h satisfying (1.5) as N tends to infinity and τ and h tend to zero.

DEFINITION. A function $u = u(t, x)$ is called a *weak solution* of (1.1) if u satisfies

$$u \in L^1([0, T] \times \boldsymbol{R}^d) \cap L^\infty([0, T] \times \boldsymbol{R}^d) \qquad \text{for all } T > 0,$$

$$\int_0^\infty dt \int_{\boldsymbol{R}^d} \{u\cdot f_t + \tfrac{1}{2}\varphi(u)u\cdot \triangle f\}dx = 0 \qquad \text{for all } f \in C_0^\infty((0, \infty) \times \boldsymbol{R}^d) \text{ and}$$

$$\text{ess lim}_{t\to 0} \int_{\boldsymbol{R}^d} |u(t, x) - u_0(x)|dx = 0.$$

THEOREM 1 (*convergence of empirical density*). *Assume* (1.2a) *and* (1.3). *Then there exists a unique weak solution* $u = u(t, x)$ *of* (1.1) *satisfying*

$$0 \leq u(t, x) \leq \|u_0\|_\infty, \qquad (t, x) \in [0, \infty) \times \boldsymbol{R}^d, \tag{1.6}$$

$$\int_{\boldsymbol{R}^d} u(t, x)dx = 1, \qquad t \geq 0, \tag{1.7}$$

$$\int_{\boldsymbol{R}^d} |x|^2 u(t, x)dx \leq \int_{\boldsymbol{R}^d} |x|^2 u_0(x)dx + d\varphi(\|u_0\|_\infty)t, \qquad t \geq 0 \quad \textit{and} \tag{1.8}$$

$$\lim_{\substack{|\varepsilon|\to 0\\ |\delta|\to 0}} \sup_{0\leq t\leq T} \int_Q |u(t + \varepsilon, x + \delta) - u(t, x)|dx = 0 \tag{1.9}$$

for all $T > 0$ *and compact set* $Q \subset \boldsymbol{R}^d$, *where* ε *belongs to* $\boldsymbol{R}$ *and* δ *belongs to* $\boldsymbol{R}^d$. *Further if we assume* (1.2c) *and*

$$\lim_{(N,\tau,h) \underset{(1.5)}{\longrightarrow} (\infty,0,0)} K^{1/\tau} E_{N,\tau,h}\left[\textstyle\sum_{x\in S_h} |h^{-d}\cdot \bar{X}_0^N(x) - \bar{u}_0(x)|^2 h^d\right] = 0 \tag{A.1}$$

for any fixed $K > 0$ *where* $\bar{u}_0(x) = u_0(x)/c_h$ *and* $c_h = \sum_{x\in S_h} u_0(x)h^d$, *then*

$$\lim_{(N,\tau,h) \underset{(1.5)}{\longrightarrow} (\infty,0,0)} E_{N,\tau,h}\left[\int_0^T dt \int_{\boldsymbol{R}^d} |\bar{X}_{\tau,h}^N(t, x) - u(t, x)|^2 dx\right] = 0 \tag{1.10}$$

holds for each $T>0$, *where* $\bar{X}^N_{\tau,h}$ *is the empirical density* (1.4).

THEOREM 2 (*propagation of chaos*). *Assume* (1.2) (= (1.2*a*) ~ (1.2*c*)), (1.3) *and* (A. 1). *Let m be a fixed positive integer and* $\bar{u}_0$ *be the one in Theorem* 1. *If*

$$\text{(A.2)} \qquad \sup_{x_1,\dots,x_m\in S_h}|P_{N,\tau,h}(X_0^{N,1}=x_1,\cdots,X_0^{N,m}=x_m)h^{-dm}-\prod_{i=1}^m \bar{u}_0(x_i)|$$

$$\longrightarrow 0 \ (as\ (N,\tau,h)\underset{(1.5)}{\longrightarrow}(\infty,0,0)),$$

then the m marginal process

$$(\{(X^{N,1}_{[t/\tau]},\cdots,X^{N,m}_{[t/\tau]}): t\geq 0\},\quad P_{N,\tau,h})$$

converges in law to an md-dimensional process

$$(\{(X^{(1)}(t),\cdots,X^{(m)}(t)): t\geq 0\},\quad P)$$

as $(N,\tau,h)\underset{(1.5)}{\longrightarrow}(\infty,0,0)$ *and the d-dimensional processes* $\{X^{(i)}(t)=(X^{(i)}_1(t),\cdots,X^{(i)}_d(t)): t\geq 0\}$ $(i=1,\cdots,m)$ *are independently and identically distributed diffusion processes satisfying*

$$\text{(1.11)} \qquad P(X^{(i)}(t)\in dx)=u(t,x)dx, \qquad (t\geq 0,\ x\in \boldsymbol{R}^d)$$

and

$$\text{(1.12)} \qquad X^{(i)}_j(t)=X^{(i)}_j(0)+\int_0^t \varphi(u(s,X^{(i)}(s)))^{1/2}\,dB^{(i)}_j(s),\ (j=1,\cdots,d),$$

where $\{(B^{(i)}_1(t),\cdots,B^{(i)}_d(t))\}$ $(i=1,\cdots,m)$ *are independent d-dimensional Brownian motions. Here the function* $u=u(t,x)$ *is the unique weak solution of* (1.1).

REMARK. (1) If the initial positions of N-particles are independently and identically distributed with the density $\bar{u}_0$ (i.e. $P_{N,\tau,h}(X_0^{N,1}=x_1,\cdots,X_0^{N,N}=x_N) = \prod_{i=1}^N(\bar{u}_0(x_i)h^d)$), then we have

$$E_{N,\tau,h}[\textstyle\sum_{x\in S_h}|h^{-d}\bar{X}^N_0(x)-\bar{u}_0(x)|^2\,dx]\leq 1/Nh^d$$

and hence the assumption (A.1) is certainly satisfied.

(2) If we can take the limit of N, τ and h satisfying (1.5) and

$$\text{(1.5')} \qquad \sup_{x:\varphi(x)d\tau h^{-2}\leq 1}\varphi'(x)xd\tau h^{-2}<c$$

for some constant $c>0$, then we can prove Theorems 1 and 2 without the assumption (1.2c).

(3) In case of $\varphi(u)=u^{\alpha-1}$ $(\alpha>1)$ and $\tau=d^{-1}h^{d(\alpha-1)+2}$, τ and h satisfy (1.5) automatically and the transition rule (M.2) is independent of τ and h: i.e.

$$\varphi(h^{-d}\cdot\bar{x}(x_i))\,d\tau h^{-2} \wedge 1 = (\bar{x}(x_i))^{\alpha-1}.$$

This is the simple case stated in §0.

§2. Difference approximation of parabolic equation

On the finite difference approach to the porous medium equation, several difference schemes to (1) in §0 with $d=1$ were studied precisely (see e.g. Mimura, Nakaki and Tomoeda [13]). In this section we solve the Cauchy problem (1.1) ($d \geq 1$) by the following difference approximation. For τ, $h>0$ let us consider the difference equation

$$\{\bar{u}_{n+1}(x) - \bar{u}_n(x)\}/\tau = \tfrac{1}{2}(\triangle_h \varphi(\bar{u}_n)\bar{u}_n)(x), \qquad (x \in S_h,\ n \geq 0), \tag{2.1}$$

where φ is a given function satisfying (1.2a) and $(\triangle_h f)(x) = \sum_{j=1}^{d}\{f(x+h_j) - 2f(x) + f(x-h_j)\}/h^2$. For a function u_0 satisfying (1.3), put

$$\bar{u}_0(x) = u_0(x)/c_h, \qquad (x \in S_h), \tag{2.2}$$

where $c_h = \sum_{x \in S_h} u_0(x)h^d$ is a normalized constant. Then we have

$$\textstyle\sum_{x\in S_h} \bar{u}_0(x)h^d = 1 \qquad \text{and} \quad \sum_{x\in S_h} |x|^2 \bar{u}_0(x)h^d < \infty. \tag{2.3}$$

Let $C_2 > 0$ be a fixed constant satisfying $C_2 < 1/db(\|u_0\|_\infty)$, where $b(u) = \varphi'(u)\cdot u + \varphi(u)$ (see (1.5)). Since $c_h \to 1$ as $h \to 0$, there exists a constant $h_0 > 0$ such that

$$\tfrac{1}{2} < c_h < 2 \qquad \text{and} \quad C_2 < 1/db(\|\bar{u}_0\|_\infty) \tag{2.4}$$

hold for all $h \in (0, h_0)$. Put

$$B = \{(\tau, h)\colon 0 < \tau \leq C_2 h^2,\quad 0 < h < h_0\}. \tag{2.5}$$

For each $(\tau, h) \in B$, let $\{\bar{u}_n(x)\}$ be a solution of (2.1)–(2.2) and $u_{\tau,h}$ be a function on $[0, \infty) \times \boldsymbol{R}^d$ defined by

$$u_{\tau,h}(t, x) = \bar{u}_{[t/\tau]}([x]_h) + \left(\frac{t}{\tau} - \left[\frac{t}{\tau}\right]\right)\{\bar{u}_{[t/\tau]+1}([x]_h) - \bar{u}_{[t/\tau]}([x]_h)\} \tag{2.6}$$

where $[x]_h = ([x_1/h]h, \cdots, [x_d/h]h)$ for $x = (x_1, \cdots, x_d) \in \boldsymbol{R}^d$. Then we can approximate the weak solution u of (1.1) by $u_{\tau,h}$ as follows.

PROPOSITION 1 (*difference approximation*). *Assume* (1.2a) *and* (1.3). *Then there exists a unique weak solution* $u = u(t, x)$ *of* (1.1) *such that for each* $T > 0$

$$\sup_{0\leq t\leq T} \int_{\boldsymbol{R}^d} |u_{\tau,h}(t, x) - u(t, x)|\,dx \longrightarrow 0 \tag{2.7}$$

holds as τ *and* h *tend to zero keeping* $(\tau, h) \in B$ *and* u *satisfies* (1.6) ~ (1.9) *in Theorem* 1.

To prove this proposition we first show the stability of the sequence $\{\bar{u}_n(x)\}$ as follows.

LEMMA 2.1 (*stability*). *Assume* (1.2a). *For each* $(\tau, h) \in B$, *let* $\bar{u}_0$ *be a bounded non-negative function on* S_h *satisfying* (2.3) *and* $b(\|\bar{u}_0\|_\infty) < 1/dC_2$, *where* $b(u) = \varphi'(u)u + \varphi(u)$. *For this* τ, h *and* $\bar{u}_0$, *let* $\{\bar{u}_n(x): x \in S_h, n \geq 0\}$ *be the solution of* (2.1) *with the initial function* $\bar{u}_0$. *Then we have*

$$0 \leq \bar{u}_n(x) \leq \|\bar{u}_0\|_\infty, \tag{2.8}$$

$$\sum_{x \in S_h} \bar{u}_n(x) h^d = 1, \tag{2.9}$$

$$\sum_{x \in S_h} |x|^2 \bar{u}_n(x) h^d \leq \sum_{x \in S_h} |x|^2 \bar{u}_0(x) h^d + d\varphi(\|\bar{u}_0\|_\infty) n\tau \tag{2.10}$$

and

$$\begin{aligned} &\sum_{j=1}^d \sum_{x \in S_h} |\bar{u}_n(x + h_j) - \bar{u}_n(x)| h^{d-1} \\ &\leq \sum_{j=1}^d \sum_{x \in S_h} |\bar{u}_0(x + h_j) - \bar{u}_0(x)| h^{d-1} \end{aligned} \tag{2.11}$$

for all $n = 0, 1, \cdots$.

PROOF. Put $\Phi(u) = \varphi(u) \cdot u$. By (2.1), we have

$$\begin{aligned} &\bar{u}_{n+1}(x) \\ &= \bar{u}_n(x) + \frac{\tau}{2} h^{-2} \sum_{j=1}^d (\Phi(\bar{u}_n(x + h_j)) - 2\Phi(\bar{u}_n(x)) + \Phi(\bar{u}_n(x - h_j))). \end{aligned} \tag{2.12}$$

By (2.3) we have (2.9) for all $n \geq 0$. Put

$$a_j^n(x) = \{\Phi(\bar{u}_n(x + h_j)) - \Phi(\bar{u}_n(x))\} / \{\bar{u}_n(x + h_j) - \bar{u}_n(x)\}$$

for $n = 0, 1, \cdots$, $x \in S_h$ and $j = 1, \cdots, d$, then (2.12) is rewritten as

$$\begin{aligned} \bar{u}_{n+1}(x) &= [1 - q\sum_{j=1}^d \{a_j^n(x) + a_j^n(x - h_j)\}] \bar{u}_n(x) \\ &+ q\sum_{j=1}^d \{a_j^n(x) \cdot \bar{u}_n(x + h_j) + a_j^n(x - h_j) \cdot \bar{u}_n(x - h_j)\}, \end{aligned} \tag{2.13}$$

where $q = \tau h^{-2}/2 \leq (2\mathrm{d} \cdot b(\|\bar{u}_0\|_\infty))^{-1}$. We note that $b(u) = \Phi'(u)$. If $0 \leq \bar{u}_n(x) \leq \|\bar{u}_0\|_\infty$ for all $x \in S_h$, then

$$0 \leq q a_j^n(x) \leq q\Phi'(\|\bar{u}_0\|_\infty) \leq 1/2d$$

for all $x \in S_h$. By (2.13) we have

$$\min\{\bar{u}_n(x), \bar{u}_n(x \pm h_j)\} \leq \bar{u}_{n+1}(x) \leq \max\{\bar{u}_n(x), \bar{u}_n(x \pm h_j)\}$$

which implies $0 \le \bar{u}_{n+1}(x) \le \|\bar{u}_0\|_\infty$ for all $x \in S_h$. Therefore we get (2.8) for all $n \ge 0$. By (2.12), (2.8) and (2.9) we have (2.10). Finally we show (2.11). Put $e_j^n(x) = \bar{u}_n(x+h_j) - \bar{u}_n(x)$. Since

$$\begin{aligned} e_j^{n+1}(x) &= [1 - 2dqa_j^n(x)]\, e_j^n(x) \\ &\quad + q\textstyle\sum_{k=1}^d \{a_j^n(x+h_k)\cdot e_j^n(x+h_k) + a_j^n(x-h_k)\cdot e_j^n(x-h_k)\}, \end{aligned}$$

we have

$$h^{d-1}\textstyle\sum_{x\in S_h} |e_j^{n+1}(x)| \le h^{d-1}\sum_{x\in S_h} |e_j^n(x)|,$$

which implies (2.11). □

LEMMA 2.2 (*compactness*). *Let U be a set of functions u: $[0, \infty) \times \boldsymbol{R}^d \to \boldsymbol{R}$ satisfying*

$$\sup_{u\in U}(\|u\|_{L^\infty([0,\infty)\times \boldsymbol{R}^d)} + \sup_{t\ge 0} \mathrm{V}(u(t,\cdot))) < \infty$$

and that

$$\left\{(u*f)(t,x) = \int_{\boldsymbol{R}^d} u(t,y)f(x-y)dy : u \in U\right\}$$

is equicontinuous for each $f \in C_0^\infty(\boldsymbol{R}^d)$, where the notation $\mathrm{V}(f)$ is defined in (1.3). *If U is an infinite set, then there exist a function u_∞: $[0, \infty) \times \boldsymbol{R}^d \to \boldsymbol{R}$ and a sequence $\{u_n\} \subset U$ such that*

$$\inf\{u_n(t,x) : x \in R^d, n \ge 1\} \le u_\infty(t,x) \le \sup\{u_n(t,x) : x \in R^d, n \ge 1\}, \quad t \ge 0,$$

$$\lim_{n\to\infty} \|u_n - u_\infty\|_{T,Q} = 0$$

and

$$\lim_{\substack{|\varepsilon|\to 0\\ |\delta|\to 0}} \|u_\infty(\cdot + \varepsilon, \cdot + \delta) - u_\infty\|_{T,Q} = 0$$

for all $T > 0$ and compact set $Q \subset \boldsymbol{R}^d$, where

$$\|u\|_{T,Q} = \sup_{0\le t\le T} \int_Q |u(t,x)|dx.$$

PROOF. Choose a function $\rho \in C_0^\infty(\boldsymbol{R}^d \to \boldsymbol{R})$ satisfying $0 \le \rho(x) \le 1$, $\int_{\boldsymbol{R}^d} \rho(x)dx = 1$ and $\mathrm{supp}(\rho) \subset [-1, 1]^d$. Put

$$\rho_n(x) = n\rho(n^{1/d}x_1, \cdots, n^{1/d}x_d)$$

for $x = (x_1, \cdots, x_d) \in \boldsymbol{R}^d$. Then $\{u * \rho_n : u \in U\}$ is uniformly bounded and equicontinuous for each $n \in \boldsymbol{N}$. Hence we can choose a sequence $\{u_n\} \subset U$ such that $\{u_n * \rho_n : n \geq T\}$ is a Cauchy sequence with respect to the norm $\| \|_{L^\infty([0,T]\times[-T,T]^d)}$ for each $T > 0$. Since

$$\begin{aligned} &\| u*\rho_n - u \|_{T,Q} \\ &\leq \sup_{0 \leq t \leq T} \int_{R^d} \rho_n(y) (\lim_{h \to 0} \textstyle\sum_{x \in S_h} |u(t, x-y) - u(t, x)| h^d) dy \\ &\leq dn^{-1/d} \cdot \sup_{0 \leq t \leq T} \mathrm{V}(u(t, \cdot)) \longrightarrow 0 \qquad (\text{as } n \to \infty) \end{aligned}$$

for any $u \in U$, $T > 0$ and compact set $Q \subset R^d$, we get the lemma by putting $u_\infty = \lim_{n \to \infty} u_n * \rho_n$. □

LEMMA 2.3 (*existence*). *Assume* (1.2a) *and* (1.3). *Then there exist a weak solution* $u = u(t, x)$ *of* (1.1) *satisfying* (1.6) ~ (1.9) *and a sequence* $\{(\tau_n, h_n)\} \subset B$ *such that* τ_n, $h_n \to 0$ *as* $n \to \infty$ *and*

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \int_{R^d} |u_{\tau_n, h_n}(t, x) - u(t, x)| dx = 0 \tag{2.14}$$

holds for each $T > 0$.

PROOF. By (2.2), (2.4), (2.6) and (2.8) ~ (2.11), we get for $(\tau, h) \in B$

$$0 \leq u_{\tau,h}(t, x) \leq \| \bar{u}_0 \|_\infty < 2 \| u_0 \|_\infty, \tag{2.15}$$

$$\int_{R^d} u_{\tau,h}(t, x) dx = 1, \tag{2.16}$$

$$\int_{R^d} |x|^2 u_{\tau,h}(t, x) dx \leq \textstyle\sum_{x \in S_h} |x|^2 \bar{u}_0(x) h^d + d\varphi(\| \bar{u}_0 \|_\infty) t, \quad t \geq 0, \tag{2.17}$$

$$\mathrm{V}(u_{\tau,h}(t, \cdot)) \leq \mathrm{V}(u_{\tau,h}(0, \cdot)) < 2\mathrm{V}(u_0) \tag{2.18}$$

and

$$\begin{aligned} &|(u_{\tau,h} * f)(t, x) - (u_{\tau,h} * f)(s, y)| \\ &\leq d^{1/2} A_f |x - y| + \tfrac{1}{2} B_f \varphi(\| \bar{u}_0 \|_\infty) |t - s|, \qquad f \in C_0^\infty(R^d), \end{aligned} \tag{2.19}$$

where $A_f = \max_{1 \leq j \leq d} \| \partial f / \partial x_j \|_\infty$ and $B_f = \sum_{j=1}^d \| \partial^2 f / \partial x_j^2 \|_\infty$. By Lemma 2.2, there exist a function u satisfying (1.6), (1.9) and a sequence $\{(\tau_n, h_n)\} \subset B$ such that τ_n, $h_n \to 0$ as $n \to \infty$ and

$$\lim_{n \to \infty} \| u_{\tau_n, h_n} - u \|_{T,Q} = 0 \tag{2.20}$$

for each $T > 0$ and compact set $Q \subset R^d$. By (2.16) and (2.17), the limiting function u satisfies (1.7) and (1.8). For each $f \in C_0^\infty((0, \infty) \times R^d)$, we have from (2.1)

$$\sum_{n=0}^{\infty} \tau \sum_{x \in S_h} \bar{u}_n(x) \{f_{n+1}(x) - f_n(x)\} \tau^{-1} h^d$$
$$= - \sum_{n=0}^{\infty} \tau \sum_{x \in S_h} \varphi(\bar{u}_n(x)) \bar{u}_n(x) \cdot \tfrac{1}{2} (\triangle_h f_n)(x) h^d,$$

where $f_n(x) = f(n\tau, x)$. Therefore we have

$$\int_0^{\infty} dt \int_{\boldsymbol{R}^d} \{u \cdot f_t + \tfrac{1}{2} \varphi(u) u \cdot \triangle f\} dx = 0,$$

which implies that the function $u = u(t, x)$ is a weak solution of (1.1). By (2.17) and (1.8), the equality (2.20) holds with $\boldsymbol{R}^d$ in place of Q. Therefore we have (2.14). □

On the uniqueness of the Cauchy problem (1.1), Brezis and Crandall [4] proved the following

LEMMA 2.4 (Brezis-Crandall). *Fix* $T > 0$ *and put* $H = [0, T] \times \boldsymbol{R}^d$. *Let* $z \in L^1(H) \cap L^{\infty}(H)$ *and* $w \in L^{\infty}(H)$. *Assume*

$$z_t - \triangle w = 0 \text{ in } \mathscr{D}'(H),$$

$$zw \geq 0 \text{ a.e. in } H,$$

$$\text{meas}\{(t, x) \in H : |w(t, x)| > \varepsilon\} < \infty$$

for each $\varepsilon > 0$, *where* meas A *is the Lebesgue measure of* A, *and*

$$\text{ess}\lim_{t \downarrow 0} \int_{\boldsymbol{R}^d} |z(t, x)| dx = 0.$$

Then $z = 0$ *a.e. on* H.

Put $z = u - v$ and $w = 2^{-1}\{\varphi(u) \cdot u - \varphi(v) \cdot v\}$ for weak solutions u and v of (1.1), then by Lemma 2.4, $u = v$ a.e. on $[0, T] \times \boldsymbol{R}^d$ for each $T > 0$. If follows that the weak solution of (1.1) is unique. By Lemma 2.3 we complete the proof of Proposition 1.

§3. Convergence of empirical density

In the previous section, we have proved (1.6) ~ (1.9) in Theorem 1. In this section we complete the proof of Theorem 1. We firstly prepare some notations. For $h > 0$ and $f, g : S_h \to \boldsymbol{R}$, put

$$\langle f, g \rangle_{(h)} = \sum_{x \in S_h} f(x) g(x) h^d, \qquad \|f\|_{(h)} = (\langle f, f \rangle_{(h)})^{1/2}.$$

Let φ be a function satisfying (1.2). For $r > 0$ put

$$\varphi_r[f](x) = \min\{\varphi(|f(x)|)\cdot r,\ 1\},$$

$$K_{h,r}(f;\, g)(x) = g(x) + (2d)^{-1}h^2\varphi_r[f](x)\cdot(\triangle_h g)(x)$$

and

$$K^*_{h,r}(f;\, g)(x) = g(x) + (2d)^{-1}h^2(\triangle_h(\varphi_r[f]\cdot g))(x).$$

Then we have

$$\langle e,\, K_{h,r}(f;\, g)\rangle_{(h)} = \langle K^*_{h,r}(f;\, e),\, g\rangle_{(h)}$$

for all functions $e, f, g\colon S_h \to \boldsymbol{R}$. Put $r = d\tau h^{-2}$. We note that the transition rule (M.2) is rewritten as

$$(3.1)\qquad \begin{aligned} &P_{N,\tau,h}(X^{N,i}_{n+1} = x\,|\,\boldsymbol{X}^N_n = \boldsymbol{x}) \\ &\quad = K_{h,r}(h^{-d}\bar{\boldsymbol{x}};\, \delta(x))(x_i) = K^*_{h,r}(h^{-d}\bar{\boldsymbol{x}};\, \delta(x_i))(x), \end{aligned}$$

for $x \in S_h$, $\boldsymbol{x} = (x_1, \cdots, x_N) \in (S_h)^N$, where $\delta(x)(y) = \delta(x, y)$ and $\bar{\boldsymbol{x}}(x) = \sum_{i=1}^N \delta(x_i, x)/N$. Let $\mathscr{B}^{N,h}_n$ be the σ-field on $\Omega_{N,h}$ generated by $\{\boldsymbol{X}^N_k\colon k \le n\}$. By (3.1), for each $f\colon S_h \to \boldsymbol{R}$ and $i = 1, \cdots, N$, the process

$$(3.2)\qquad \{f(X^{N,i}_n) - \textstyle\sum_{k=0}^{n-1} K_{h,r}(h^{-d}\bar{\boldsymbol{X}}^N_k;\, f)(X^{N,i}_k)\colon n \ge 0\}$$

is a $\mathscr{B}^{N,h}_n$-martingale on $(\Omega_{N,h},\, P_{N,\tau,h})$.

For each $(\tau, h) \in B$ (see (2.5)) we note that $r = d\tau h^{-2} \le dC_2$. We will use the following two inequalities later

$$(3.3)\qquad \sup_{0\le r\le dC_2} |\varphi_r[f](x)f(x) - \varphi_r[g](x)g(x)| \le C(\varphi)|f(x) - g(x)|,$$

$$(3.4)\qquad \sup_{0\le r\le dC_2} |\varphi_r[f](x) - \varphi_r[g](x)| \le D(\varphi)|f(x) - g(x)|^\rho,$$

for $f, g\colon S_h \to [0, \infty)$ and $x \in S_h$, where

$$C(\varphi) = \sup_{x>0}\{\varphi'(x)x/\varphi(x)\} + 1$$

and

$$D(\varphi) = \{\sup_{0<x<1}\varphi'(x)x^{1-\rho}dC_2 + \sup_{x\ge 1}\varphi'(x)x/\varphi(x)\}/\rho.$$

The inequality (3.3) is obtained by the assumption (1.2a), (1.2c), and the inequality (3.4) is obtained by (1.2a) $\sim$ (1.2c) and the inequality

$$\int_a^b x^{\rho-1}\,dx \le \frac{1}{\rho}|b - a|^\rho.$$

The following lemma is a basic lemma to prove Theorems 1 and 2.

LEMMA 3.1 (*basic lemma*). *Assume* (1.2a) *and* (1.2c). *Let* τ, h, $\bar{u}_0$ *and* $\bar{u}_n$ *be* *those of Lemma* 2.1. *Let* μ *be a finite Markov measure on* $\Omega_{N,h}$ *satisfying* (M. 1) *and* (M.2) *with* μ *in place of* $P_{N,\tau,h}$. *Then we have*

$$\int \| h^{-d}\bar{X}_n^N - \bar{u}_n\|_{(h)}^2 d\mu \le (K_0)^n \int \| h^{-d}\bar{X}_0^N - \bar{u}_0\|_{(h)}^2 d\mu + 2|\mu|(K_0)^n/Nh^d,$$

where $K_0 = 2 + 9C(\varphi)^2$ *and* $|\mu| = \mu(\Omega_{N,h})$.

PROOF. Put $r = d\tau h^{-2}$, then we note $r \le dC_2 \le 1/b(\|\bar{u}_0\|_\infty)$, where $b(u) = \varphi'(u)u + \varphi(u)$. Further by (2.8) we have

$$r \le 1/b(\|\bar{u}_n\|_\infty) \le 1/\varphi(\|\bar{u}_n\|_\infty) \qquad \text{and} \quad \varphi_r[\bar{u}_n](x) = \varphi(\bar{u}_n(x))r.$$

Therefore, by (2.1), we have

$$\bar{u}_{n+1}(x) = K_{h,r}^*(\bar{u}_n;\, \bar{u}_n)(x) \tag{3.5}$$

for all $n = 0, 1, \cdots$ and $x \in S_h$. By (M.1) and (3.1) with μ in place of $P_{N,\tau,h}$ we have

$$\begin{aligned} &\mu(X_{n+1}^{N,\sigma(1)} = x_1, \cdots, X_{n+1}^{N,\sigma(m)} = x_m) \\ &\quad = \int (\textstyle\prod_{i=1}^m K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,\sigma(i)}))(x_i))\, d\mu \end{aligned} \tag{3.6}$$

for all integers $m \in [1, N]$, $1 \le \sigma(1) < \cdots < \sigma(m) \le N$, $n \ge 0$ and $x_1, \cdots, x_m \in S_h$. Since the map $g \mapsto K_{h,r}^*(f;\, g)(x)$ is linear for each function $f\colon S_h \to [0, \infty)$, we get from (3.6) with $m = 1$

$$\int \langle h^{-d}\bar{X}_{n+1}^N, f\rangle_{(h)}\, d\mu = \int \langle K_{h,r}^*(h^{-d}\bar{X}_n^N;\, h^{-d}\bar{X}_n^N), f\rangle_{(h)}\, d\mu. \tag{3.7}$$

By (3.6) with $m = 2$ or 1, we have

$$\begin{aligned} &\int \langle \{h^{-d}\bar{X}_{n+1}^N\}^2, f\rangle_{(h)}\, d\mu \\ &\quad = \int \langle \{K_{h,r}^*(h^{-d}\bar{X}_n^N;\, h^{-d}\bar{X}_n^N)\}^2, f\rangle_{(h)}\, d\mu \\ &+ \int \langle N^{-2}h^{-2d}\textstyle\sum_{i=1}^N K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,i}))\{1 - K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,i}))\}, f\rangle_{(h)}\, d\mu. \end{aligned} \tag{3.8}$$

By the definition of $K_{h,r}^*$, we have

$$\begin{aligned} &\textstyle\sum_{x\in S_h} K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,i}))(x)\{1 - K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,i}))(x)\} \\ &\quad = \textstyle\sum_{x\in S_h} \delta(X_n^{N,i}, x)\varphi_r[h^{-d}\bar{X}_n^N](x)\{2 - (3/d)\varphi_r[h^{-d}\bar{X}_n^N](x)\}. \end{aligned} \tag{3.9}$$

Therefore using (3.7) with $f=\bar{u}_n$, (3.8) with $f\equiv 1$, (3.9) and (3.5), we have

$$\int \| h^{-d}\bar{X}^N_{n+1} - \bar{u}_{n+1}\|^2_{(h)}d\mu$$

$$= \int \| K^*_{h,r}(h^{-d}\bar{X}^N_n;\, h^{-d}\bar{X}^N_n) - K^*_{h,r}(\bar{u}_n;\, \bar{u}_n)\|^2_{(h)}d\mu$$

$$+ N^{-1}h^{-d}\int \textstyle\sum_{x\in S_h}\bar{X}^N_n(x)\cdot\varphi_r[h^{-d}\bar{X}^N_n](x)\{2-(3/d)\varphi_r[h^{-d}\bar{X}^N_n](x)\}\,d\mu.$$

Using the inequalities $(\sum_{i=1}^m |x_i|)^2 \le m\sum_{i=1}^m |x_i|^2$ and (3.3), we finally have

$$\int \| h^{-d}\bar{X}^N_{n+1} - \bar{u}_{n+1}\|^2_{(h)}d\mu \le \{2+9C(\varphi)^2\}\int \| h^{-d}\bar{X}^N_n - \bar{u}_n\|^2_{(h)}d\mu + 2|\mu|/Nh^d,$$

as was to be proved. □

PROOF OF THEOREM 1. By Lemma 3.1 with $\mu = P_{N,\tau,h}$, we have

$$E_{N,\tau,h}[\| h^{-d}\bar{X}^N_n - \bar{u}_n\|^2_{(h)}] \le (K_0)^n E_{N,\tau,h}[\| h^{-d}\bar{X}^N_0 - \bar{u}_0\|^2_{(h)}] + 2(K_0)^n/Nh^d, \tag{3.10}$$

where $K_0 = 2+9C(\varphi)^2$. By (1.4), (2.6) and (3.10), we get

$$E_{N,\tau,h}\Big[\int_0^{[T/\tau]\tau} dt\int_{\mathbf{R}^d} |\bar{X}^N_{\tau,h}(t,x) - u_{\tau,h}([t/\tau]\tau, x)|^2 dx\Big] \le (K_0)^{(T/\tau)+1}(E_{N,\tau,h}[\textstyle\sum_{x\in S_h}|h^{-d}\bar{X}^N_0(x) - \bar{u}_0(x)|^2 h^d] + 2/Nh^d)\tau. \tag{3.11}$$

By the assumption (A.1), the right hand side of (3.11) converges to zero as $(N,\tau,h)\xrightarrow[(1.5)]{}(\infty,0,0)$ for all $T>0$. By Proposition 1 we have (1.10) in Theorem 1. □

§4. Propagation of chaos as $N\to\infty$

In this section we consider the propagation of chaos for the Markov system of N-particles $\{(X^{N,1}_n,\cdots,X^{N,N}_n): n\ge 0\}$ on $(\Omega_{N,h}, P_{N,\tau,h})$ as N tends to infinity with fixed $(\tau,h)\in B$ (see (2.5)). Let $\bar{u}_0(x)$ and $\bar{u}_n(x)$ be those of Lemma 2.1. By Lemma 3.1 and (2.8), if $h^{-d}\bar{X}^N_0(x)$ converges to $\bar{u}_0(x)$ as $N\to\infty$, then the empirical density $h^{-d}\bar{X}^N_n(x)$ converges to $\bar{u}_n(x)$ ($\le \|\bar{u}_0\|_\infty$) and so

$$\varphi(h^{-d}\bar{X}^N_n(x))\,d\tau h^{-2}\wedge 1 \longrightarrow \varphi(\bar{u}_n(x))\,d\tau h^{-2}.$$

This means the convergence of the transition probability of each particle (see (M.2) in §1). Therefore each process $\{X^{N,i}_n: n\ge 0\}$ converges in law to the following Markov chain $\{Y_n\}$ on $(\Omega_h, P_{\tau,h})$.

Let $\Omega_h = \{y = (y_0, y_1, \cdots): y_n \in S_h\}$ be a path space and Y_n be a function on Ω_h defined by $Y_n(y) = y_n$. Let $P_{\tau,h}$ be a Markov measure on Ω_h satisfying

$$(4.1) \qquad P_{\tau,h}(Y_{n+1} = x \pm h_j | Y_n = x) = \varphi(\bar{u}_n(x))\tau h^{-2}/2, \qquad (j = 1, \cdots, d),$$

$$P_{\tau,h}(Y_{n+1} = x | Y_n = x) = 1 - \varphi(\bar{u}_n(x))d\tau h^{-2}$$

and

$$(4.2) \qquad P_{\tau,h}(Y_0 = x) = \bar{u}_0(x)h^d$$

for all $n = 0, 1, \cdots$ and $x \in S_h$. By (2.1) and the Markov property of $P_{\tau,h}$, we have

$$(4.3) \qquad P_{\tau,h}(Y_n = x) = \bar{u}_n(x)h^d$$

for all $n = 0, 1, \cdots$ and $x \in S_h$. We prepare the following Proposition 2, which estimate the rate of convergence for the propagation of chaos as $N \to \infty$, for the proof of Theorem 2 in the next section.

PROPOSITION 2. *Assume* (1.2). *Let* $\bar{u}_0(x)$, τ *and* h *be those of Lemma* 2.1. *Then for each integers* $1 \leq m \leq N$, $\ell \geq 0$, $n_0 = 0 < \cdots < n_\ell$ *and functions* $f_k^i: \boldsymbol{R}^d \to [0, 1]$ $(i = 1, \cdots, m,\ k = 0, \cdots, \ell)$, *we have*

$$(4.4) \qquad |E_{N,\tau,h}[\textstyle\prod_{k=0}^{\ell} \prod_{i=1}^{m} f_k^i(X_{n_k}^{N,i})] - \prod_{i=1}^{m} E_{\tau,h}[\prod_{k=0}^{\ell} f_k^i(Y_{n_k})]|$$

$$\leq \textstyle\sum_{x_1,\ldots,x_m \in S_h} \{\prod_{i=1}^{m} f_0^i(x_i)\} | P_{N,\tau,h}(X_0^{N,1} = x_1, \cdots, X_0^{N,m} = x_m) - \prod_{i=1}^{m} \{\bar{u}_0(x_i)h^d\}|$$

$$+ \textstyle\sum_{k=1}^{\ell} 2mD(\varphi)K^{n_k} \cdot h^{-d\gamma}(E_{N,\tau,h}[\| h^{-d} \cdot \bar{X}_0^N - \bar{u}_0 \|_{(h)}^2] + 4/Nh^d)^\gamma,$$

where $K = (K_0)^\gamma$, $\gamma = \rho/2$, $K_0 = 2 + 9C(\varphi)^2$ *and* $C(\varphi)$, $D(\varphi)$ *are positive constants defined in* (3.4), (3.5).

To prove this proposition, we prepare the following

LEMMA 4.1. *Assume* (1.2). *Let* τ, h, $\bar{v}_n$ *and* μ *are those of Lemma* 3.1 *with* $\bar{v}_n$ *in place of* $\bar{u}_n$. *Let* $v_0^1(x), \cdots, v_0^m(x)$ *be non-negative bounded functions on* S_h. *Let* $v_n^i(x)$ $(i = 1, \cdots, m)$ *be the sequence defined by the following linear difference equation*

$$(4.5) \qquad \{v_{n+1}^i(x) - v_n^i(x)\}/\tau = \tfrac{1}{2}(\triangle_h \varphi(\bar{v}_n)v_n^i)(x), \qquad (x \in S_h,\ n \geq 0\},$$

with the initial function v_0^i. *Then we have*

$$(4.6) \qquad \textstyle\sum_{x_1,\ldots,x_m \in S_h} | \mu(X_n^{N,1} = x_1, \cdots, X_n^{N,m} = x_m) - \prod_{i=1}^{m} \{v_n^i(x_i)h^d\}|$$

$$\leq \textstyle\sum_{k_1,\ldots,x_m \in S_h} | \mu(X_0^{N,1} = x_1, \cdots, X_0^{N,m} = x_m) - \prod_{i=1}^{m} \{v_0^i(x_i)h^d\}|$$

$$+ \sum_{\ell=0}^{n-1} 2mD(\varphi) |\mu|^{(2-\rho)/2} \left(\int \| h^{-d} \bar{X}_\ell^N - \bar{v}_\ell \|_{(h)}^2 d\mu \right)^\gamma h^{-d\gamma}$$

where $|\mu| = \mu(\Omega_{N,h})$.

PROOF. By the same argument as (3.5), (4.5) is rewritten as

$$\begin{aligned} v_{n+1}^i(x)h^d &= K_{h,r}^*(\bar{v}_n;\, v_n^i h^d)(x) \\ &= \textstyle\sum_{\sigma\in\{0,\pm h_1,\dots,\pm h_d\}} a_r(\bar{v}_n;\, \sigma)(x)\; v_n^i(x+\sigma)h^d \end{aligned} \tag{4.7}$$

for $x \in S_h$, $n \geq 0$, where $r = d\tau h^{-2}$,

$$a_r(f;\, 0)(x) = 1 - \varphi_r[f](x) \quad \text{and} \quad a_r(f;\, \pm h_j)(x) = \varphi_r[f](x \pm h_j)/2d.$$

Then we have

$$\begin{aligned} &\textstyle\sum_{x_1,\dots,x_m\in S_h} \left| \int \prod_{i=1}^m K_{h,r}^*(\bar{v}_n;\, \delta(X_n^{N,i}))(x_i)\, d\mu - \prod_{i=1}^m K_{h,r}^*(\bar{v}_n;\, v_n^i h^d)(x_i) \right| \\ &\leq \textstyle\sum_{x_1,\dots,x_m} \sum_{\sigma_1,\dots,\sigma_m\in\{0,\pm h_1,\dots,\pm h_d\}} \{\prod_{i=1}^m a_r(\bar{v}_n;\, \sigma_i)(x_i)\} \\ &\quad \times |\mu(X_n^{N,1} = x_1 + \sigma_1, \cdots, X_n^{N,m} = x_m + \sigma_m) - \prod_{i=1}^m \{v_n^i(x_i+\sigma_i)h^d\}| \\ &= \textstyle\sum_{x_1,\dots,x_m} |\mu(X_n^{N,1} = x_1, \cdots, X_n^{N,m} = x_m) - \prod_{i=1}^m \{v_n^i(x_i)h^d\}|. \end{aligned} \tag{4.8}$$

On the other hand, using trianglar inequalities successively, we have from (3.4)

$$\begin{aligned} &\textstyle\sum_{x_1,\dots,x_m\in S_h} \left| \int \prod_{i=1}^m K_{h,r}^*(h^{-d}\bar{X}_n^N;\, \delta(X_n^{N,i}))(x_i)\, d\mu \right. \\ &\qquad\qquad \left. - \int \prod_{i=1}^m K_{h,r}^*(\bar{v}_n;\, \delta(X_n^{N,i}))(x_i) d\mu \right| \\ &\leq \textstyle\sum_{i=1}^m \sum_{x_i\in S_h} 2 \int |\varphi_r[h^{-d}\bar{X}_n^N](x_i) - \varphi_r[\bar{v}_n](x_i)|\delta(X_n^{N,i},\, x_i)\, d\mu \\ &\leq 2mD(\varphi)|\mu|^{(2-\rho)/2} \left(\int \| h^{-d}\bar{X}_n^N - \bar{v}_n \|_{(h)}^2\, d\mu \right)^{\gamma} h^{-d\gamma}. \end{aligned} \tag{4.9}$$

By (3.6) and (4.7) $\sim$ (4.9) we have

$$\begin{aligned} &\textstyle\sum_{x_1,\dots,x_m\in S_h} |\mu(X_{n+1}^{N,1} = x_1, \cdots, X_{n+1}^{N,m} = x_m) - \prod_{i=1}^m \{v_{n+1}^i(x_i)h^d\}| \\ &\quad \leq \textstyle\sum_{x_1,\dots,x_m\in S_h} |\mu(X_n^{N,1} = x_1, \cdots, X_n^{N,m} = x_m) - \prod_{i=1}^m \{v_n^i(x_i)h^d\}| \\ &\qquad + 2mD(\varphi)|\mu|^{(2-\rho)/2} \left(\int \| h^{-d}\bar{X}_n^N - \bar{v}_n \|_{(h)}^2\, d\mu \right)^{\gamma} h^{-d\gamma}, \end{aligned}$$

as was to be proved. □

PROOF OF PROPOSITION 2. The idea of this proof is based on Uchiyama [18]. Put

$$F_\ell = \sum_{x_1,\dots,x_m\in S_h} |E_{N,\tau,h}[\{\prod_{k=0}^{\ell}\prod_{i=1}^{m} f_k^i(X_{n_k}^{N,i})\}\prod_{i=1}^{m}\delta(x_i, X_{n_\ell}^{N,i})]$$
$$- \prod_{i=1}^{m} E_{\tau,h}[\{\prod_{k=0}^{\ell} f_k^i(Y_{n_k})\}\delta(x_i, Y_{n_\ell})]|$$

and

$$G_\ell = 2mD(\varphi)K^{n_\ell}\cdot h^{-d\gamma}(E_{N,\tau,h}[\|h^{-d}\bar{X}_n^N - \bar{u}_0\|_{(h)}^2] + 4/Nh^d)^\gamma.$$

To prove (4.4) we show that

$$F_\ell \le F_{\ell-1} + G_\ell \tag{4.10}$$

for all $\ell = 1, 2, \cdots$. Let μ be a finite Markov measure on $\Omega_{N,h}$ satisfying (M.1) and (M.2) with the initial distribution

$$\mu(X_0^N = \boldsymbol{x}) = E_{N,\tau,h}[\{\prod_{k=0}^{\ell-1}\prod_{i=1}^{m} f_k^i(X_{n_k}^{N,i})\}\prod_{i=1}^{N}\delta(X_{n_{\ell-1}}^{N,i}, x_i)]$$

for $\boldsymbol{x} = (x_1, \cdots, x_N)\in(S_h)^N$. Then $|\mu| = \mu(\Omega_{N,h}) \le 1$. Put

$$v_0^i(x) = E_{\tau,h}[\{\prod_{k=0}^{\ell-1} f_k^i(Y_{n_k})\}\delta(Y_{n_\ell-1}, x)]h^{-d}$$

and

$$\bar{v}_0(x) = \bar{u}_{n_{\ell-1}}(x),$$

where $\bar{u}_n(x)$ is the solution of (2.1) with the initial function $\bar{u}_0(x)$. Let $\bar{v}_n(x)$ (resp. $v_n^i(x)$) be the solution of (2.1) (resp. (4.5)) with the initial function $\bar{v}_0(x)$ (resp. $v_0^i(x)$). By the Markov property and the uniqueness of the solution of (4.5), we have

$$E_{N,\tau,h}[\{\prod_{k=0}^{\ell}\prod_{i=1}^{m} f_k^i(X_{n_k}^{N,i})\}\prod_{i=1}^{m}\delta(x_i, X_{n_\ell}^{N,i})]$$
$$= \int \prod_{i=1}^{m}\{f_\ell^i(X_n^{N,i})\delta(x_i, X_n^{N,i})\}\,d\mu$$

and

$$\prod_{i=1}^{m} E_{\tau,h}[\{\prod_{k=0}^{\ell} f_k^i(Y_{n_k})\}\delta(x_i, Y_{n_\ell})] = \prod_{i=1}^{m}(f_\ell^i(x_i)v_n^i(x_i)h^d)$$

where $n = n_\ell - n_{\ell-1}$. By Lemma 4.1 and Lemma 3.1, we have

$$F_\ell \le \sum_{x_1,\dots,x_m\in S_h}|\mu(X_n^{N,1} = x_1, \cdots, X_n^{N,m} = x_m) - \prod_{i=1}^{m}\{v_n^i(x_i)h^d\}|$$
$$\le \sum_{x_1,\dots,x_m\in S_h}|\mu(X_0^{N,1} = x_1, \cdots, X_0^{N,m} = x_m) - \prod_{i=1}^{m}\{v_0^i(x_i)h^d\}|$$
$$+ 2mD(\varphi)\sum_{k=0}^{n-1}\left(\int \|h^{-d}\bar{X}_k^N - \bar{v}_k\|_{(h)}^2\,d\mu\right)^\gamma h^{-d\gamma}$$
$$\le F_{\ell-1} + 2mD(\varphi)K^{n_\ell}\cdot h^{-d\gamma}(E_{N,\tau,h}[\|h^{-d}\cdot\bar{X}_0^N - \bar{u}_0\|_{(h)}^2] + 4/Nh^d)^\gamma,$$

which implies (4.10) and therefore (4.4). Thus Proposition 2 has been proved. □

§5. Proof of Theorem 2

In this section we prove Theorem 2 as a limit theorem of probability measures by applying the random walk approach to a Brownian motion. For each integer m (≥ 1), let W^m be the metric space of all continuous functions $w: [0, \infty) \to \boldsymbol{R}^{dm}$ with the distance $d(w, \hat{w}) = \sum_{n=1}^{\infty} 2^{-n}\{\sup_{0 \leq t \leq 2^n} |w(t) - \hat{w}(t)| \wedge 1\}$ and $\mathscr{F}^m$ be the σ-field generated by all cylinder sets in W^m. Let $\mathscr{P}(W^m)$ be the space of all probability measures on $(W^m, \mathscr{F}^m)$ with the topology of weak convergence. Let $X_{(\tau)}^{N|m}$ be the W^m-valued random variable on $(\Omega_{N,h}, P_{N,\tau,h})$ such that $X_{(\tau)}^{N|m}$ is the polygonal function whose value at a point $t \geq 0$ is given by

$$X_{(\tau)}^{N|m}(t) = (X_{(\tau)}^{N,1}(t), \cdots, X_{(\tau)}^{N,m}(t)), \tag{5.1}$$

where

$$X_{(\tau)}^{N,i}(t) = X_{[t/\tau]}^{N,i} + ((t/\tau) - [t/\tau])\{X_{[t/\tau]+1}^{N,i} - X_{[t/\tau]}^{N,i}\}.$$

Let $P_{N|m,\tau,h}$ be the probability measure on $(W^m, \mathscr{F}^m)$ such that $P_{N|m,\tau,h}(A) = P_{N,\tau,h}(X_{(\tau)}^{N|m} \in A)$ for all $A \in \mathscr{F}^m$. Let B_m be the set of (N, τ, h) satisfying (1.5) with $N \geq m$ and $(\tau, h) \in B$ (see (2.5)).

LEMMA 5.1(*tightness*). *For each sequence* $\{(N_\nu, \tau_\nu, h_\nu)\} \subset B_m$ *satisfying* $N_\nu \to \infty$ *and* τ_ν, $h_\nu \to 0$ *as* $\nu \to \infty$, *the family of the probability measures*

$$\{P_{N_\nu|m,\tau_\nu,h_\nu}: \nu = 1, 2, \cdots\}$$

is tight in $\mathscr{P}(W^m)$.

PROOF. We write $N_\nu = N$, $\tau_\nu = \tau$ and $h_\nu = h$. For each $M > 0$, put

$$C_{\nu,M} = \sum_{\substack{|x|^2 \leq M^2/m \\ x \in S_h}} \bar{u}_0(x) h^d$$

and

$$\varepsilon_{\nu,M} = |P_{N,\tau,h}(|X_0^{N,i}|^2 < M^2/m \text{ for all } i = 1, \cdots, m) - \{C_{\nu,M}\}^m|.$$

Then, by (A.2) and (1.3), we have

$$\lim_{\nu \to \infty} \varepsilon_{\nu,M} = 0 \quad \text{and} \quad \lim_{\nu \to \infty} C_{\nu,M} = \int_{|x|^2 \leq M^2/m} u_0(x)\, dx \equiv C_{\infty,M}$$

for each $M > 0$. Therefore we have

$$\limsup_{\nu \to \infty} P_{N,\tau,h}(|X_{(\tau)}^{N|m}(0)| > M) \leq 1 - \{C_{\infty,M}\}^m, \tag{5.2}$$

which converges to zero as $M \to \infty$.

Next we show the equicontinuity of the trajectory. By (3.2) with $f(x) = |x$

$- X_k^{N,i}|^4$ or $f(x) = |x - X_k^{N,i}|^2$, we have for $n > k \geq 0$ and $i = 1, \cdots, m$

(5.3) $$E_{N,\tau,h}[|X_n^{N,i} - X_k^{N,i}|^4]$$
$$= E_{N,\tau,h}[\textstyle\sum_{\ell=k}^{n-1} \frac{h^2}{2d} \varphi_r[h^{-d}\bar{X}_\ell^N](X_\ell^{N,i})\{(8 + 4d)|X_\ell^{N,i} - X_k^{N,i}|^2 + 2dh^2\}]$$

and

(5.4) $$E_{N,\tau,h}[|X_n^{N,i} - X_k^{N,i}|^2] = E_{N,\tau,h}[\textstyle\sum_{\ell=k}^{n-1} \frac{h^2}{2d} \varphi_r[h^{-d}\bar{X}_\ell^N](X_\ell^{N,i}) \cdot 2d]$$
$$\leq \xi + \varphi(\|\bar{u}_0\|_\infty)(n - k)d\tau,$$

where $r = d\tau h^{-2}$ and

$$\xi = \textstyle\sum_{\ell=k}^{n-1} h^2 E_{N,\tau,h}[|\varphi_r[h^{-d}\bar{X}_\ell^N](X_\ell^{N,i}) - \varphi_r[\bar{u}_\ell](X_\ell^{N,i})|].$$

For each $t > s \geq 0$, put

$$I = (E_{N,\tau,h}[|X_{[t/\tau]}^{N,i} - X_{[s/\tau]}^{N,i}|^4])^{1/2},$$

then by (5.3), (5.4), (3.4), (3.10), (A.1), (1.5) and Hölder's inequality we have

$$I^2 \leq \xi_1 I + \xi_2 + (2 + d)\varphi(\|\bar{u}_0\|_\infty)^2(t - s)^2 d$$

for some negligible constants ξ_1, $\xi_2 > 0$. It follows that

(5.5) $$\limsup_{\nu\to\infty} E_{N,\tau,h}[|X_{[t/\tau]}^{N,i} - X_{[s/\tau]}^{N,i}|^4] \leq (4 + 2d)\varphi(\|u_0\|_\infty)^2(t - s)^2 d.$$

By the definition (5.1) we have for each T, $\varepsilon > 0$

$$P_{N,\tau,h}(\max_{\substack{0 \leq s < t \leq T \\ |t-s|<\delta}} |X_{(\tau)}^{N|m}(t) - X_{(\tau)}^{N|m}(s)|^2 > \varepsilon^2)$$
$$\leq \textstyle\sum_{i=1}^m P_{N,\tau,h}(\max_{\substack{0 \leq s < t \leq T \\ |t-s|<\delta}} |X_{[t/\tau]}^{N,i} - X_{[s/\tau]}^{N,i}|^2 + 8h^2 > \varepsilon^2/m)$$
$$\leq \textstyle\sum_{i=1}^m \sum_{j=0}^J P_{N,\tau,h}(\max_{k_j \leq \ell \leq n_j} |X_\ell^{N,i} - X_{k_j}^{N,i}|^2 > \varepsilon'^2),$$

where $\varepsilon' = \{(\varepsilon^2/m) - 8h^2\}^{1/2}$, $k_j = [j\delta/2\tau]$, $n_j = [(j + 2)\delta/2\tau]$ and $J = [2T/\delta] - 1$. By the martingale inequality and (5.5) we have

(5.6) $$\limsup_{\nu\to\infty} P_{N,\tau,h}(\max_{\substack{t,s\in[0,T] \\ |t-s|<\delta}} |X_{(\tau)}^{N|m}(t) - X_{(\tau)}^{N|m}(s)| > \varepsilon)$$
$$\leq \limsup_{\nu\to\infty} \textstyle\sum_{i=1}^m \sum_{j=0}^J \varepsilon'^{-4} E_{N,\tau,h}[|X_{n_j}^{N,i} - X_{k_j}^{N,i}|^4]$$
$$\leq m(2T/\delta)(m^2/\varepsilon^4)(4 + 2d)\varphi(\|u_0\|_\infty)^2\delta^2 d \downarrow 0 \ \ (\text{as } \delta \downarrow 0)$$

for each $T>0$ and $\varepsilon>0$. Then the tightness of the family $\{P_{N_\nu|m,\tau_\nu,h_\nu}\}$ follows from (5.2) and (5.6). □

By Lemma 5.1 there exist a probability measure P on W^m and a sequence $\{(N_\nu, \tau_\nu, h_\nu)\} \subset B_m$ such that $P_{N_\nu|m,\tau_\nu,h_\nu}$ converges to P weakly as $\nu \to \infty$ and $N_\nu \to \infty$, $\tau_\nu \to 0$, $h_\nu \to 0$. For each $t \geq 0$ and $i = 1, \cdots, m$, let $X^{(i)}(t)$ be the function on W^m defined by $X^{(i)}(t, w) = w^{(i)}(t)$ for every $w = \{(w^{(1)}(t), \cdots, w^{(m)}(t)): t \geq 0\} \in W^m$. For each $t_0 = 0 < \cdots < t_\ell$ and $\tau > 0$, put $n_k = [t_k/\tau]$ $(k = 0, \cdots, \ell)$. By Proposition 2, (A.1) and (A.2) we see that, for each $f^i_k \in C_0(\boldsymbol{R}^d \to [0, 1])$ $(i = 1, \cdots, m,\ k = 0, \cdots, \ell)$, each of the following terms

$$|E_{N,\tau,h}[\prod_{k=0}^{\ell} \prod_{i=1}^{m} f^i_k(X^{N,i}_{n_k})] - \prod_{i=1}^{m} E_{\tau,h}[\prod_{k=0}^{\ell} f^i_k(Y_{n_k})]|$$

and

$$|E_{\tau,h}[\prod_{k=0}^{\ell} f^i_k(Y_{n_k})] - E_{N,\tau,h}[\prod_{k=0}^{\ell} f^i_k(X^{N,i}_{n_k})]| \quad (i = 1, \cdots, m)$$

converges to zero as $(N, \tau, h) \underset{(1.5)}{\longrightarrow} (\infty, 0, 0)$. By the weak convergence of the probability measures $\{P_{N_\nu|m,\tau_\nu,h_\nu}: \nu \geq 0\}$, we have

$$E[\prod_{i=1}^{m} \prod_{k=0}^{\ell} f^i_k(X^{(i)}(t_k))] = \prod_{i=1}^{m} E[\prod_{k=0}^{\ell} f^i_k(X^{(i)}(t_k))],$$

which implies

$$P((X^{(1)}, \cdots, X^{(m)}) \in dw_1 \times \cdots \times dw_m) = \prod_{i=1}^{m} P(X^{(i)} \in dw_i).$$

To prove that the d-dimensional processes $X^{(i)}$ $(i = 1, \cdots, m)$ are identically distributed, we will show that distribution of $X^{(i)}$ is characterized by the following nonlinear martingale problem (cf. Funaki [6]). Let $W = W^1$ be the space of all continuous functions $w: [0, \infty) \to \boldsymbol{R}^d$ and $\mathscr{F} = \mathscr{F}^1$ be the σ-field generated by all cyclinder sets in W. Let P_i be a probability measure on $(W, \mathscr{F})$ defined by $P_i(A) = P(\{(w_1, \cdots, w_m): w_i \in A\})$ for all $A \in \mathscr{F}$. Let $\mathscr{F}^{(i)}_t$ be the σ-field in W generated by $\{X^{(i)}(s): 0 \leq s \leq t\}$ and all P_i-null sets. Then we have the following

LEMMA 5.2 (*martingale problem*). *For each* $i \in \{1, \cdots, m\}$ *and* $t \geq 0$, *we get*

$$P_i(X^{(i)}(t) \in dx) = u(t, x)\,dx, \qquad (x \in \boldsymbol{R}^d) \tag{5.7}$$

where $u = u(t, x)$ *is the unique weak solution of* (1.1). *Further for* $f \in C_0^\infty([0, \infty) \times \boldsymbol{R}^d \to \boldsymbol{R})$ *the process*

$$\left\{ f(t, X^{(i)}(t)) - \int_0^t L(u; f)(s, X^{(i)}(s))\,ds : t \geq 0 \right\} \tag{5.8}$$

is an $\mathscr{F}^{(i)}_t$*-martingale on* $(W, \mathscr{F}^{(i)}, P_i)$, *where*

$$L(u;f)(s,x)=f_t(s,x)+\tfrac{1}{2}\varphi(u(s,x))(\triangle f)(s,x)$$

and $\triangle$ *is the d-dimensional Laplacian with respect to the variable* $x\in\boldsymbol{R}^d$.

PROOF. Firstly we show (5.7). Fix $g\in C_0(\boldsymbol{R}^d\to[0,1])$ and $t\geq 0$. By (A.1), (A.2), (4.3) and Proposition 2 with $m=\ell=1$ and $n=[t/\tau]$, we have

$$\begin{aligned}&|E_{N,\tau,h}[g(X^{N,i}_{[t/\tau]})]-\textstyle\sum_{x\in S_h}g(x)\bar{u}_{[t/\tau]}(x)h^d|\\&\leq\textstyle\sum_{x\in S_h}g(x)h^d|P_{N,\tau,h}(X^{N,i}_0=x)h^{-d}-\bar{u}_0(x)|\\&\quad+2D(\varphi)K^{t/\tau}h^{-d\gamma}(E_{N,\tau,h}[\|h^{-d}\cdot\bar{X}^N_0-\bar{u}_0\|^2_{(h)}]+4/Nh^d)^\gamma,\end{aligned}$$

which vanishes as $(N,\tau,h)\xrightarrow[(1.5)]{}(\infty,0,0)$. By the weak convergence of the probability measures $\{P_{N_\nu|m,\tau_\nu,h_\nu}\}$ and Proposition 1 we have

$$E_i[g(X^{(i)}(t))]=E[g(X^{(i)}(t))]=\int_{\boldsymbol{R}^d}g(x)u(t,x)dx,$$

which implies (5.7). Next we show the martingale property. For $g\in C_b(\boldsymbol{R}^d\to\boldsymbol{R})$ and $t>s\geq 0$, we show

$$\begin{aligned}(5.9)\qquad&E_i[f(t,X^{(i)}(t))g(X^{(i)}(s))]\\&=E_i[\{f(s,X^{(i)}(s))+\int_s^t L(u;f)(\theta,X^{(i)}(\theta))d\theta\}g(X^{(i)}(s))].\end{aligned}$$

By (3.2) we have for $n>k\geq 0$

$$\begin{aligned}&E_{N,\tau,h}[f(n\tau,X^{N,i}_n)g(X^{N,i}_k)]\\&=E_{N,\tau,h}[\{f(k\tau,X^{N,i}_k)+\textstyle\sum_{\ell=k}^{n-1}(f((\ell+1)\tau,X^{N,i}_{\ell+1})-f(\ell\tau,X^{N,i}_{\ell+1}))\\&\quad+\textstyle\sum_{\ell=k}^{n-1}(2d)^{-1}h^2\varphi_r[h^{-d}\bar{X}^N_\ell](X^{N,i}_\ell)(\triangle_h f)(\ell\tau,X^{N,i}_\ell)\}g(X^{N,i}_k)],\end{aligned}$$

where $r=d\tau h^{-2}$. Put $n=[t/\tau]$ and $k=[s/\tau]$. By (3.4) and Theorem 1 we have

$$\begin{aligned}&E_{N,\tau,h}[|\textstyle\sum_{\ell=k}^{n-1}(2d)^{-1}h^2\varphi_r[h^{-d}\bar{X}^N_\ell](X^{N,i}_\ell)-\sum_{\ell=k}^{n-1}\frac{1}{2}\varphi(u(\ell\tau,X^{N,i}_\ell))\tau|]\\&\qquad\longrightarrow 0\ (\text{as } (N,\tau,h)\xrightarrow[(1.5)]{}(\infty,0,0)).\end{aligned}$$

Suppose $N=N_\nu$, $\tau=\tau_\nu$ and $h=h_\nu$. By the weak convergence of $\{P_{N_\nu|m,\tau_\nu,h_\nu}\}$ we have

$$\begin{aligned}&E_i[f(t,X^{(i)}(t))g(X^{(i)}(s)]\\&=\lim_{\nu\to\infty}E_{N,\tau,h}[\{f(k\tau,X^{N,i}_k)+\textstyle\sum_{\ell=k}^{n-1}\tau f_t(\ell\tau,X^{N,i}_{\ell+1})\end{aligned}$$

$$+ \sum_{\ell=k}^{n-1} \tfrac{1}{2}\varphi(u(\ell\tau, X_\ell^{N,i}))\tau(\triangle_h f)(\ell\tau, X_\ell^{N,i})\} g(X_k^{N,i})]$$

$$= E_i[\{f(s, X^{(i)}(s)) + \int_s^t f_t(\theta, X^{(i)}(\theta))\,d\theta$$

$$+ \int_s^t \tfrac{1}{2}\varphi(u(\theta, X^{(i)}(\theta)))(\triangle f)(\theta, X^{(i)}(\theta))d\theta\} g(X^{(i)}(s))],$$

which implies (5.9). By the same method as above, we get the same equation as (5.9) with $g_1(X^{(i)}(s_1))\cdots g_p(X^{(i)}(s_p))$ in place of $g(X^{(i)}(s))$ for all integer $p \geq 1$, non-negative numbers $s_1 < \cdots < s_p = s < t$ and functions $g_1, \cdots, g_p \in C_b(\boldsymbol{R}^d \to \boldsymbol{R})$. Hence we have

$$E_i[f(t, X^{(i)}(t))|\mathscr{F}_s^{(i)}] = f(s, X^{(i)}(s)) + \int_s^t L(u; f)(\theta, X^{(i)}(\theta))d\theta,$$

as was to be proved. □

To prove the uniqueness (in the law sense) of the nonlinear martingale problem (5.7)–(5.8), we show the following

LEMMA 5.3 (*Markov property*). *The process* $X^{(i)} = \{X^{(i)}(t)\}$ *on* $(W, \mathscr{F}, P_i; \mathscr{F}_t^{(i)})$ *is a Markov process with the generator*

$$\{\mathscr{G}_t^i = \tfrac{1}{2}\varphi(u(t, x))\,\triangle : t \geq 0\}. \tag{5.10}$$

PROOF. To prove the Markov property of $X^{(i)}$, we will show

$$\begin{aligned} &E_i[f(X^{(i)}(t_0))g_0(X^{(i)}(s_0))\cdots g_\ell(X^{(i)}(s_\ell))] \\ &\quad = E_i[E_i[f(X^{(i)}(t_0))|X^{(i)}(s_0)]\, g_0(X^{(i)}(s_0))\cdots g_\ell(X^{(i)}(s_\ell))] \end{aligned} \tag{5.11}$$

for each integer $\ell \geq 1$, real numbers $t_0 > s_0 > \cdots > s_\ell \geq 0$ and functions $f \in C_0^\infty(\boldsymbol{R}^d \to \boldsymbol{R})$, $g_0, \cdots, g_\ell \in L^1(\boldsymbol{R}^d \to [0, 1])$. For each $t \geq 0$, let $v_1(t, dx)$ and $v_2(t, dx)$ be measures on $\boldsymbol{R}^d$ defined by

$$\int_{\boldsymbol{R}^d} f(x)v_1(t, dx) = E_i[f(X^{(i)}(t + s_0))g_0(X^{(i)}(s_0))\cdots g_\ell(X^{(i)}(s_\ell))]$$

and

$$\int_{\boldsymbol{R}^d} f(x)v_2(t, dx) = E_i[E_i[f(X^{(i)}(t + s_0))|X^{(i)}(s_0)]\, g_0(X^{(i)}(s_0))\cdots g_\ell(X^{(i)}(s_\ell))].$$

By (5.7) we see that $v_1(t, dx)$ and $v_2(t, dx)$ have densities $v_1(t, x)$ and $v_2(t, x)$ satisfying

$$0 \leq v_1(t, x),\ v_2(t, x) \leq u(t + s_0, x),$$

where $u(t, x)$ is the unique weak solution of (1.1). It follows from (5.8) that $v_1 = v_1(t, x)$ and $v_2 = v_2(t, x)$ satisfy the following linear differential equation

$$\frac{\partial v}{\partial t}(t, x) = \frac{1}{2}\triangle(u(t + s_0, x)v(t, x))$$

in the distribution sense. By the definition of v_1 and v_2 we have $v_1(0, x) = v_2(0, x)$. Put $z(t, x) = v_1(t, x) - v_2(t, x)$ and $w(t, x) = 2^{-1}\varphi(u(t + s_0, x))z(t, x)$. Then, by Lemma 2.4, $z(t, x) = 0$ a.e. on $[0, T] \times \boldsymbol{R}^d$ for all $T > 0$. Hence we have $v_1(t_0 - s_0, x) = v_2(t_0 - s_0, x)$ a.e. $x \in \boldsymbol{R}^d$, which implies (5.11). Thus the Markov property has been proved. By (5.8) with $f(t, x) = f(x) \in C_0^\infty(\boldsymbol{R}^d)$, the generator of the process $X^{(i)}$ is

$$\begin{aligned}(\mathscr{G}_t^i f)(x) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{E_i[f(X^{(i)}(t + \varepsilon)) \mid X^{(i)}(t) = x] - f(x)\}\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E_i\left[\int_t^{t+\varepsilon}\tfrac{1}{2}\varphi(u(s, X^{(i)}(s))(\triangle f)(X^{(i)}(s))\,ds \mid X^{(i)}(t) = x\right]\\ &= \tfrac{1}{2}\varphi(u(t, x))(\triangle f)(x),\end{aligned}$$

as was to be proved. □

By the martingale representation theorem (see e.g. Ikeda and Watanabe [7] p. 90), the d-dimensional diffusion process $\{X^{(i)}(t) = (X_1^{(i)}(t), \cdots, X_d^{(i)}(t))\}$ satisfies the stochastic differential equation (1.12).

Finally we note that the limit of the probability measures $\{P_{N|m,\tau,h}\}$ as N tends to infinity and τ, h tend to zero satisfying (1.5) is unique in $\mathscr{P}(W^m)$, because if there exists a probability measure $\hat{P}$ on $(W^m, \mathscr{F}^m)$ as a limit of the probability measures $\{P_{N|m,\tau,h}\}$ along some subsequence of B_m, then the distribution

$$\hat{P}((X^{(1)}, \cdots, X^{(m)}) \in dw_1 \times \cdots \times dw_m) = \textstyle\prod_{i=1}^m \hat{P}(X^{(i)} \in dw_i)$$

is determined by the same generator (5.10) and therefore coincides with the distribution

$$\textstyle\prod_{i=1}^m P_i(X^{(i)} \in dw_i) = P((X^{(1)}, \cdots, X^{(m)}) \in dw_1 \times \cdots \times dw_m).$$

Thus we complete the proof of Theorem 2.

§6. Remarks

A) *Self-similar diffusion process.* Let u_0 be a continuous function on $\boldsymbol{R}^d$ satisfying the condition (1.3). Fix $\alpha > 1$. Let $X = \{X(t)\}$ be a d-dimensional diffusion process satisfying (4) and (5) in §0 with the initial density $u(0, x)$

$= u_0(x)$. Put $\beta = (d(\alpha - 1) + 2)^{-1}$ and $X_k(t) = k^{-\beta} \cdot X(kt)$ for $k > 0$. Then the process $X_k = \{X_k(t)\}$ converges in law to a d-dimensional diffusion process $X_\infty = \{X_\infty(t)\}$ satisfying (4) and (5) with $u_{(\alpha)}$ in place of u and $X_\infty(0) = \mathbf{0}$ $(\in \boldsymbol{R}^d)$ with probability 1, where $u_{(\alpha)}$ is Barenblatt's function ([2]) described by

$$u_{(\alpha)}(t, x) = L^{-1} t^{-d\beta} (\{1 - |x|^2 (Jt)^{-2\beta}\}_+)^{1/(\alpha-1)},$$

$$J = \kappa A^{-\alpha+1}, \; L = \kappa^{d\beta} A^{2\beta}, \; \kappa = \alpha/\beta(\alpha - 1),$$

$$A = \Gamma\left(\frac{\alpha}{\alpha - 1}\right)\{\Gamma(1/2)\}^d / \Gamma\left(\frac{\alpha}{\alpha - 1} + \frac{1}{2}\right) \quad \text{and} \quad \{x\}_+ = \max\{x, 0\}.$$

The function $u_{(\alpha)}$ satisfies the d-dimensional porous medium equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \triangle (u^\alpha)$$

in the domain $\{(t, x) \in (0, \infty) \times \boldsymbol{R}^d \colon |x| < (Jt)^\beta\}$ and

$$\int_{\boldsymbol{R}^d} u_{(\alpha)}(t, x)\, dx = 1, \qquad (t > 0).$$

The limiting diffusion process X_∞ is self-similar with the exponent β: i.e. $X_\infty(kt) \sim k^\beta X_\infty(t)$ for all $k, t > 0$. This limit theorem follows from the analytic results for the weak solution u of (1) (see Friedman-Kamin [5] and Veron [19]).

B) *Self-similar sequence of Markov measures.* In case of $\varphi(u) = u^{\alpha-1}$ $(\alpha > 1)$, the transition rule (M.2) is independent of τ, h if and only if $\tau = ah^{d(\alpha-1)+2}$ for some constant $a > 0$. If $P_{N,\tau,h}(X_0^{N,1} = \cdots = X_0^{N,N} = \mathbf{0}) = 1$ for all N, τ and h, then

$$P_{N,\tau,h}(hA) = P_{N,a,1}(A)$$

holds for all $\tau = ah^{d(\alpha-1)+2}$, $h > 0$, $N \geq 1$, $a > 0$ and $A \in \mathscr{B}(\Omega_{N,1})$, where $hA = \{h\omega = (h\omega_0, h\omega_1, \cdots) \colon \omega = (\omega_0, \omega_1, \cdots) \in A\}$ and $h\omega_n = (h\omega_n^1, \cdots, h\omega_n^N) \in (S_h)^N$ for $\omega_n = (\omega_n^1, \cdots, \omega_n^N) \in (S_1)^N$. In this sense the sequence of the Markov measures

$$\{P_{N,\tau,h} \colon \tau = ah^{d(\alpha-1)+2}, \; h > 0\}$$

may be called *self-similar.* In the case $(\varphi(u) = u^{\alpha-1}, X_0^{N,i} = \mathbf{0})$, if each process $\{X_{[t/\tau]}^{N,i} \colon t \geq 0\}$ on $(\Omega_{N,h}, P_{N,\tau,h})$ converges in law as $N \to \infty$ and $\tau, h \to 0$ satisfying $\tau = ah^{d(\alpha-1)+2}$, then the limiting process is self-similar with the exponent $\beta = (d(\alpha - 1) + 2)^{-1}$. We think that the limiting process is the same process $X_\infty = \{X_\infty(t)\}$ as above.

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