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Topology of moduli space of certain $SU(2)$ connections of degree 2 over S^4

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1. Introduction

Let $SU(2) \to P_k \to S^4$ be the principal $SU(2)$ bundle of degree $c_2(P_k) = k$ and let $\mathscr{A}_k$(resp. $\mathscr{B}_k$) be the set of anti-self-dual connections (resp. $SU(2)$ connections) over P_k. The restricted gauge group (consisting automorphisms which are the identity on the base point $\infty \in S^4$) acts on $\mathscr{A}_k$ and $\mathscr{B}_k$. We define M_k and $\mathscr{M}_k$ to be the orbit space of $\mathscr{A}_k$ and $\mathscr{B}_k$ by the restricted gauge group respectively. M_k is called the framed moduli space of instantons of degree k and $\mathscr{M}_k$ the framed moduli space of $SU(2)$ connections of degree k.

M_k is described by the linear algebra known as the ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction [2]. In the ADHM construction, the following three conditions are imposed: (1) symmetric condition, (2) rank condition and (3) reality condition. The space obtained by imposing the conditions (1) and (2) only is denoted by $\hat{M}_k$ and we have inclusions i_1 and i_2:

$$M_k \xrightarrow{i_1} \hat{M}_k \xrightarrow{i_2} \mathscr{M}_k.$$

We see easily that $M_1 = \hat{M}_1$. So, we shall compare the topology of M_2 with that of $\hat{M}_2$.

THEOREM A. *$\hat{M}_2$ is connected, $\pi_1(\hat{M}_2) = \mathbf{Z}_2$ and $\pi_2(\hat{M}_2) = \mathbf{Z}$. Moreover, $i_{1*}: H_*(M_2; \mathbf{Z}_2) \to H_*(\hat{M}_2; \mathbf{Z}_2)$ is an isomorphism.*

It is known that M_1 is diffeomorphic to $SO(3) \times \mathbf{R}^5$ [2]. The topology of M_2 is studied in [6] and the structures of $H_*(M_2; \mathbf{Z}_2)$ and $H^*(M_2; \mathbf{Z}_2)$ are completely determined in [8]. It is known also that $\pi_1(M_k) = \mathbf{Z}_2$ for all k [7]. Appropriate modifications of Hurtubise's proof might show that $\pi_1(\hat{M}_k) = \mathbf{Z}_2$ hold for all k.

This paper is organized as follows. In §2 we shall review the ADHM construction and give the precise definition of $\hat{M}_k$ as $\hat{F}_k/O(k)$. In §3 we shall construct non-trivial elements in $H_*(\hat{M}_2; \mathbf{Z}_2)$ by using the methods of Boyer and Mann [4] and as an application, we shall estimate the $\mathbf{Z}_2$ coefficient Betti numbers of $\hat{M}_2$ from below. In §4 we shall prove a proposition which determines $H_*(\hat{F}_2; \mathbf{Z}_2)$. In §5 and 6 we shall prove Theorem A by using the results of § 3 and 4.

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2. The ADHM construction

Let $M_{m,n}(\mathbf{H})$ be the set of $m \times n$ quaternion matrices. We define $\hat{F}_k$ to be

$$\hat{F}_k = \left\{ \begin{pmatrix} \Lambda \\ B \end{pmatrix} ; \Lambda \in M_{1,k}(\mathbf{H}),\ B \in M_{k,k}(\mathbf{H}) \text{ such that the following (2.2) and (2.3) are satisfied} \right\}. \tag{2.1}$$

(2.2) (Symmetric condition) B is a symmetric matrix.

(2.3) (Rank condition) For any $x \in \mathbf{H}$ the rank of the matrix $\begin{pmatrix} \Lambda \\ B - xE \end{pmatrix}$ is equal to k where E is the identity matrix.

The group $O(k)$ acts on $\hat{F}_k$ (from the right) by the formula

$$\begin{pmatrix} \Lambda \\ B \end{pmatrix} \cdot T = \begin{pmatrix} \Lambda T \\ T^{-1} B T \end{pmatrix}, \qquad T \in O(k).$$

For an element $\begin{pmatrix} \Lambda \\ B \end{pmatrix}$ of $\hat{F}_k$, the corresponding connection $j\begin{pmatrix} \Lambda \\ B \end{pmatrix}$ on the one point compactification S^4 of $\mathbf{H}$ is described over $x \in \mathbf{H}$ by the following $A(x)$.

$$A(x) = \sigma(x) U(x)^* dU(x) \sigma(x) + \sigma(x)^{-1} d\sigma(x) \tag{2.4}$$

where

$$U(x) = [\Lambda(B - x)^{-1}]^* \text{ and } \sigma(x) = [1 + U(x)^* U(x)]^{-1/2}.$$

We define F_k to be

$$F_k = \left\{ \begin{pmatrix} \Lambda \\ B \end{pmatrix} \in \hat{F}_k ; \text{the following (2.6) is satisfied} \right\}. \tag{2.5}$$

(2.6) (Reality condition) $\Lambda^* \Lambda + B^* B$ is real ($* =$ transpose conjugate). Then according to [2] we can state

THEOREM 2.7. *j gives a diffeomorphism of $\hat{F}_k/O(k)$ onto the image $\hat{M}_k$ of j. Moreover the restriction of j to $F_k/O(k)$ gives the diffeomorphism of $F_k/O(k)$ onto M_k.*

REMARKS 2.8. (1) *We see $M_1 = \hat{M}_1$ because* (2.6) *is satisfied automatically*

in this case.
(2) *We define three symplectic 2-forms* ω_i, ω_j *and* ω_k *on* $\hat{F}_k$ *to be* $\omega_i = \mathrm{Re}\,\mathbf{i}(dA \wedge dA^* + (1/2)\,\mathrm{Tr}(dB \wedge dB^*))$ (ω_j *and* ω_k *are defined similarly*). *Then the corresponding moment maps* μ_i, μ_j *and* μ_k *with respect to the action of* $O(k)$ *on* $\hat{F}_k$ *are given by* $\mu_i = \mathrm{Re}\,\mathbf{i}(A^*A + B^*B)$ (μ_j *and* μ_k *are given similarly*). *Hence we see* $F_k = \mu_i^{-1}(0) \cap \mu_j^{-1}(0) \cap \mu_k^{-1}(0)$.

Before we end this section, we state a proposition which is proved by direct computations.

PROPOSITION 2.9. *The action of* $O(2)$ *on* $\hat{F}_2$ *is free.*

3. Construction of non-trivial elements in $H_*(\hat{M}_2; Z_2)$

In what follows we identify $\mathscr{M}_k$ with $\Omega_k^3 S^3$ because they are homotopically equivalent [3]. So, $\mathscr{M}_k$ admits a natural loop sum and a C_4-structure.

On the other hand Boyer and Mann introduced a loop sum $*: M_k \times M_l \to M_{k+l}$ in [4]. In fact, fixing an element δ of $(0, 1)$, we first define a map $\phi_\delta: M_k \times M_l \to \hat{M}_{k+l}$ by

$$\phi_\delta((b_1), (b_2)) = \begin{pmatrix} (\delta/\|b_1\|)A_1 & (\delta/\|b_2\|)A_2 \\ E + (\delta/\|b_1\|)B_1 & 0 \\ 0 & -E + (\delta/\|b_2\|)B_2 \end{pmatrix} \tag{3.1}$$

where $(b_1) = \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ and $(b_2) = \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$. $\|b_1\|$ is the norm of b_1 regarded as an element of the vector space $M_{1,k}(\mathbf{H}) \times M_{k,k}(\mathbf{H})$. $\|b_2\|$ is defined similarly. Note that the right hand of (3.1) satisfies the rank condition (2.3). If we take δ small enough, then we see that ϕ_δ is a map into the Taubes tubular neighborhood [10]. Hence we can regard ϕ_δ as a map into M_{k+l}. This map is the loop sum constructed by Boyer and Mann. In the same way they introduced a C_4-structure in $\coprod_k M_k$.

We can define a loop sum $*: \hat{M}_k \times \hat{M}_l \to \hat{M}_{k+l}$ by the same formula as (3.1). The fact that our loop sum is compatible with i_1 is obvious and compatible with i_2 is proved in the same way as in the proof of [4]. We can also introduce a C_4-structure in $\coprod_k \hat{M}_k$ in the same way as [4] so as to be compatible with i_1 and i_2.

Let $z_q (q = 1, 2, 3)$ be the generators of $H_q(M_1; \mathbf{Z}_2)$. By operating the Araki-Kudo operations Q_1, Q_2, Q_3, which are defined by the C_4-structure [5], iteratedly on z_q, and then by computing the loop sums of such elements, Boyer and Mann obtained new elements of $H_*(M_k; \mathbf{Z}_2)$. In the case $k = 2$, the result is as follows.

PROPOSITION 3.2 ([4]). *$H_*(M_2; \mathbf{Z}_2)$ contains the following non-trivial elements.*

q	1	2	3	4
$H_q(M_2; \mathbf{Z}_2)$	$z_1 * [1]$	$z_1^2 \ z_2 * [1]$	$Q_1(z_1) \ z_1 * z_2 \ z_3 * [1]$	$Q_2(z_1) \ z_2^2 \ z_1 * z_3$

5	6	7	8	9
$Q_1(z_2) \ z_2 * z_3 \ Q_3(z_1)$	$z_3^2 \ Q_2(z_2)$	$Q_3(z_2) \ Q_1(z_3)$	$Q_2(z_3)$	$Q_3(z_3)$

REMARK 3.3. *By using the results of* [6], *it is known* [8] *that the elements of Proposition* 3.2 *generate $H_*(M_2; \mathbf{Z}_2)$ and the following relations hold.*

(3.4) $$Q_1(z_1) + z_1 * z_2 + z_3 * [1] = 0.$$

(3.5) $$Q_2(z_1) = z_1 * z_3.$$

(3.6) $$Q_1(z_2) + z_2 * z_3 + Q_3(z_1) = 0.$$

Now we shall estimate the $\mathbf{Z}_2$ coefficient Betti numbers of $\hat{M}_2$ from below. The result will be needed to prove Theorem A. As in the case of M_2, we obtain new elements of $H_*(\hat{M}_2; \mathbf{Z}_2)$ and the result is as follows.

PROPOSITION 3.7. *The image of the elements of Proposition* 3.2 *are elements of $H_*(\hat{M}_2; \mathbf{Z}_2)$ all of which are non-trivial and differ to each other in $H_*(\hat{M}_2; \mathbf{Z}_2)$ except for $i_{1*}Q_2(z_1) = i_{1*}(z_1 * z_3)$.*

PROOF. First of all it is known [4] that all the elements of Proposition 3.2 are non-trivial and differ to each other in $H_*(\Omega_2^3 S^3; \mathbf{Z}_2)$ except for $(i_2 \cdot i_1)_* Q_2(z_1) = (i_2 \cdot i_1)_*(z_1 * z_3)$. Hence they are non-trivial and differ to each other except for $i_{1*}Q_2(z_1) = i_{1*}(z_1 * z_3)$ by (3.5). This completes the proof of Proposition 3.7.

Now we see the following corollary by Proposition 3.7.

COROLLARY 3.8. *Put $b_q = \dim_{\mathbf{Z}_2} H_q(\hat{M}_2; \mathbf{Z}_2)$. Then we have*

$$\begin{cases} b_q \geq 1 & q = 0, 1, 8, 9 \text{ and} \\ b_q \geq 2 & 2 \leq q \leq 7. \end{cases}$$

4. Computation of $H_*(\hat{F}_2; Z_2)$

In this section we shall determine $H_*(\hat{F}_2; \mathbf{Z}_2)$. The result is as follows. Hereafter all homology groups and cohomology groups are with $\mathbf{Z}_2$ coefficients.

PROPOSITION 4.1. *$H_*(\hat{F}_2)$ is given as follows.*

$$H_q(\hat{F}_2) = \begin{cases} \mathbf{Z}_2 & q = 0, 3, 7, 10 \\ 0 & \text{otherwise.} \end{cases}$$

We write (λ_1, λ_2) instead of Λ and $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for B. We define open subsets U and V of $\hat{F}_2$ by

$$(4.2) \quad U = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix} \in \hat{F}_2 ; \lambda_1 \neq 0 \right\} \text{ and } V = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix} \in \hat{F}_2 ; \lambda_2 \neq 0 \right\}.$$

Note that $\hat{F}_2 = U \cup V$ by (2.3). We shall prove Proposition 4.1 by using the Mayer-Vietoris exact sequence of the pair $\{U, V\}$.

First we shall determine $H_*(U)$. The following lemma is proved by direct computation taking $x = c - b\lambda_1^{-1}\lambda_2$ in (2.3).

LEMMA 4.3. $U = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix} ; \lambda_1 \neq 0 \textit{ and the following } (4.4) \textit{ is satisfied} \right\}.$

$$(4.4) \qquad b(\lambda_1^{-1}\lambda_2)^2 + (a - c)\lambda_1^{-1}\lambda_2 - b \neq 0.$$

We define open subsets U_1 and U_2 of U by

(4.5)

$$U_1 = \left\{ \begin{pmatrix} \Lambda \\ B \end{pmatrix} \in U ; \| \lambda_1^{-1}\lambda_2 \| < 2/3 \right\} \text{ and } U_2 = \left\{ \begin{pmatrix} \Lambda \\ B \end{pmatrix} \in U ; \| \lambda_1^{-1}\lambda_2 \| > 1/3 \right\}.$$

We write $\mathbf{H}^*$ for $\mathbf{H} - \{0\}$. The following lemma is easily proved by using Lemma 4.3.

LEMMA 4.6. (1) *A homeomorphism* $f_1 \colon U_1 \to \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \{q \in \mathbf{H}; \|q\| < 2/3\}$ *is defined by* $\begin{pmatrix} \Lambda \\ B \end{pmatrix} \mapsto (\lambda_1, a, c, b(\lambda_1^{-1}\lambda_2)^2 + (a - c)\lambda_1^{-1}\lambda_2 - b, \lambda_1^{-1}\lambda_2)$.
(2) *A homeomorphism* $f_2 \colon U_2 \to \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \{q \in \mathbf{H}; \|q\| > 1/3\}$ *is defined by* $\begin{pmatrix} \Lambda \\ B \end{pmatrix} \mapsto (\lambda_1, a, b, b(\lambda_1^{-1}\lambda_2)^2 + (a-c)\lambda_1^{-1}\lambda_2 - b, \lambda_1^{-1}\lambda_2)$ *and* $f_2(U_1 \cap U_2) = \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \{q \in \mathbf{H}; 1/3 < \|q\| < 2/3\}$.

Next we shall determine the generators of $H_*(U_1)$. Let $\sigma \in H_3(S^3)$ be the fundamental class. We define a map $u \colon (S^3)^2 \to U_1$ to be the composite of the following maps, defined by $l_1(z_1, z_2) = (z_1, 0, 0, z_2, 0)$, (4.7) $$(S^3)^2 \xrightarrow{l_1} \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \{q \in \mathbf{H};\ \|q\| < 2/3\} \underset{\simeq}{\xrightarrow{f_1^{-1}}} U_1.$$

We define α_1 and α_2 by $\alpha_1 = u_*(\sigma \otimes 1)$ and $\alpha_2 = u_*(1 \otimes \sigma)$ respectively. Then α_1 and α_2 generate $H_3(U_1)$. Similarly we define the generator A of $H_6(U_1)$ by $A = u_*(\sigma \otimes \sigma)$.

It is easy to see that the inclusion $j_2: U_1 \cap U_2 \to U_2$ is a homotopy equivalence. Hence by considering the Mayer-Vietoris exact sequence of the pair $\{U_1, U_2\}$, we see the following

PROPOSITION 4.8. *Let $m: U_1 \to U$ be the inclusion. Then $m_*: H_*(U_1) \to H_*(U)$ is an isomorphism. In particular,*

$$H_q(U) = \begin{cases} \mathbf{Z}_2 & q = 0, 6 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

Similarly we can determine $H_*(V)$ and their generators.

PROPOSITION 4.9.

$$H_q(V) = \begin{cases} \mathbf{Z}_2 & q = 0, 6 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, $\tilde{m}_*\tilde{\alpha}_1, \tilde{m}_*\tilde{\alpha}_2$ are generators of $H_3(V)$ and $\tilde{m}_*\tilde{A}$ is the generator of $H_6(V)$ for the inclusion $\tilde{m}: V_1 \to V$ and some generators $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{A}$ defined in the same way as before.

The computation of $H_*(U \cap V)$ is easy. In fact the map $h: U \cap V \to \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \mathbf{H}^*$ defined by the same formula as f_1 is a homeomorphism. We define a map $\theta: (S^3)^3 \to U \cap V$ to be the composite of the following maps, defined by $l_2(z_1, z_2, z_3) = (z_1, 0, 0, z_2, 2z_3)$,

(4.10) $$(S^3)^3 \xrightarrow{l_2} \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \mathbf{H}^* \underset{\simeq}{\xrightarrow{h^{-1}}} U \cap V.$$

We define $\omega_1 = \theta_*(\sigma \otimes 1 \otimes 1)$, $\omega_2 = \theta_*(1 \otimes \sigma \otimes 1)$, $\omega_3 = \theta_*(1 \otimes 1 \otimes \sigma)$, $\Omega_1 = \theta_*(1 \otimes \sigma \otimes \sigma)$, $\Omega_2 = \theta_*(\sigma \otimes 1 \otimes \sigma)$, $\Omega_3 = \theta_*(\sigma \otimes \sigma \otimes 1)$, and $\Omega = \theta_*(\sigma \otimes \sigma \otimes \sigma)$. Then

PROPOSITION 4.11.

$$H_q(U \cap V) = \begin{cases} \mathbf{Z}_2 & q = 0, 9 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 3, 6 \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, ω_1, ω_2 *and* ω_3 *are generators of* $H_3(U \cap V)$, Ω_1, Ω_2 *and* Ω_3 *are generators of* $H_6(U \cap V)$, *and* Ω *is a generator of* $H_9(U \cap V)$.

LEMMA 4.12. *Let* $s_1: U \cap V \to U$ *and* $s_2: U \cap V \to V$ *be the inclusions. Then, we have that* $s_{1*} \oplus s_{2*}(\omega_1) = (m_*\alpha_1, \tilde{m}_*\tilde{\alpha}_1)$, $s_{1*} \oplus s_{2*}(\omega_2) = (m_*\alpha_2, \tilde{m}_*\tilde{\alpha}_2)$, $s_{1*} \oplus s_{2*}(\omega_3) = (0, \tilde{m}_*\tilde{\alpha}_1)$, $s_{1*} \oplus s_{2*}(\Omega_1) = (0, \tilde{m}_*\tilde{A})$, $s_{1*} \oplus s_{2*}(\Omega_2) = (0, 0)$ *and* $s_{1*} \oplus s_{2*}(\Omega_3) = (m_*A, \tilde{m}_*\tilde{A})$.

PROOF. Let $g_1: V_1 \to \mathbf{H}^* \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}^* \times \{q \in \mathbf{H};\ \|q\| < 2/3\}$ be the homeomorphism, defined by $(\lambda_2, a, c, b(\lambda_2^{-1}\lambda_1)^2 + (c-a)\lambda_2^{-1}\lambda_1 - b, \lambda_2^{-1}\lambda_1)$, in the same way as f_1. Then we see that $g_1 \cdot (h|U \cap V_1)^{-1} \cdot l_2(z_1, z_2, z_3) = (2z_1z_3, 0, 0, -(1/4)z_2z_3^{-2}, (1/2)z_3^{-1})$. This implies easily Lemma 4.12, because the inclusions $V_1 \to V$ and $U \cap V_1 \to U \cap V$ are homology equivalences.

Now we know the map $s_{1*} \oplus s_{2*}: H_*(U \cap V) \to H_*(U) \oplus H_*(V)$ is injective in dimension 3 and surjective in dimension 6 by Lemma 4.12. Proposition 4.1 follows from the Mayer-Vietoris exact sequence of the pair $\{U, V\}$.

5. Determination of $H_*(\hat{M}_2; Z_2)$

In this section we shall prove the homological statement of Theorem A. The homotopical statement is proved in §6. Note that we have the following principal bundle and double covering by Proposition 2.9.

$$SO(2) \longrightarrow \hat{F}_2 \longrightarrow \hat{F}_2/SO(2). \tag{5.1}$$

$$\mathbf{Z}_2 \longrightarrow \hat{F}_2/SO(2) \longrightarrow \hat{M}_2. \tag{5.2}$$

We shall compute $H_*(\hat{M}_2)$ in the following manner. First we compute $H^*(\hat{F}_2/SO(2))$ by using the Serre spectral sequence of (5.1). Next we compute $H^*(\hat{M}_2)$ by using the Gysin sequence of (5.2).

In order to compute $H^*(\hat{F}_2/SO(2))$, we need one technical fact.

PROPOSITION 5.3. *Let* $w_2(\hat{F}_2)$ *be the second Stiefel-Whitney class of* (5.1). *Then* $w_2(\hat{F}_2)^2$ *is non-trivial.*

This proposition is a direct consequence of the following

LEMMA 5.4. *Let* $w_2(F_2)$ *be the second Stiefel-Whitney class of the principal bundle*

$$SO(2) \longrightarrow F_2 \longrightarrow F_2/SO(2). \tag{5.5}$$

Then $w_2(F_2)^2$ *is non-trivial.*

Lemma 5.4 is proved by using the following facts with the Serre spectral sequence of (5.5).

PROPOSITION 5.6 ([8], [9]). (1) $H^*(M_2) = \mathbf{Z}_2[u, v]/(u^4, v^4)$ *where* $\deg u = 1$ *and* $\deg v = 2$.
(2) F_2 *is connected and simply-connected.*

Now by using Propositions 4.1 and 5.3, we can prove the following proposition by a standard argument on the Serre spectral sequence associated to (5.1).

PROPOSITION 5.7. $H^*(\hat{F}_2/SO(2))$ *is given by the following table.*

q	0	1	2	3	4	5	6	7	8	9
H^q	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$
gen.	1		x	y	x^2	xy	x^3	x^2y		x^3y

Finally we shall prove the homological statement of Theorem A. Because of Propositions 3.7 and 5.6, it suffices to show the following

PROPOSITION 5.8.

$$H^q(\hat{M}_2) = \begin{cases} \mathbf{Z}_2 & q = 0, 1, 8, 9 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & 2 \leq q \leq 7 \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. We consider the Gysin exact sequence of (5.2), that is,

$$\longrightarrow H^q(\hat{M}_2) \xrightarrow{p^*} H^q(\hat{F}_2/SO(2)) \xrightarrow{\nu} H^q(\hat{M}_2) \xrightarrow{\cup w_1} H^{q+1}(\hat{M}_2) \longrightarrow$$

where w_1 is the first Stiefel-Whitney class of (5.2) and $p: \hat{F}_2/SO(2) \to \hat{M}_2$ is the projection of (5.2).

Step 1. $H^1(\hat{M}_2) = \mathbf{Z}_2$. In fact, we know $H^1(\hat{F}_2/SO(2)) = 0$ by Proposition 5.7. Hence, $H^0(\hat{M}_2) \xrightarrow{\cup w_1} H^1(\hat{M}_2)$ is an isomorphism.

Step 2. $H^2(\hat{M}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. In fact, $H^1(\hat{F}_2/SO(2)) = 0$ and $H^2(\hat{F}_2/SO(2)) = \mathbf{Z}_2$ by Proposition 5.7 and $H^1(\hat{M}_2) = \mathbf{Z}_2$ by Step 1. Hence we see $H^2(\hat{M}_2)$ is whether $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. But we know $b_2 \geq 2$ by Corollary 3.8. Hence Step 2 holds. Note that this argument shows that there exists $\varphi \in H^2(\hat{M}_2)$ which satisfies $p^*\varphi = x$.

Step 3. $H^q(\hat{M}_2) = 0$ for $q \geq 10$. In fact, we see $H^q(\hat{M}_2) \simeq H^{10}(\hat{M}_2)$ for $q \geq 10$. As $\hat{M}_2$ is a finite dimensional manifold, we see $H^q(\hat{M}_2) = 0$ for $q > \dim \hat{M}_2$. Hence Step 3 holds.

We can easily determine $H^q(\hat{M}_2)$ for $9 \geq q \geq 3$ similarly as in Step 2 by using Corollary 3.8 and Proposition 5.7 in the decreasing order of q. This will complete the proof of Proposition 5.8. But the details are omitted.

6. $\pi_q(\hat{M}_2)$ for $0 \leq q \leq 2$

LEMMA 6.1. *$\pi_q(\hat{F}_2) = 0$ for $0 \leq q \leq 2$.*

PROOF. We use the notation of §4. It is clear that $U \cap V$ is a Zariski open set of $\hat{F}_2$ such that the real codimension of the complement in $\hat{F}_2$ equals to 4. Hence general position argument shows that $\pi_q(U \cap V) \simeq \pi_q(\hat{F}_2)$ for $0 \leq q \leq 2$. Note that $U \cap V$ is homotopically equivalent to $(S^3)^3$. Hence Lemma 6.1 holds.

Now by using the homotopy exact sequence of the principal bundle

$$O(2) \longrightarrow \hat{F}_2 \longrightarrow \hat{M}_2, \tag{6.2}$$

we can prove the homotopical statement of Theorem A.

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