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Asymptotic behavior of a biological model with time delays

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1. Introduction

In this paper we study the scalar delay differential equation

$$(1.1) \qquad x'(t) = \sum_{i=1}^{n} b_i x(t - r_i)(1 - ax(t)) - cx(t)$$

where $r_i > 0$, $b_i > 0$ $(i = 1, 2, \cdots, n)$, $a > 0$, $b = \sum_{i=1}^{n} b_i > 0$, $c \geq 0$. If we take $n = 1$, $a = 1$ and $c > 0$, this equation becomes the epidemic model given by K. L. Cooke in [4, 5], that is

$$x'(t) = bx(t - \tau)(1 - x(t)) - cx(t).$$

K. L. Cooke made the following assumptions on his model (in this paper we also use them.):

(a) The infection is transmitted to man by a vector, such as a mosquito. Susceptible persons receive the infection from infectious vectors, and susceptible vectors receive the infection from infectious persons.

(b) The human population in the community under consideration is fixed, hence we are interested in the solution $x(t)$ of (1.1) which obeys $0 \leq x(t) \leq 1$. The infection in humans does not result in death or isolation.

(c) When a susceptible vector is infected by a person, there is a fixed time during which the infectious agent develops in the vector. At the end of this time the vector can infect a susceptible human.

(d) Infected humans have a constant recovery rate c. Note that the time during which the infectious agent develops in vectors of different species may be different, so the following model which is a special case of (1.1) may be more reasonable

$$(1.2) \qquad x'(t) = \sum_{i=1}^{n} b_i x(t - r_i)(1 - x(t)) - cx(t) \qquad t \geq 0$$

where $b_i > 0$, $r_i > 0$ $(i = 1, 2, \cdots, n)$, $c > 0$ are constants. On the other hand, when $c = 0$ and $n = 1$, (1.1) is the Logistic model given by K. Gopalsamy in [7].

In this paper, we study more general model

$$x'(t) = \int_{-r}^{0} x(t+s)\,d\eta(s)(1 - ax(t)) - cx(t), \qquad t \geq 0 \tag{1.3}$$

where $a > 0$, $b = \operatorname*{Var}_{[-r,0]} \eta > 0$, $c \geq 0$ and $r > 0$ are constants, and $\eta(s)$ is nondecreasing on $[-r, 0]$ and may be discontinuous.

Instead of solutions satisfying $0 \leq x(t) \leq 1$, we are interested in the positive solution $x(t)$ of (1.1).

2. Preliminary results

Consider the delay differential equation (1.1) with initial condition

$$x_0(t) = \varphi(t), \quad -r \leq t \leq 0 \tag{2.1}$$

where $\varphi(t)$ is continuous on $[-r, 0]$ and $\varphi(t) \geq 0$, $r = \max\{r_1, r_2, \cdots, r_n\}$. At first, we treat the case $c > 0$ and we have the following result:

LEMMA 2.1 Consider the problem (1.1)–(2.1) and assume that $\varphi(t) \geq 0$ for $t \in [-r, 0]$ and $\varphi(t) \not\equiv 0$. Suppose $\omega > 0$ and $x(t)$ exists on $[-r, \omega]$. Then the following results hold:

(a) $x(t) \geq 0$ for all $t \in [0, \omega]$;
(b) $x(t)$ is bounded;
(c) Any nontrivial solution $x(t)$ is positive.

Proof. (a) If it is not true, then the set $S = \{t: t > 0 \text{ and } x(t) < 0\}$ is not empty. Let $m = \inf S$, we have $m \geq 0$, $x(m) = 0$ and $x(t) \geq 0$ for $t \in [-r, 0]$. For any $\varepsilon > 0$ which satisfies $0 < \varepsilon < r_* = \min\{r_1, r_2, \cdots, r_n\}$, there exists a $t_\varepsilon \in (0, \varepsilon)$ such that $x(m + t_\varepsilon) < 0$. From (1.1) we conclude that

$$x'(m + t_\varepsilon) = \sum_{i=1}^{n} b_i x(m + t_\varepsilon - r_i)(1 - ax(m + t_\varepsilon) - cx(m + t_\varepsilon)) > 0. \tag{2.2}$$

So there exists a left neighborhood N of $m + t_\varepsilon$ with

$$x(m + t) \leq x(m + t_\varepsilon) \tag{2.3}$$

$$x'(m + t) > 0 \qquad \text{for } t \in N \tag{2.4}$$

Let $t^* = \inf\{t: t \in N\}$. Clearly $t^* > 0$, $x(m + t^*) < 0$ and $x'(m + t^*) = 0$. However, from (1.1) we have

$$x'(m + t^*) = \sum_{i=1}^{n} b_i x(m + t^* - r_i)(1 - ax(m + t^*)) - cx(m + t^*) > 0 \tag{2.5}$$

and this leads to a contradiction. Hence the set S is empty, i.e. $x(t) \geq 0$ for $t \geq 0$.

(b) Suppose $x(t)$ is a solution of the problem (1.1)–(2.1). We will prove $x(t)$ is bounded. If it is not true, then there exists a time t_0 such that

$$x(t_0) = M > \max\{\sup_{-r \leq t \leq 0} x(t),\ 1/a\} \quad \text{and} \quad x'(t_0) > 0 \tag{2.6}$$

for a suitable constant $M > 1/a$. Noting $x(t_0 - r_i) \geq 0$, $i = 1, 2, \cdots, n$ and from (1.1) we have $x'(t_0) < 0$, which is a contradiction. Hence $x(t)$ is bounded.

(c) Let $x(t)$ be a nontrival solution of the problem (1.1)–(2.1). Now using the results (a) and (b), we can find a constant $M > 0$ such that

$$x'(t) \geq -(abM + c)\, x(t) \tag{2.7}$$

As $x(t)$ is a nontrivial solution of (1.1), we may chose a $t_1 > 0$ such that $x(t_1) > 0$ and

$$x(t) \geq x(t_1) \exp\{-(abM + c)(t - t_1)\} > 0, \qquad t > t_1.$$

Now we consider the following RFDE

$$x'(t) = f(x_t) \qquad t \geq 0 \tag{2.8}$$

where $x_t \in C = C([-r, 0], R^n)$.

To discuss asymptotic behavior of (1.1), we pick some results from the book [2].

THEOREM 2.1 ([2]. p. 280] Suppose that $f: C \to R$ and v is a continuously differentiable function and $G \subseteq C$ is a closed and positively invariant set with respect to (2.8) with property that

$$v'(\varphi) \leq 0 \text{ for all } \varphi \in G \text{ such that } v(\varphi(0)) = \max_{-r \leq s \leq 0} v(\varphi(s)).$$

Then for any $\varphi \in G$ such that $x(\varphi)$ is bounded on $(-r, +\infty)$, we have $\Omega(\varphi) \subseteq M_v(G) \subseteq E_v(G)$. Hence $x_t(\varphi) \to M_v(G)$ as $t \to \infty$, here we define

$$E_v(G) = \{\varphi \in G: \max_{-r \leq s \leq 0} v(x(t+s, \varphi)) = \max_{-r \leq s \leq 0} v(\varphi(s)) \text{ for all } t \geq 0\}$$

and denote $M_v(G)$ the largest subset of $E_v(G)$ that is invariant with respect to (2.8).

3. Some results

THEOREM 3.1 If $b > c > 0$, then every eventually positive solution $x(t)$ of

(1.1) satisfies $x(t) \to (b-c)/ab$ as $t \to +\infty$.

THEOREM 3.2 If $b \leq c$ and $c > 0$, then any eventually positive solution $x(t)$ of (1.1) satisfies $x(t) \to 0$ as $t \to +\infty$.

COROLLARY 1 If $b > c > 0$, then any eventually positive solution of (1.2) satisfies $\lim_{t\to\infty} x(t) = 1 - c/b$. If $b \leq c$ and $c > 0$, then any eventually positive solution of (1.2) satisfies $x(t) \to 0$ as $t \to \infty$. That is when the infectious rate b is greater than the recovery rate c, infectious persons and susceptible persons will tend to an equilibrium for large time. If the infectious rate b is less than or equal to the recovery rate c, the disease will disappear.

THEOREM 3.3 Suppose $c = 0$ in (1.3). Then any eventually positive solution $x(t)$ satisfies $\lim_{t\to\infty} x(t) = 1/a$ if $a > 0$.

THEOREM 3.4 Suppose $c > 0$ and $b = \operatorname{Var}_{[-r,0]} \eta > 0$ in (1.3). Then every eventully positive solution $x(t)$ of (1.3) satisfies $\lim_{t\to\infty} x(t) = \dfrac{b-c}{ab}$ if $b > c$ and $\lim_{t\to\infty} x(t) = 0$ if $b \leq c$.

COROLLARY 2 Suppose $c = 0$ in (1.2). Then any eventually positive solution $x(t)$ of (1.2) satisfies $\lim_{t\to\infty} x(t) = 1$. That is, if recovery rate is zero, then all susceptible persons will be infected.

4. Some preliminary knowledges for Theorem 2.1

In the equation (2.8), we suppose that $f: C \to R$ and f satisfies a local Lipschitz condition.

DEFINITION 1. An element $\psi \in C$ belongs to the ω-limit set $\Omega(\varphi)$ of φ, if $x(\varphi)$ is defined on $(-r, +\infty)$ and there is a sequence $\{t_n\} \to \infty$ with $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$.

Clearly $\Omega(\varphi)$ is connected.

DEFINITION 2. *A* set $M \subseteq C$ is positively invariant if for each $\varphi \in M$, $x_t(\varphi) \in M$ for all $t \geq 0$.

Thus, if v is a continuously differentiable function, then

$$v'(\varphi)|_{(2.8)} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(\varphi(0)) f_i(\varphi)$$

is a functional even though v is a function.

If v is a continuously differentiable function and $G \subseteq C$, define

$$E_v(G) = \{\varphi \in G: \max_{-r \leq s \leq 0} v(x(t+s, \varphi)) = \max_{-r \leq s \leq 0} v(\varphi(s)) \text{ for all } t \geq 0\}$$

and let $M_v(G)$ denote the largest subset of $E_v(G)$ that is invariant with respect to (2.8).

Notice that for a continuously differentiable function v and any $\varphi \in E_v(G)$, we have

$$v'[x_t(\varphi)] = 0, \text{ where } t > 0 \text{ satisfied } v[x(t, \varphi)] = \max_{-r \leq s \leq 0} v[\varphi(s)].$$

(v must attain a relative maximum for such t.)

5. Proof of main results

(1). The proof of Theorem 3.1:

Let $y = x - (b-c)/ab$. From (1.1) we have

$$y'(t) = -by(t) + (c/b)\sum_{i=1}^{n} b_i y(t-r_i) - ay(t)\sum_{i=1}^{n} b_i y(t-r_i) \tag{5.1}$$

As we are only interested in the initial condition $\varphi(s)$ where $\varphi(s) \geq 0$ for $s \in [-r, 0]$ and $x(t) \geq 0$ for equation (1.1), we only consider the initial condition $\varphi(s)$ with $\varphi(s) \geq -(b-c)/ab$ and $y(t) \geq -(b-c)/ab$ for model (5.1). Let $G = \{\varphi(s) \in C([-r, 0] \to R): \varphi(s) \geq (c-b)/ab, -r \leq s \leq 0\}$, $v(y) = y^2/2$. From Lemma 2.1, we have that $G \subseteq C$ is a closed and positively invariant set with respect to (5.1). We will argue that

$$M_v(G) = \{\varphi_1(s) \equiv 0, \ \varphi_2(s) \equiv (c-b)/ab, \text{ for } s \in [-r, 0]\}$$

Clearly, $\varphi_1(s)$ and $\varphi_2(s)$ belong to $M_v(G)$, so $M_v(G)$ is not empty. For any $\varphi \in E_v(G)$, we have $\varphi \in G$ and $\|\varphi\| = \|y_t(\varphi)\|$ for all $t \geq 0$. Let $Y(t) = y(\varphi)(t)$ and $t^* > 0$ such that $|y(t^*)| = \|y_{t^*}(\varphi)\|$. we conclude that

$$v'[y_{t^*}(\varphi)]|_{(5.1)} = -by^2(t^*) + (c/b)\sum_{i=1}^{n} b_i y(t^*) y(t^* - r_i) - ay^2(t^*)\sum_{i=1}^{n} b_i y(t^* - r_i) = 0 \tag{5.2}$$

Noting that $|y(t^* - r_i)| \leq |y(t^*)|$, from (5.2) we can obtain that $\sum_{i=1}^{n} b_i y(t^* - r_i) \leq 0$.

If $\sum_{i=1}^{n} b_i y(t^* - r_i) = 0$, from (5.2) we have $y(t^*) = 0$ and $y_{t^*}(\varphi) \equiv 0$ and $y_{t^*}(\varphi) \in C$. As $\varphi \in E_v(G)$, we have $y_t(\varphi) = 0$ for $t \geq t^*$. So $\varphi \equiv 0$ and $\varphi_1(s) \equiv 0$ and $\varphi_1(s) \in E_v(G)$.

If $\sum_{i=1}^{n} b_i y(t^* - r_i) < 0$ and $y(t^*) \neq 0$, from (5.2) we have

$$|by(t^*)| = |\sum_{i=1}^{n} b_i y(t^* - r_i)| \cdot |c/b - ay(t^*)|. \tag{5.3}$$

As $y(t^* - r_i) \geq (c-b)/ab$, we have $b + a\sum_{i=1}^{n} b_i y(t^* - r_i) \geq c > 0$. From (5.2) we may conclude that

$$(c/b) y(t^*) \cdot \sum_{i=1}^{n} b_i y(t^* - r_i) = y^2(t^*) \cdot (b + a \sum_{i=1}^{n} b_i y(t - r_i)) \tag{5.4}$$

and $y(t^*) < 0$. On the other hand, we have

$$\sum_{i=1}^{n} b_i y(t^* - r_i) \leq \sum_{i=1}^{n} b_i |y(t^* - r_i)| \leq b|y(t^*)|.$$

Here, there are two different cases:

(a) the case $|\sum_{i=1}^{n} b_i y(t^* - r_i)| < b|y(t^*)|$;

From (5.3) we have $1 < |c/b - ay(t^*)| = c/b - ay(t^*)$ and $y(t^*) < (c-b)/ab$ which contradics to $y(t) \geq (c-b)/ab$.

(b) the case $|\sum_{i=1}^{n} b_i y(t^* - r_i)| = b|y(t^*)|$.

From (5.3) we have $1 = c/b - ay(t^*)$, hence $y(t^*) = (c-b)/ab$. From (c) of Lemma 2.1, we know $y_{t^*}(\varphi) = (c-b)/ab$ and $y_t(\varphi) \equiv (c-b)/ab$ for $t \geq t^*$. As $\varphi \in E_v(G)$, we conclude that $\varphi \equiv (c-b)/ab$, that is $\varphi_2(s) \equiv \dfrac{c-b}{a \cdot b}$ Hence

$$M_v(G) = E_v(G) = \{\varphi_1(s) \equiv 0,\ \varphi_2(s) \equiv (c-b)/ab, \text{ for } -r \leq s \leq 0\}.$$

On the other hand, if $\max_{-r \leq s \leq 0} v(\varphi(s)) = v(\varphi(0))$, then we have $\|\varphi\| = |\varphi(0)|$ and

$$v'(\varphi)|_{(5.1)} = -b\varphi^2(0) + (c/b)\sum_{i=1}^{n} b_i \varphi(0)\varphi(-r_i) - a\varphi^2(0)\sum_{i=1}^{n} b_i \varphi(-r_i)$$

$$\leq [-b + c - a \sum_{i=1}^{n} b_i \varphi(-r_i)] \cdot \|\varphi\|^2$$

As $\varphi(-r_i) \geq (c-b)/ab$, hence $v'_{(5.1)}(\varphi) \leq 0$ for $\varphi \in G$. From Theorem 2.1, we conclude that for any $\varphi \in G$, $y_t(\varphi) \to \Omega(\varphi) \subseteq M_v(G)$ as $t \to +\infty$. Because $\Omega(\varphi)$ is connected, we have $\Omega(\varphi) = \varphi_1(s)$ or $\Omega(\varphi) = \varphi_2(s)$ for any given $\varphi \in G$. And hence $y_t(\varphi) \to \varphi_1(s)$ or $y_t(\varphi) \to \varphi_2(s)$ as $t \to +\infty$. To complete the proof of our theorem, we need argue that any eventually positive solution $x(t)$ of (1.1) does not tend to zero as t tends to infinity.

Suppose $x(t)$ is an eventually positive solution of (1.1) satisfying $\lim_{t\to\infty} x(t) = 0$. Let $w(t) = x(t) + \sum_{i=1}^{n} b_i \int_{t-r_i}^{t} x(s)\,ds$. Clearly, $w(t) \to 0$ as $t \to +\infty$, but $w(t) > 0$ and $w'(t) = [b - c - a \sum_{i=1}^{n} b_i x(t-r_i)]x(t) > 0$ for large $t > 0$, this leads to a contradiction. Hence for any $\varphi \in G$, we have $y_t(\varphi) \to 0$ as $t \to \infty$ as long as $y_t(\varphi) \not\equiv (c-b)/ab$, that is $x(t) \to (b-c)/ab$ as $t \to \infty$.

Note: The proof of Theorem 3.2 is similar to that of Theorem 3.1. In this case we have $M_v(G) = \{\varphi_1(s) \equiv 0, \text{ for } -r \leq s \leq 0\}$.

Before proving Theorem 3.3, we give a Lemma at first.

LEMMA 5.1 Suppose $\lim_{t\to\infty} x(t) = a$ exists and $x'(t)$ is continuous uniformly on $[0, +\infty)$, then $\lim_{t\to\infty} x'(t) = 0$.

PROOF. If the conclusion is not true, then there are a $\varepsilon_0 > 0$ and a sequence $\{t_n\} \to \infty$ $(n = 1, 2, \cdots)$ as $n \to +\infty$ such that $|x'(t_n)| > \varepsilon_0$. Noting that $x'(t)$ is uniformly continuous on $(0, +\infty)$, we may choose a $\delta > 0$ such that as long as $|t_1 - t_2| \leq \delta$ we have $|x'(t_1) - x'(t_2)| < \varepsilon_0/2$. On the other hand, we have

$$|x(t_n + \delta) - x(t_n)| = \delta |x'(\xi_n)| \qquad \text{for some } \xi_n \in (t_n, t_n + \delta)$$

and $|x'(\xi_n) - x'(t_n)| < \varepsilon_0/2$. Hence

$$|x'(\xi_n)| \geq |x'(t_n)| - |x'(t_n) - x'(\xi_n)| > \varepsilon_0/2$$

and $|x'(t_n + \delta) - x(t_n)| > \delta\varepsilon_0/2$. We obtain a contradiction by letting $n \to \infty$ and this completes the proof of the lemma.

(2). The proof of Theorem 3.3:

Suppose $x(t)$ is an eventually positive solution of equation (1.3) with $c = 0$.

(a) If there is a $T > 0$ such that $x(t) \geq 1/a$ for all $t > T$, then from (1.3) we have $x'(t) \leq 0$ for $t \geq T$ and hence $\lim_{t\to\infty} x(t)$ exists, say $\lim_{t\to\infty} x(t) = A$. From

(1.3) we conclude that $\lim_{t\to\infty} x'(t)$ exists and $x'(t)$ is continuous uniformly on $[0, +\infty)$. By Lemma 5.1 we obtain $0 = Ab(1 - aA)$. Hence $A = 0$ or $A = 1/a$. Similarly to the proof of Theorem 3.1, we may argue that $A \neq 0$, here we choose $w(t)$ as

$$w(t) = x(t) + a\int_{-r}^{0}\left[\int_{t}^{t-\theta} x(s+\theta)\,ds\right]d\eta(\theta),$$

and we only have $A = 1/a$.

(b) If there is a $T > 0$ such that $0 < x(t) \leq 1/a$ for all $t > T$, then from (1.3) we have $x'(t) \geq 0$. Hence $\lim_{t\to\infty} x(t) = B$ exists. Similarly to (a), we may conclude $B = 1/a$.

(c) If $x(t)$ is oscillatory about the equilibria $x^* = 1/a$, then we can choose a sequence $\{t_n\} \to +\infty$ as $n \to \infty$ such that $x'(t_n) = 0$ and $\lim_{t\to\infty} x(t_n) = \overline{\lim_{t\to\infty}}\, x(t)$. From (1.3) we obtain that

$$0 = \int_{-r}^{0} x(t_n + s)\,d\eta(s)(1 - ax(t_n))$$

and $x(t_n) = 1/a$. Hence $\overline{\lim_{t\to\infty}}\, x(t) = 1/a$. Similarly we may prove that $\underline{\lim_{t\to\infty}}\, x(t) = 1/a$ and we obtain that $\lim_{t\to\infty} x(t) = 1/a$. This completes the proof of our theorem.

Note: The proof of Theorem 3.4 is similar to that of Theorem 3.1 and Theorem 3.2. In this case we have that

$$(5.5)\qquad y'(t) = -by(t) + (c/b)\int_{-r}^{0} y(t+\theta)\,d\eta(\theta) - ay(t)\int_{-t}^{0} y(t+\theta)\,d\eta(\theta)$$

$$v'[y_{t^*}(\varphi)]|_{(5.5)} = -by^2(t^*) + (c/b)y(t^*)\int_{-r}^{0} y(t^*+\theta)\,d\eta(\theta) - ay^2(t^*)\int_{-r}^{0} y(t^*+\theta)\,d\eta(\theta) = 0$$

and

$$v'(\varphi)|_{(5.5)} = -b\varphi^2(0) + (c/b)\varphi(0)\int_{-r}^{0} \varphi(\theta)\,d\eta(\theta) - a\varphi^2(0)\int_{-r}^{0} \varphi(\theta)\,d\eta(\theta)$$
$$\leq [-b + c - a\int_{-r}^{0} \varphi(\theta)\,d\eta(\theta)]\cdot\|\varphi\|.$$

Here we omit the details of the proofs.

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