HIROSHIMA MATH. J.
22 (1992), 125–127

Infinite families of non-principal prime ideals

P. RIBENBOIM, F.R.S.C.

(Received December 25, 1990)

Abstract. Bouvier [2] has shown that if A is a Krull domain, then the set of non-principal prime ideals of height one of A is either empty or infinite. Here we prove a similar result under less restrictive hypotheses.

Let A be a domain and let $\mathscr{P}$ be a family of prime ideals of A, satisfying the following conditions:

1) $\bigcap_{n=0}^{\infty} A_P P^n = 0$ for every $P \in \mathscr{P}$.

2) $\bigcap_{P \in \mathscr{P}} A_P = A$.

3) If $f \in A$, $f \neq 0$, there exist only finitely many ideals $P \in \mathscr{P}$ such that $f \in P$.

4) If $n > 1$ and $P_1, \ldots, P_n$ are distinct prime ideals in the family $\mathscr{P}$, there exists f such that $f \in P_1 \backslash P_1^2$, $f \notin P_2 \cup \cdots \cup P_n$.

For example, we may take $\mathscr{P}$ to be the family of prime ideals of height one of a Krull domain, as it was done by Bouvier.

Following suggestions of W. Heinzer, we indicate other examples of domains and families of prime ideals satisfying conditions (1)–(4).

The family of all prime ideals of height one of a noetherian domain, in which every principal ideal has no embedded primes, satisfies the conditions (1)–(4). These are precisely the prime ideals of height one of domains satisfying the S-sub-2-condition of Serre, and include the Cohen-Macaulay domains.

In a still unpublished paper, Barucci, Gabelli & Roitman [1] study the semi-Krull domains, introduced earlier by Matlis [3]: the family of prime ideals of height one satisfies also conditions (1)–(4).

We note that condition (4) implies:

4′) If $P, P' \in \mathscr{P}$, $P \neq P'$, then P, P' are incomparable by inclusion.

LEMMA. *If P is a prime ideal of A satisfying condition* (1) *and $A_P P$ is a principal ideal, then A_P is the ring of a discrete valuation of height one of the field of quotients of A.*

PROOF. If $x \in A_P$, $x \neq 0$, let $v_P(x) = n$ be the unique integer such that $x \in A_P P^n \backslash A_P P^{n+1}$; let also $v_P(0) = \infty$. If $x, y \in A_P$ it is obvious that $v_P(x + y) \geq$

$\min\{v_P(x), v_P(y)\}$. If $v_P(x) = n$, $v_P(y) = m$, if $A_P P = A_P t$, then $x = rt^n$, $y = st^m$, with $r, s \in A_P \backslash A_P t$. Hence $rs \notin A_P P$, $xy = rst^{n+m}$, and this proves that $v_P(xy) = n + m$.

Therefore, v_P may be extended canonically to a discrete valuation of height one, still denoted v_P, of the field of quotients of A_P. Finally, if $x, y \in A_P$, $y \neq 0$ and $v_P(x/y) \geq 0$, then $x = rt^n$, $y = st^m$, with $n \geq m$, $r, s \in A_P \backslash A_P P$. Hence r, s are invertible elements of the ring A_P and $x/y = (rt^{n-m})/s \in A_P$. This proves the lemma. ■

PROPOSITION. *Assume that $\mathscr{P}$ contains a non-principal prime ideal P_1 such that $A_{P_1}P_1$ is principal and P_1^2 is a primary ideal. Then $\mathscr{P}$ contains infinitely many nonprincipal ideals.*

PROOF. We assume that $P_1, \ldots, P_n$ are the only non-principal ideals in the family $\mathscr{P}$.

Let $\mathscr{Q} = \mathscr{P} \backslash \{P_1, \ldots, P_n\}$; a priori it is not excluded that $\mathscr{Q}$ be empty.

If $Q \in \mathscr{Q}$ then Q is a principal ideal; let $g_Q \in A$ be a generator of $Q : Q = Ag_Q$. Let f be such that $f \in P_1 \backslash P_1^2$, $f \notin P_2 \cup \cdots \cup P_n$, hence $f \neq 0$. It follows from (1) that for every $P \in \mathscr{P}$ there exists a unique integer $n_P(f) \geq 0$ such that $f \in A_P P^{n_P(f)} \backslash A_P P^{n_P(f)+1}$.

It follows from (3) that $n_P(f) = 0$, except for finitely many ideals $P \in \mathscr{P}$, because $A_P P \cap A = P$.

Let $h = f/g$ where

$$g = \begin{cases} 1 & \text{if } \mathscr{Q} = \varnothing \\ \prod_{Q \in \mathscr{Q}} g_Q^{n_Q(f)} & \text{if } \mathscr{Q} \neq \varnothing \end{cases}$$

(this is a finite product).

If $Q \in \mathscr{Q}$, it follows from (4′) that $g_Q \notin P_i$ for $i = 1, \ldots, n$. Hence $h \in A_{P_1}P_1$ and $h \in A_{P_i} \backslash A_{P_i} P_i$ for $i = 2, \ldots, n$. If $Q, Q' \in \mathscr{Q}$, $Q \neq Q'$, then we have also $g_{Q'} \notin Q$, by (4′). Hence, for every $Q \in \mathscr{Q}$

$$\prod_{\substack{Q' \in \mathscr{Q} \\ Q' \neq Q}} \frac{1}{g_{Q'}^{n_{Q'}(f)}} \in A_Q ;$$

by definition $f \in A_Q g_Q^{n_Q(f)} \backslash A_Q g_Q^{n_Q(f)+1}$, hence

$$\frac{f}{g_Q^{n_Q(f)}} \in A_Q \backslash A_Q Q .$$

Therefore $h \in A_Q$. From (2) it follows that $h \in A$, hence $h \in A_{P_1}P_1 \cap A = P_1$, i.e. $Ah \subseteq P_1$.

Now we show that $Ah = P_1$, which is contrary to the hypothesis. Let

$k \in P_1$, hence

$$\frac{k}{h} = \frac{k}{f} \prod_{Q \in \mathcal{Q}} g_Q^{n_Q(f)} \in \left(\bigcap_{i=2}^{n} A_{P_i} \right) \cap \left(\bigcap_{Q \in \mathcal{Q}} A_Q \right),$$

because

$$\frac{f}{g_Q^{n_Q(f)}} \notin A_Q Q .$$

Moreover, $\prod_{Q \in \mathcal{Q}} g_Q^{n_Q(f)} \in A$ and we show that $k/f \in A_{P_1}$. Indeed, $f \in P_1 \setminus P_1^2$, hence $f \notin A_{P_1} P_1^2$, otherwise $f \in A_{P_1} P_1^2 \cap A = P_1^2$, because P_1^2 is a primary ideal. By hypothesis $A_{P_1} P_1$ is a principal ideal. By the lemma, we may consider the valuation v_{P_1} associated to P_1, we have $v_{P_1}(k) \geq 1$, $v_{P_1}(f) = 1$, hence $v_{P_1}(k/f) \geq 0$, thus $k/f \in A_{P_1}$. It follows that $k/h \in A_{P_1}$, hence from (2), $k/h \in A$. This shows that $P_1 = Ah$, a contradiction. ■

References

[1] V. Barucci, S. Gabelli & M. Roitman, On semi-Krull domains, preprint, 1990.

[2] A. Bouvier, Sur les idéaux premiers de hauteur 1 d'un anneau de Krull, C. R. Acad. Sci. Paris **284** (1977), 727–729.

[3] E. Matlis, Some properties of commutative ring extensions, Ill. J. Math. **31** (3) (1987), 374–418.

Department of Mathematics and Statistics
Queen's University
Kingston, Ontario, Canada K7L 3N6