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On the differential equation $y'' + p(t)|y'|\operatorname{sgn} y + q(t)y = 0$

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0. Introduction. Let $y = y(t)$ be a nontrivial solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{0.1}$$

on the interval I and $t_1, t_2 \in I$ be consecutive zeros of $y(t)$ such that $t_1 < t_2$ and $|p(t)| \le M_1$, $|q(t)| \le M_2$ for $t_1 \le t \le t_2$. Then the well-known inequality of de la Vallée Poussin [16] states that for $h = t_2 - t_1$ the relation

$$\frac{1}{2}M_2h^2 + 2M_1h > 1$$

holds. There were several attempts to sharpen this inequality (see for references in [10] pp. 375–376) and Z. Opial [12] has established the optimal inequality of this form

$$M_2h^2 + 2M_1h \ge \pi^2. \tag{0.2}$$

Recently J.H.E. Cohn [2] has found another inequality

$$h \ge 2\int_0^\infty \frac{ds}{M_2s^2 + M_1s + 1}. \tag{0.3}$$

We shall see that this inequality is sharper than (0.2). The Cohn's proof is a skillful application of some differential inequlity which has the flavour of a particular Sturmian comparison theorem. Just this is the direction in which we shall proceed in this paper to obtain (0.3) and similar results.

It is well-known that Sturm [15] worked out his theorems for differential equations of the self-adjoint form

$$(r(t)y')' + q(t)y = 0 \tag{0.4}$$

(see [5] or [6]). Kamke [7] gave a new proof of Sturmian theorems by using Prüfer transformation and his method made possible the extension of the Sturmian theorems to half-linear second order differential equations of the form

$$(r(t)y')' + q(t)f(y, r(t)y') = 0 \tag{0.5}$$

(see [1], [3] and the references therein) where $f(y, z)$ is some homogeneous function of first degree.

In order to be precise we mention that under Sturmian theorems we mean two groups of theorems in general. To the first group belongs only one theorem on two linearly independent solutions of *one* differential equation stating the interlacing property of the zeros of the solutions. To the second group belong three theorems comparing special solutions of *two* differential equations

$$(r_i(t)y_i')' + q_i(t)y_i = 0, \qquad r_i(t) > 0,\ i = 1, 2.$$

The first Sturmian comparison theorem compares the zeros of the solutions $y_i(t)$, $(i = 1, 2)$. The second one compares the functions $r_i y_i'/y_i$. The third one—which is properly due to Watson ([17] p. 518, see also [4], [9])—compares the solutions $y_1(t)$, $y_2(t)$ under the condition $r_1(t) = r_2(t)$.

The main difference between (0.1) and (0.4) is the fact that in the "common" case (0.4) we have one term besides the second order expression $(ry')'$ while in (0.1) there are two terms. By using the notation $\tilde{r}(t) = \exp(\int_{t_0}^t p(s)\,ds)$, $\tilde{q}(t) = q(t)\tilde{r}(t)$ we can transform (0.1) into (0.4) with coefficients $\tilde{r}(t)$, $\tilde{q}(t)$, resp. and imposing the necessary restrictions on $\tilde{q}$, $\tilde{r}$ we can get the corresponding version of the comparison theorems (see [11]). The two terms in (0.1) cause the difficulty which prevents a complete extension of Sturmian comparison theorems to (0.1). However a partial extension is still possible. In [8] H.G. Kaper and M.K. Kwong succeeded in proving the second and the third comparison theorems under the extra restriction $y_i(t)y_i'(t) > 0$ on I.

Keeping in mind the Cohn's proof and the results of H.G. Kaper and M.K. Kwong we find that a proper setting of the problem should be a theory of half-linear second order differential equations of the form

$$y'' + p(t)|y'|\operatorname{sgn} y + q(t)y = 0. \tag{0.6}$$

These differential equations are no more linear but they preserve many properties of linear equations. In Section 1 we extend the Sturmian theorems to (0.6). In Section 2 we shall derive the inequalities

$$h\left[\sqrt{M_2} + \frac{1}{\pi}M_1\right] > \pi \qquad \text{if } M_1 > 0 \tag{0.7}$$

$$M_2 \max_{t_1 \le t \le t_2} y^2(t) = M_2 y^2(t_1') \ge y'^2(t_1)e^{-M_1(t_1' - t_1)} \tag{0.8}$$

for the solution $y(t)$ mentioned at the beginning by application of the third comparison theorem. The inequalities above are sharp and the sharpness of (0.7) lies between (0.2) and (0.3).

1. Sturmian theorems. Let a, b be real constants, $a > 0$ such that

$$a\tau^2 + b\tau + 1 > 0 \qquad \text{for all } \tau \geq 0 \tag{1.1}$$

and denote by $\hat{\pi} = \hat{\pi}(a, b)$ the integral

$$\hat{\pi}(a, b) = 2\int_0^\infty \frac{d\tau}{a\tau^2 + b\tau + 1} \tag{1.2}$$

$$= \begin{cases} \dfrac{2}{\sqrt{4a - b^2}}\left[\pi - 2\arctan\left(\dfrac{b}{\sqrt{4a - b^2}}\right)\right] & \text{for } |b| < 2\sqrt{a} \\ \dfrac{2}{\sqrt{a}} & \text{for } b = 2\sqrt{a} \\ \dfrac{2}{\sqrt{b^2 - 4a}}\log\dfrac{b + \sqrt{b^2 - 4a}}{b - \sqrt{b^2 - 4a}} & \text{for } b > 2\sqrt{a}. \end{cases}$$

Let $S = S(\varphi)$ be the solution of the differential equation

$$S'' + b|S'|\operatorname{sgn} S + aS = 0 \qquad \text{for } -\infty < \varphi < \infty \tag{1.3}$$

with the initial conditions

$$S(0) = 0, \quad S'(0) = 1. \tag{1.4}$$

By (1.3) the function $T(\varphi) = S(\varphi)/S'(\varphi)$ is a solution of

$$T' = aT^2 + b|T| + 1, \tag{1.5}$$

hence

$$\int_0^{T(\varphi)} \frac{d\tau}{a\tau^2 + b|\tau| + 1} = \varphi, \tag{1.6}$$

which uniquely defines the funcin $T(\varphi)$ on the interval $(-\hat{\pi}/2, \hat{\pi}/2)$ with the properties that $T(\varphi)$ is strictly increasing there and

$$T(-\varphi) = -T(\varphi) \quad \text{and} \quad \lim_{\varphi \to \pm(\hat{\pi}/2 - 0)} T(\varphi) = \pm\infty.$$

We extend this domain of definition of $T(\varphi)$ to $\mathbf{R} \setminus \bigcup_{k=-\infty}^{\infty}\left\{\left(k + \frac{1}{2}\right)\hat{\pi}\right\}$ by the relation $T(\varphi + \hat{\pi}) = T(\varphi)$. Hence $\lim_{\varphi \to \pm\hat{\pi}/2} S'(\varphi) = 0$ and by (1.3) $S(\varphi)$ has a local maximum at $\varphi = \hat{\pi}/2$ and a local minimum at $\varphi = -\hat{\pi}/2$. Clearly $S(-\varphi) = -S(\varphi)$ on $[-\hat{\pi}/2, \hat{\pi}/2]$. With the aid of $T(\varphi)$ the functions $S(\varphi)$ has the representation

$$S(\varphi)=\frac{T(\varphi)}{\sqrt{aT^2(\varphi)+b|T(\varphi)|+1}}\,e^{-\frac{b}{2}\varphi}\qquad\text{for } -\hat{\pi}/2\le\varphi\le\hat{\pi}/2. \tag{1.7}$$

Formula (1.7) can be proved directly by using the relations $T=S/S'$ and (1.4), (1.5).

For later purpose we need the particular value $S(\hat{\pi}/2)$ from (1.7):

$$S\left(\frac{\hat{\pi}}{2}\right)=M(\mathrm{a},b)=\frac{1}{\sqrt{a}}\exp\left(-\frac{b\hat{\pi}(a,b)}{4}\right). \tag{1.8}$$

Finally we extend the domain of definition of $S(\varphi)$ from $[-\hat{\pi}/2,\hat{\pi}/2]$ to $(-\infty,\infty)$ by the relation

$$S(\varphi+\hat{\pi})=-S(\varphi).$$

Hence the function $S(\varphi)$ is periodic with the period $2\hat{\pi}$ and $S(\varphi)=0$ only for $\varphi=k\hat{\pi}$, $k=0,\ \pm 1,\ \pm 2,\dots$, where $S'(k\hat{\pi})=(-1)^k$ and $S'(\varphi)=0$ only at $\varphi=\left(k+\frac{1}{2}\right)\hat{\pi}$ with the particular values $S\left(\left(k+\frac{1}{2}\right)\hat{\pi}\right)=(-1)^k M(a,b)$. We should observe there that $S(\varphi)$ and $S'(\varphi)$ are not vanishing anywhere at the same time.

We then define $\tilde{T}(\varphi)$ by

$$\tilde{T}(\varphi)=\frac{S'(\varphi)}{S(\varphi)}\qquad\text{for } \varphi\neq k\hat{\pi}\quad k=0,\ \pm 1,\ \pm 2,\dots \tag{1.9}$$

Clearly, the functin $\tilde{T}(\varphi)$ is periodic with period $\hat{\pi}$, continuous and strictly decreasing on $(0,\hat{\pi})$. Let us introduce the polar coordinates φ, $\varrho(-\infty<\varphi<\infty$, $\varrho\ge 0)$ on the plane (x_1, x_2) as follows

$$\begin{aligned} x_1&=\varrho S'(\varphi)\\ x_2&=\varrho S(\varphi)\end{aligned} \tag{1.10}$$

The particular value $\varrho=0$ belongs to the origin (0, 0) of the plane (x_1, x_2). In this case the value of φ may be taken arbitrary. For other points of the plane (x_1, x_2) $\varrho>0$ and there is a unique value of φ_0 on $[0, 2\hat{\pi})$ such that (1.10) holds with $\varphi=\varphi_0$. In case $x_2=0$ we have $\varphi_0=0$ or $\hat{\pi}$ according to $x_1>0$ or $x_1<0$ and $\varrho=|x_1|$. Similarly, in case $x_1=0$ we obtain $\varphi_0=\hat{\pi}/2$ or $3\hat{\pi}/2$ according to $x_2>0$ or $x_2<0$ and $\varrho=|x_2|/M(a,b)$. Finally when $x_1x_2\neq 0$ we have $T(\varphi_0)=x_2/x_1$, and so there are two values $\tilde{\varphi}$ and $\tilde{\varphi}+\hat{\pi}$ such that $\tilde{\varphi}\in(0,\hat{\pi})$, $T(\tilde{\varphi})=T(\tilde{\varphi}+\hat{\pi})=x_2/x_1$. Since $S(\tilde{\varphi})S(\tilde{\varphi}+\hat{\pi})<0$ we choose the proper value of φ_0 from $\tilde{\varphi}$, $\tilde{\varphi}+\hat{\pi}$ for which $x_2S(\varphi_0)>0$. To accomplish the procedure we take $\varrho=x_2/S(\varphi_0)$.

If we drop the restriction on φ we have infinitely many pairs of polar coordinates to every given point $(x_1, x_2) \neq (0, 0)$, namely $(\varrho, \varphi_0 + 2k\hat{\pi})$ with $k = 0, \pm 1, \pm 2, \ldots$.

We observe that the polar transformation defined by (1.10) is smooth, locally invertible for $\varrho > 0$, $\varphi \in \mathbf{R}$ because the determinant of the Jacobian $\partial(\varrho, \varphi)/\partial(x_1, x_2)$ is

$$\begin{vmatrix} \dfrac{\partial\varrho}{\partial x_1} & \dfrac{\partial\varrho}{\partial x_2} \\ \dfrac{\partial\varphi}{\partial x_1} & \dfrac{\partial\varphi}{\partial x_2} \end{vmatrix} = \frac{1}{\varrho\Delta(\varphi)}$$

where

$$\Delta(\varphi) = aS^2(\varphi) + b|S(\varphi)S'(\varphi)| + [S'(\varphi)]^2 \tag{1.11}$$

and $\Delta(\varphi)$ is periodic with the period $\hat{\pi}$ and by (1.11) $0 < \min_{0 \le \varphi \le \hat{\pi}} \Delta(\varphi) \le \Delta(\varphi) \le \max_{0 \le \varphi \le \hat{\pi}} \Delta(\varphi) < \infty$.

Now we can pass over to the generalization of the Prüfer transformation. Let us consider the half-linear differential equation

$$(r(t)y')' + p(t)|y'|\operatorname{sgn} y + q(t)y = 0 \tag{1.12}$$

where the functions $p(t)$, $q(t)$, $r(t)$ are piece-wise continuous on I and $r(t) > 0$. A function $y = y(t)$ is a solution of (1.12) if y is continuous, piece-wise continuously differentiable, $r(t)y'(t)$ is continuous, piece-wise differentiable satisfying (1.12) piece-wise (see [13], pp. 26).

Let us introduce the continuous functions $\varrho(t)$, $\varphi(t)$ as polar coordinates by

$$\begin{aligned} y &= \varrho(t)S(\varphi(t)) \\ r(t)y' &= \varrho(t)S'(\varphi(t)). \end{aligned} \tag{1.13}$$

Then $\varrho = \varrho(t)$, $\varphi = \varphi(t)$ satisfy the first order differential system

$$\varphi' = \frac{1}{r}\Delta_1(\varphi) + \frac{p}{r}\Delta_2(\varphi) + q\Delta_3(\varphi) \tag{1.14}$$

$$\varrho' = \varrho \frac{\dfrac{a}{r} - q + \operatorname{sgn}(S'(\varphi)S(\varphi))\dfrac{b-p}{r}}{\Delta(\varphi)} S(\varphi)S'(\varphi), \tag{1.15}$$

where

$$\begin{aligned} &\Delta_1(\varphi) = \frac{[S'(\varphi)]^2}{\Delta(\varphi)}, \quad \Delta_2(\varphi) = \frac{|S(\varphi)S'(\varphi)|}{\Delta(\varphi)}, \quad \Delta_3(\varphi) = \frac{[S(\varphi)]^2}{\Delta(\varphi)} \\ &\Delta_1(\varphi) + b\Delta_2(\varphi) + a\Delta_3(\varphi) = 1 \end{aligned} \tag{1.16}$$

and $\Delta(\varphi)$ is given in (1.11). To a fixed solution $y(t)$ the radial part $\varrho(t)$ is fixed while $\varphi(t)$ can be also $\varphi(t)+2k\hat{\pi}$ for any $k=0, \pm 1, \pm 2, \ldots$. This means that for $\varphi(t)$ we have to give further information about how to choose the proper value of k. We mention here that for the solution $-y(t)$ we have the same radial part $\varrho(t)$ while for the second coordinate we have $\varphi(t) \pm \hat{\pi}$.

The functions $\Delta_i(\varphi)$ $(i=1, 2, 3)$ in (1.16) are Lipschitzian. Hence any initial value problem to (1.14) with initial condition $\varphi(t^0)=\varphi^0$ has a unique solution $\varphi(t)$. Inserting this already known function $\varphi(t)$ into the differential equation (1.15) of ϱ we have

$$\varrho(t)=\varrho(t^0)\exp\left(\int_{t^0}^{t}\frac{\frac{a}{r}-q+\operatorname{sgn}\left(S(\varphi)S'(\varphi)\right)\frac{b-p}{r}}{\Delta(\varphi)}S(\varphi)S'(\varphi)\right)$$

i.e. the function $\varrho(t)$ is also unique; it remains finite and positive for all $t \in I$. A singularity will occur if either $r(t)\to 0$, or $|q(t)|\to\infty$ or $|p(t)|\to\infty$ at some endpoint of I. Moreover by (1.13) we find that $y(t)$ and $y'(t)$ are not vanishing simultaneously. This uniqueness of $\varrho(t)$, $\varphi(t)$ implies also the uniqueness of the solutions of (1.12) with initial conditions $y(t^0)=y^0$, $r(t^0)y'(t^0)=Y^0$, $|y^0|+|Y^0|>0$.

Suppose that the solution $y(t)$ has consecutive zeros $t_0, t_1, t_2, \ldots$. We may assume $y(t)>0$ on (t_0, t_1) because the solutions $y(t)$ and $-y(t)$ have the same zeros. By (1.13) $\varphi(t_i)=i\hat{\pi}$, $i=0, 1, 2, \ldots$. By (1.14), (1.16) $\varphi'(t_i)=1/r(t_i)>0$ hence the function $\varphi(t)$ takes on the values $0, \hat{\pi}, 2\hat{\pi}, \ldots$ only once and we have for the zeros of $y(t)$ the relations

(1.17) $\quad \varphi(t_i)=i\pi,\ i\hat{\pi}<\varphi(t)<(i+1)\hat{\pi} \qquad \text{for } t_i<t<t_{i+1},\ i=0, 1, 2, \ldots.$

As a consequence of this relation the zeros of a solution $y(t)$ are not accumulating at an inner point of I.

LEMMA 1.1. *Let $J \subset I$ be a closed interval. Then the number of the zeros of any solution of* (1.12) *on J is bounded.*

PROOF. Let $y=y(t)$ be a solution with consecutive zeros at $t_0, t_1, \ldots, t_N$ on J. Let $\varphi(t)$ be the corresponding polar function satisfying (1.17). According to the sign of b we have three possibilities.

If $b>0$ then by (1.14), (1.17) we have

$$\hat{\pi}=\int_{t_i}^{t_{i+1}}\varphi'\,dt<\int_{t_i}^{t_{i+1}}\max\left\{\frac{1}{r}, \frac{p}{br}, \frac{q}{a}\right\}$$

hence

$$N\hat{\pi} < \int_J \max\left\{\frac{1}{r}, \frac{p}{br}, \frac{q}{a}\right\}.$$

If $b < 0$ we have similarly

$$N\hat{\pi} < \int_J \max\left\{\frac{1}{r}, -\frac{p}{br}, \frac{q}{a}\right\}.$$

If $b = 0$ then by (1.16) we get

$$\Delta_2 \leq \frac{1}{2}(\Delta_1 + \Delta_3),$$

consequently

$$N\hat{\pi} < \int_J \max\left\{\frac{1 + \frac{1}{2}|p|}{r}, \frac{|p|}{2r} + \frac{q}{a}\right\}.$$

In all cases we have established an effective upper bound of N which proves the lemma.

In the theory of linear differential equations an important role is played by the linear independence of solutions. This notion can be extended to the half-linear differential equations, too. We say that two solutions y, $\tilde{y}$ of (1.12) are linearly *dependent* if there is a constant $c \neq 0$ such that $\tilde{y}(t) = cy(t)$ for all $t \in I$. If there is no such constant the solutions y and $\tilde{y}$ are linearly *independent*.

To any pair of solutions y, $\tilde{y}$ of (1.12) we assign the Wronskian W as usual

$$W = W(t) = W(t; y, \tilde{y}) = r(y'\tilde{y} - y\tilde{y}') \tag{1.18}$$

Clearly, the function $W(t)$ is continuous and piece-wise continuously differentiable. Moreover, if y and $\tilde{y}$ are linearly dependent then $W \equiv 0$. The converse statement is also true.

LEMMA 1.2. *A pair of solutions of* (1.12) *is linearly dependent if and only if* $W \equiv 0$ *on* I.

PROOF. The only thing to be proved is that if $W(t^0) = 0$ at some point $t^0 \in I$ then $W(t) \equiv 0$ on I. Let us consider the system of equations

$$\begin{aligned} y(t^0)x_1 - \tilde{y}(t^0)x_2 &= 0 \\ r(t^0)y'(t^0)x_1 - r(t^0)\tilde{y}'(t^0)x_2 &= 0. \end{aligned} \tag{1.19}$$

Since $W(t^0)=0$ there exists a solution (x_1, x_2) such that $|x_1|+|x_2|>0$. We claim that $x_1 x_2 \neq 0$. Suppose the contrary. If $x_1=0$ then $x_2 \neq 0$ and (1.19) implies $\tilde{y}(t^0)=r(t^0)\tilde{y}'(t^0)=0$ which is impossible. Hence $x_1 \neq 0$. Similarly we have $x_2 \neq 0$.

Let $c=x_1/x_2$. Then by (1.19) the solutions $\tilde{y}$ and cy satisfy the same initial conditions at $t=t^0$ hence $\tilde{y}=cy$ for all $t\in I$. Consequently $W(t)\equiv 0$ which proves the lemma.

Now we announce our extension of Sturmian theorem. For the sake of convenience we introduce the notation I_{t^0} by $I_{t^0}=I\cap(t^0, \infty)$.

THEOREM 1.1. *Let $y(t)$, $\tilde{y}(t)$ be two linearly independent solutions of* (1.12). *Suppose $y(t^0)\geq 0$, $\tilde{y}(t^0)>0$ and $W(t^0)>0$ for some $t^0\in I$ where the Wronskian $W(t)$ is defined by* (1.18). *Denote by $t_1, t_2,\dots$ the consecutive zeros of $y(t)$ on I_{t^0} and similarly by $\tilde{t}_1, \tilde{t}_2,\dots$ the ones of $\tilde{y}(t)$. Then $\tilde{t}_1<t_1<\tilde{t}_2<t_2<\dots$.*

PROOF. Define the polar functions ϱ, φ of y and $\tilde{\varrho}, \tilde{\varphi}$ of $\tilde{y}$, resp. by (1.13). The conditions $y(t^0)\geq 0$, $\tilde{y}(t^0)>0$ imply that we can choose $0\leq\varphi(t^0)\leq\hat{\pi}$, $0<\tilde{\varphi}(t^0)<\hat{\pi}$. Then the condition $W(t^0)>0$ can be written by (1.18) as

$$[S'(\varphi)S(\tilde{\varphi})-S(\varphi)S'(\tilde{\varphi})]_{t=t^0}>0. \tag{1.20}$$

We claim that

$$0\leq\varphi(t^0)<\tilde{\varphi}(t^0)<\hat{\pi}. \tag{1.21}$$

Since $S'(\hat{\pi})=-1$ the possibility $\varphi(t^0)=\hat{\pi}$ is excluded by (1.20). If $\varphi(t^0)=0$ there is nothing to prove in (1.9). If $0<\varphi(t^0)<\hat{\pi}$ then on using (1.9) the inequality in (1.20) can be written as $\tilde{T}(\varphi(t^0))>\tilde{T}(\tilde{\varphi}(t^0))$. Since $\tilde{T}(\varphi)$ is strictly decreasing on $(0,\hat{\pi})$ we conclude (1.21). The uniqueness of the solutions of (1.15) with given initial condition implies the inequality $\varphi(t)<\tilde{\varphi}(t)$ for all $t\in I$. We have also $\varphi(t_i)=\tilde{\varphi}(t_i)=i\hat{\pi}$, hence by (1.17) $\tilde{t}_i<t_i$.

Using (1.16) we can see that $\bar{\varphi}(t)=\varphi(t)+\hat{\pi}$ is also a solution of (1.14), hence by (1.21) we obtain $\tilde{\varphi}(t)<\bar{\varphi}(t)$, consequently $\tilde{\varphi}(\tilde{t}_{i+1})=(i+1)\hat{\pi}<\varphi(\tilde{t}_{i+1})+\hat{\pi}$ and again by (1.17) $t_i<\tilde{t}_{i+1}$ which completes the proof of our theorem.

The importance of Theorem 1.1 lies in the fact that it makes possible the classification of differential equations of the form (1.12) into two classes. A solution $y(t)$ of (1.12) is *nonoscillatory* if it has finitely many zeros on I. Then by Theorem 1.1 every solution of (1.12) has only finitely many zeros and we say that (1.12) is nonoscillatory. On the other hand if there is a solution which has infinitely many zeros then by Lemma 1.1 the zeros are tending to one or both endpoints of I. Again by Theore 1.1 every solution has infinitely many zeros and differential equation (1.12) is called *oscillatory* one.

Now we shall deal with *comparison* theorems for differential equations of the form

$$(Eq_i) \qquad (r_i(t)y')' + p_i(t)|y'| \operatorname{sgn} y + q_i(t)y = 0 \qquad i = 1, 2$$

Then the differential equation Eq_2 is a *Sturmian majorant* of Eq_1 on I if the inequalities

$$(1.22) \qquad r_1(t) \geq r_2(t) > 0, \quad \frac{p_1(t)}{r_1(t)} \leq \frac{p_2(t)}{r_2(t)}, \quad q_1(t) \leq q_2(t) \qquad \text{on } I$$

hold.

The first Sturmian comparison theorem can be generalized as follows.

THEOREM 1.2. *Let Eq_2 be a Sturmian majorant of Eq_1 on I_{t^0} and let $y_i(t)$, $i = 1, 2$, be solutions of Eq_i with the initial conditions*

$$(1.23) \qquad \begin{gathered} y_1(t^0) = y_2(t^0) = y^0 \geq 0 \\ r_1(t^0)y_1'(t^0) = r_2(t^0)y_2'(t^0) = Y^0 \end{gathered}$$

at some $t^0 \in I$ with $|y^0| + |Y^0| > 0$. Suppose $y_1(t)$ has consecutive zeros $t_1, t_2, \ldots, t_n$ on I_{t^0}. Then $y_2(t)$ has consecutive zeros $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n$ on I_{t^0} such that the inequalities $\hat{t}_1 \leq t_1, \hat{t}_2 \leq t_2, \ldots, \hat{t}_n \leq t_n$ hold.

PROOF. Let φ_1, φ_2 be defined by (1.13) as polar coordinates belonging to the solutions $y_1(t)$, $y_2(t)$, resp. such that $\varphi_1(t^0)$, $\varphi_2(t_0) \in [0, \hat{\pi})$. Then (1.23) yields immediately $\varphi_1(t^0) = \varphi_2(t^0)$. By (1.14) φ_i satisfies a differential equation of the form

$$\varphi_i' = G_i(t, \varphi_i),$$

where

$$G_i(t, \varphi) = \frac{1}{r_i}\Delta_1(\varphi) + \frac{p_i}{r_i}\Delta_2(\varphi) + q_i\Delta_3(\varphi).$$

By (1.16) and (1.22) we have $G_1(\varphi) \leq G_2(\varphi)$ for every $\varphi \in \mathbf{R}$ hence the standard theory of differential inequalities (see e.g. [5], p. 27) implies $\varphi_1(t) \leq \varphi_2(t)$ on I_{t^0}. By (1.17) we have for $t = t_i$ $(i = 1, 2, \ldots, n)$ $i\hat{\pi} \leq \varphi_2(t_i)$. By the continuity there exists a value $\hat{t}_i \in (t^0, t_i]$ where $\varphi_2(\hat{t}_i) = i\hat{\pi}$. Again by (1.17) the value $\hat{t}_i$ is unique which completes the proof.

REMARK 1.1. The severe restrictions in (1.23) can be relaxed if we combine the statements of Theorem 1.1 and Theorem 1.2 as follows: Let

$$(1.24) \qquad \begin{gathered} y_1(t^0) \geq 0, \ y_2(t^0) > 0 \text{ and} \\ r_1(t^0)y_1'(t^0)y_2(t^0) - y_1(t^0)r_2(t^0)y_2'(t^0) > 0. \end{gathered}$$

Then under the conditions of Theorem 1.2 the inequalities $\hat{t}_1 < t_1$, $\hat{t}_2 < t_2, \dots$ $\hat{t}_n < t_n$ hold.

PROOF. Let $\tilde{y}(t)$ be the solution of Eq_1 with the initial conditions

$$\tilde{y}(t^0) = y_2(t^0), \quad r_1(t^0)\tilde{y}'(t^0) = Y^0 = r_2(t^0)y_2'(t^0).$$

Then $W(t^0; y, \tilde{y}) > 0$ and by Theorem 1.1 $\tilde{y}(t)$ has consecutive zeros $\tilde{t}_1, \tilde{t}_2, \dots$ $\tilde{t}_n$ on I_{t^0} satisfying the inequalities $\tilde{t}_1 < t_1, \tilde{t}_2 < t_2, \dots \tilde{t}_n < t_n$. Now we can apply Theorem 1.2 to $\tilde{y}(t)$, $y_2(t)$ satisfying the conditions (1.23), therefore we have for the zeros of $y_2(t)$ $\hat{t}_1 \le \tilde{t}_1, \dots \hat{t}_n \le \tilde{t}_n$, finally $\hat{t}_1 < t_1, \dots \hat{t}_n < t_n$ as stated.

THEOREM 1.3. *Suppose Eq_2 is a Sturmian majorant of Eq_1 and $y_i(t)$ is a solution of Eq_i $(i = 1, 2)$ satisfying the relations* (1.24) *at some $t^0 \in I$. Let (c, d) be an interval such that $c \ge t^0$ and y_1, y_2 has the same number of zeros on $[t^0, c]$ and no zeros on (c, d). Then*

$$\frac{r_1(t)y_1(t)}{y_1(t)} > \frac{r_2(t)y_2(t)}{y_2(t)} \qquad \text{on } (c, d). \tag{1.25}$$

PROOF. As in Remark 1.1 we define the solution $\tilde{y}(t)$ of Eq_1 and by (1.13) the corresponding function $\tilde{\varphi}(t)$. We consider also the functions $\varphi_1(t)$, $\varphi_2(t)$ as in the proof of Theorem 1.2. Then we find $0 \le \varphi_1(t^0) < \tilde{\varphi}(t^0) \le \varphi_2(t^0) < \hat{\pi}$. Using the differential inequality and the uniqueness for the solutions of (1.14) we obtain $\varphi_1(t) < \tilde{\varphi}(t) \le \varphi_2(t)$. Suppose $y_1(t)$, $y_2(t)$ have exactly k zeros on $(t^0, c]$ $(k = 0, 1, \dots)$. Then by (1.17) and by the assumptions about $[c, d]$ we obtain

$$k\hat{\pi} < \varphi_1(t) < \varphi_2(t) < (k+1)\hat{\pi} \qquad \text{on } (c, d),$$

consequently $\tilde{T}(\varphi_1(t)) > \tilde{T}(\varphi_2(t))$ on (c, d) which is equivalent to (1.25) due to (1.13).

The next comparison theorem was found by Watson (see [16], p. 518).

THEOREM 1.4. *Let Eq_2 be a Sturmian majorant of Eq_1 and suppose $r_1(t) = r_2(t)$. Let $y_i(t)$ be a solution of Eq_i $(i = 1, 2)$ satisfying the initial conditions* (1.23). *If the solution $y_2(t)$ has no zero on (t^0, d) and $y^0 + Y^0\delta > 0$ for sufficiently small $\delta > 0$ then the relations*

$$\frac{y_1'(t)}{y_1(t)} \ge \frac{y_2'(t)}{y_2(t)}, \quad y_1(t) \ge y_2(t) \qquad \text{on } (t^0, d) \tag{1.26}$$

hold.

PROOF. As in the proof of Theorem 1.2 we introduce functions $\varphi_1(t)$, $\varphi_2(t)$ and we find $0 \le \varphi_1(t) \le \varphi_2(t) \le \hat{\pi}$ on $[t^0, d] \in I_{t^0}$. Using the function $\tilde{T}(\varphi)$ we obtain the first inequality in (1.26). Suppose first that $y^0 > 0$ in (1.23). Then by integration we obtain for $t \in [t^0, d)$

$$\log \frac{y_1(t)}{y_1(t^0)} \ge \log \frac{y_2(t)}{y_2(t^0)},$$

i.e. the second inequality in (1.26) is also valid. In the second case when $y^0 = 0$, $Y^0 > 0$ in (1.23) we obtain by integration over $[t^0 + \delta, t]$ with sufficiently small $\delta > 0$ that

$$\delta \frac{y_1(t)}{y_1(t^0 + \delta)} \ge \delta \frac{y_2(t)}{y_2(t^0 + \delta)}.$$

Applying the mean value theorem we find $y_1(t^0 + \delta) = \delta y_1'(t_\delta)$, $y_2(t^0 + \delta) = \delta y_2'(\bar{t}_\delta)$ where $t^0 < t_\delta, \bar{t}_\delta < t^0 + \delta$. Since the functions $r_i(t) y_i(t)$ $(i = 1, 2)$ are continuous we get

$$\lim_{\delta \to +0} \frac{y_i(t^0 + \delta)}{\delta} = \frac{Y^0}{r(t^0 + 0)} > 0$$

thus we derive again the inequality $y_1(t) \ge y_2(t)$ which completes the proof.

REMARK 1.2. The theorems above concern the behaviour of the solutions on some right neighbourhood on an inner point t^0 in I. The statements can be extended to the left neighbourhood of t^0, too, by making the observation that the function $\hat{y}(t) = y(2t^0 - t)$ is a solution of the differential equation (1.12) if we replace the functions $r(t)$, $p(t)$, $q(t)$ by $\hat{r}(t) = r(2t^0 - t)$, $\hat{p}(t) = p(2t^0 - t)$, $\hat{q}(t) = q(2t^0 - t)$ and we take into consideration the change $\hat{y}'(t) = -y'(2t^0 - t)$.

In applications we often have to deal with singular differential equations and don't have any explicitly given initial value problem at any inner point of I like in (1.23) but we have some restriction at one endpoint of I. Such situation was considered by Szegő [14] and we are going to extend his result to our half-linear diffential equations.

THEOREM 1.5. *Let $I = (\alpha, \beta)$ and suppose in (Eq_i) that $r_1 = r_2 = r$, $p_1 = p_2 = p$ and the differential equations Eq_i are not oscillatory at the endpoint α. Let $y_i(t)$ be a solution of Eq_i $(i = 1, 2)$ such that*

$$\lim_{t \to \alpha+} w(t) = 0, \tag{1.27}$$

where $w(t) = r(t)[y_1'(t)y_2(t) - y_1(t)y_2'(t)]$. *Let* Eq_2 *be a Sturmian majorant of* Eq_1. *For the sake of simplicity we may assume that* $y_i(t) > 0$ $(i = 1, 2)$ *on some right neighbourhood* U *of* α *and* $q_1(t) < q_2(t)$ *there.*

In case $p(t) \neq 0$ *on* U *we assume in addition that*

$$\operatorname{sgn} y_1'(t) = \operatorname{sgn} y_2'(t) = \varepsilon, \text{ where } \varepsilon \in \{1, -1\},\ t \in U, \text{ and } \int_\alpha \frac{p(t)}{r(t)} dt \text{ exists.}$$

Let all the consecutive zeros of $y_1(t)$—*if any*—*be denoted by* $t_1, t_2, \ldots$ *and similarly the ones of* $y_2(t)$ *by* $\hat{t}_1, \hat{t}_2, \ldots$. *Then the relations*

$$\hat{t}_i < t_i, \qquad i = 1, 2, \ldots$$

hold.

Proof. Let the function $u(t)$ be defined on I by $u(t) = \exp\left(\int_\alpha^t \frac{p}{r}\right)$. Differentiating the function $w(t)$ we obtain

$$w' + \varepsilon \frac{p}{r} w = (q_2 - q_1) y_1 y_2, \quad \text{or} \quad (u^\varepsilon w)' = (q_2 - q_1) y_1 y_2 u^\varepsilon.$$

The continuity of $w(t)$, $u(t)$ and the assumption (1.27) implies

$$u^\varepsilon w = \int_\alpha^t (q_2 - q_1) y_1 y_2 u^\varepsilon,$$

hence $w(t) > 0$ on U. Choosing a value t^0 in U we obtain the desired result as in Remark 1.1 which completes the proof.

Remark 1.3. The requirement (1.27) in Theorem 1.5 can be relaxed to $\lim_{t \to \alpha+} w(t) \geq 0$ as in [14]. The case $\lim_{t \to \alpha+} w(t) > 0$ is easier to prove because now the function $w(t)$ is continuous on $[\alpha, \beta)$ hence $w(t^0) > 0$ provided t^0 is sufficiently near to α. Then we can proceed via Remark 1.1 to obtain the inequalities $\hat{t}_i < t_i$.

Remark 1.4. A similar statement can be formulated for the right endpoint β of I in Theorem 1.5 requiring $\lim_{t \to \beta-} w(t) \leq 0$ obtaining the relations..., $t_2 < \hat{t}_2$, $t_1 < \hat{t}_1$.

2. Applications. We consider the solution $y(t)$ of (0.1) with consecutive zeros t_1, t_2. We may suppose that $y(t) > 0$ for $t_1 < t < t_2$. Clearly, $y(t)$ is also a solution of the half-linear differential equation

$$(2.1) \qquad y'' + \tilde{p}(t)|y'|\operatorname{sgn} y + q(t)y = 0,$$

where $\tilde{p}(t) = p(t)\operatorname{sgn}(y(t)y'(t))$. By our assumptions the differential equation

$$(2.2) \qquad Y'' + M_1|Y'|\operatorname{sgn} Y + M_2 Y = 0$$

is a Sturmian majorant of (2.1). Moreover by (1.3) the function $Y(t) = y'(t_1)S(t-t_1)$ is a solution of (2.2) where $y'(t_1) > 0$. To apply Theorem 1.4 to (2.1), (2.2) we choose $t^0 = t_1$, $\hat{\pi} = \hat{\pi}(M_2, M_1)$, $d = t_1 + \hat{\pi}$ and we obtain the inequalities

$$(2.3) \qquad \frac{y'(t)}{y(t)} \geq \frac{S'(t-t_1)}{S(t-t_1)}, \quad y(t) \geq y'(t_1)S(t-t_1) \quad \text{for } t_1 < t < t_2.$$

Therefore it follows immediately that $t_2 \geq t_1 + \hat{\pi}$ or $h = t_2 - t_1 \geq \hat{\pi}$ which is the Cohn's result due to (0.3), (1.2).

Let t^* be defined by

$$(2.4) \qquad \max_{t_1 \leq t \leq t_2} y(t) = y(t^*), \quad t^* \in (t_1, t_2).$$

By definition of t^* we have $y'(t^*) = 0$ and the first inequality in (2.3) implies

$$(2.5) \qquad t^* - t_1 \geq \frac{1}{2}\hat{\pi}(M_2, M_1),$$

and the second inequality implies

$$y(t^*) \geq y'(t_1)S\left(\frac{\hat{\pi}}{2}\right).$$

Hence by (1.8), (2.4) we have

$$(2.6) \qquad M_2 y^2(t^*) = M_2 \max_{t_1 \leq t \leq t_2} y^2(t) \geq y'^2(t_1)\exp\left(-\frac{M_1\hat{\pi}(M_2, M_1)}{2}\right)$$

Combining (2.5), (2.6) we have the inequality in (0.8).

Another consequence of the second ineuality in (2.3) can be the following:

$$(2.7) \qquad \int_{t_1}^{t_2} y^2(t)\,dt \geq y'^2(t_1)\int_{t_1}^{t_1+\hat{\pi}} S^2(t-t_1)\,dt.$$

To evaluate the integral on the right hand side we multiply the differential equation (1.3) by $S(\varphi)$ and integrate over $[0, \hat{\pi}]$. Taking into consideration (1.4) and the value of $S(\hat{\pi}/2)$ from (1.8) we obtain

$$\int_0^{\hat{\pi}} S'^2(\varphi)\,d\varphi - a\int_0^{\hat{\pi}} S^2(\varphi)\,d\varphi = bS^2\left(\frac{\hat{\pi}}{2}\right).$$

To determine the first integral here we multiply (1.3) by $|S'(\varphi)|$ and integrate over $[0, \hat{\pi}]$:

$$b\int_0^{\hat{\pi}} S'^2(\varphi)\,d\varphi = 1 - aS^2\left(\frac{\hat{\pi}}{2}\right).$$

The last two equations give the value of the integral on the right hand side of (2.7) and we obtain by (1.8)

(2.8)

$$M_2\int_{t_1}^{t_2} y^2(t)\,dt \geq y'^2(t_1)\left[\frac{1 - e^{-M_1\frac{\hat{\pi}(M_2,M_1)}{2}}}{M_1} - \frac{M_1}{M_2}e^{-M_1\frac{\hat{\pi}(M_2,M_1)}{2}}\right], \quad M_1 \neq 0.$$

Let us observe that if we introduce the new variable $s = \sqrt{a}\tau$ in (1.2) we get

$$\hat{\pi}(a, b) = \frac{1}{\sqrt{a}}\hat{\pi}(1, 2\sigma), \qquad \text{where } \sigma = \frac{1}{2}\frac{M_1}{\sqrt{M_2}}. \tag{2.9}$$

Therefore the relation (2.8) can be written in the form

$$M_2^{3/2}\int_{t_1}^{t_2} y^2(t)\,dt \geq y'^2(t_1)\,\mu(\sigma) \tag{2.10}$$

where

$$\mu(\sigma) = \begin{cases} \pi & \text{for } \sigma = 0 \\ \dfrac{1 - e^{-\sigma\hat{\pi}(1,2\sigma)}}{2\sigma} - 2\sigma e^{-\sigma\hat{\pi}(1,2\sigma)} & \text{for } \sigma > 0. \end{cases}$$

Inequality (2.10) is concerned with L^2-norm of $y(t)$ on $[t_1, t_2]$. Similar estimates hold also for L^p-norm of $y(t)$ with $p > 0$.

We have to prove the inequality in (0.7). Since $h \geq \hat{\pi}(M_2, M_1)$ it is sufficient to show by (2.9) that

$$H(\sigma) = \left(1 + \frac{1}{\hat{\pi}}\sigma\right)\int_0^\infty \frac{ds}{s^2 + 2\sigma s + 1} \geq \frac{\pi}{2} \qquad \text{for } \sigma \geq 0. \tag{2.11}$$

Since $H(0) = \pi/2$ the inequality (2.11) is sharp. Differentiating (2.11) with respect to σ we find

$$H'(\sigma) = \frac{2}{\pi}\int_0^\infty \frac{s^2 - \pi s + 1}{(s^2 + 2\sigma s + 1)^2}\,ds.$$

A direct calculation shows that $H'(0) = 0$. Now we claim that the function

$$G(\sigma) = \left(1 + \frac{2}{\pi}\sigma\right)^2 H'(\sigma)$$

is strictly increasing. Indeed, by differentiation we get

$$G'(\sigma) = \frac{4}{\pi}\left(1 + \frac{2}{\pi}\sigma\right)\int_0^\infty \frac{(s^2 - \pi s + 1)^2}{(s^2 + 2\sigma s + 1)^3}\,ds > 0,$$

hence $G(\sigma) > G(0) = 0$ for $\sigma > 0$. Consequently $H'(\sigma) > 0$ for $\sigma > 0$ and the equality in (2.8) holds if and only if $\sigma = M_1/2\sqrt{M_2} = 0$, i.e. when $M_1 = 0$.

Opial's inequality (0.2) provides the following lower estimate:

$$h \geq \frac{\pi^2}{M_1 + \sqrt{M_1^2 + \pi^2 M_2}} = h_1.$$

It is easy to show that $h_1 < \pi/(\sqrt{M_2} + M_1/\pi)$ for $M_1 > 0$, i.e. our lower bound in (0.7) is better than the Opial's bound.

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