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Geometry of minimum contrast

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1. Introduction

Such concepts as information, entropy, divergence, energy and so on play an important role in mathematical sciences to research random phenomena. This paper tries a unified approach to measurement of these notions, in particular the geometrical structure induced by a contrast function. In the mathematical formulation a contrast function ρ on a manifold M is defined by the first requirement for distance: $\rho(x, y) \geq 0$ with equality if and only if $x = y$, see Eguchi [2] for various examples. A simple example is found in

$$\rho_1(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{n+1} p_i(\log p_i - \log q_i)$$

on the n-simplex $\mathscr{S} = \{\boldsymbol{p} = (p_1, \ldots, p_{n+1}) \colon \sum_{i=1}^{n+1} p_i = 1, 0 < p_i < 1\}$. This function is called the Kullback information in the context that $\boldsymbol{p}$ and $\boldsymbol{q}$ are the vectors of probabilities for $n + 1$ disjoint events, see [2] for other examples and construction for ρ. Thus a contrast function is generally not assumed to be symmetric as seen in ρ_1.

We discuss on the manifold M instead of $\mathscr{S}$ on the assumption of finite dimensionality because we wish to investigate contrast functions or functionals over not only $\mathscr{S}$ but also a general space of probability measures. A new geometry on M by means of ρ is presented: a Riemannian g, a pair (∇, ∇^*) of torsion-free connections and a pair (D, D^*) of second-order differentials. The asymmetry of ρ leads to different two connections ∇ and ∇^* such that $1/2\ (\nabla + \nabla^*)$ is the Riemannian connection. Lauritzen [3] calls (M, g, T) a statistical manifold, where T is the third order tensor representing the difference between ∇ and ∇^*. In general such a pair (∇, ∇^*) is called conjugate in the sense that if M is curvature-free with respect to ∇, then M is also curvature-free with respect to ∇^*. Nagaoka and Amari [6] extended a notion of locally Euclidean space: If M is curvature-free with respect to ∇, then there exists a pair of local coordinates (x^i, U) and (x_i^*, V) such that

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_j^*}\right) = \delta_i{}^j \qquad \text{(Kronecker's delta)}$$

on $U \cap V$. In Section 2 we present a further conjugacy property introduced by new operators (D, D^*) related with ρ. It is shown that the two operators D and D^* generate tensors $B(X, Y)$ and $B^*(X, Y)$ of which antisymmetric parts are the Riemannian curvature tensors with respect to ∇ and ∇^*, respectively. Section 3 investigates the case of a Riemannian space (M, g). A contrast function $\rho_0(x, y)$ on M is naturally defined by the squared arc-length of a geodesic curve connecting x with y in M. We give a formula of geometric quantities g_0, (∇_0, ∇_0^*) and (D_0, D_0^*) by ρ_0. Section 4 gives the induced form of the geometry by ρ into a submanifold $\tilde{M}$. Let x be in $M-\tilde{M}$. We consider minimization of ρ from x to $\tilde{M}$. For a fixed point $\tilde{x}$ of $\tilde{M}$ we denote $L_{\tilde{x}}$ the space of points from which minimization of ρ into $\tilde{M}$ are given at $\tilde{x}$. If for any x there exists a unique minimizer $\tilde{x}$ of the function $\rho(x, \cdot)$, M is decomposed into a foliation $M=\cup\{L_{\tilde{x}}: \tilde{x}\in\tilde{M}\}$. We call $L_{\tilde{x}}$ a minimum contrast leaf and we investigate the second fundamental tensor of $L_{\tilde{x}}$. It is shown that the tensor of $L_{\tilde{x}}$ vanishes at $\tilde{x}$.

2. Geometry associated with a contrast function

Let M be a C^∞-manifold of dimension d. Let $\mathfrak{X}(M)$ be the space of vector fields on M and $\mathfrak{F}(M)$ the space of C^∞-differentiable functions on M. We call $\rho: M\times M\to\mathbf{R}$ a contrast function if $\rho(x, y)\geq 0$ for all x and y in M with equality if and only if $x=y$. Eguchi [2] introduced three classes of *W-type*, *M-type* and *S-type* in all the contrast functions on a space of probability distributions. In this paper it is assumed that ρ is a C^∞-function on $M\times M$ and that

$$X_x X_x \rho(x, y)|_{y=x} > 0$$

for all nonzero X in $\mathfrak{X}(M)$ and $x\in M$. We will show that the assumption determines the main order of ρ (see the last paragraph in this section). Throughout this paper we use the standard notation in Kobayashi and Nomizu [3] in addition to the following notation on partial differentials:

$$\rho(X_1\cdots X_n | Y_1\cdots Y_m)(z) = (X_1)_x\cdots(X_n)_x(Y_1)_y\cdots(Y_m)_y\rho(x, y)|_{x=z, y=z}$$

for $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ in $\mathfrak{X}(M)$. A Riemannian metric g on M is defined by

$$g(X, Y) = -\rho(X|Y).$$

In effect the bilinearity of g holds by definition. Since the contrast $\rho(x, y)$ has a minimum 0 when $x=y$, we see $\rho(Y|\cdot)=0$ for any $Y\in\mathfrak{X}(M)$. Moreover, applying X to $\rho(Y|\cdot)=0$ we have

$$\rho(XY|\cdot) = -\rho(X|Y).$$

Thus from the assumption we get $g(X, X) > 0$ for all $X \neq 0$ in $\mathfrak{X}(M)$. The symmetry follows from $g(X, Y) - g(Y, X) = -\rho([X, Y]|\cdot) = 0$. Accordingly g is well-defined as a metric tensor with the expressions

$$g(X, Y) = \rho(XY|\cdot) = \rho(\cdot|XY).$$

Next we define a pair (∇, ∇^*) of covariant differentials as follows:

$$g(\nabla_X Y, Z) = -\rho(XY|Z) \quad \text{and} \quad g(\nabla^*_X Y, Z) = -\rho(Z|XY)$$

for all $Z \in \mathfrak{X}(M)$. Here $\nabla_X Y$ and $\nabla^*_X Y$ are determined by the conditions that the above quantities are satisfied for all Z. By definition the mapping $(X, Y) \to \nabla_X Y$ is bilinear. Noting that

$$\begin{aligned} g(\nabla_{fX} Y, Z) &= -\rho((fX)Y|Z) = g(f\nabla_X Y, Z), \\ g(\nabla_X fY, Z) &= -\rho(X(fY)|Z) = -\rho((Xf)Y + f(XY)|Z) \\ &= g((Xf)Y + f\nabla_X Y, Z) \end{aligned}$$

for all $f \in \mathfrak{F}(M)$ and all $Z \in \mathfrak{X}(M)$, we have

$$\nabla_{fX} Y = f\nabla_X Y \quad \text{and} \quad \nabla_X fY = (Xf)Y + f\nabla_X Y. \tag{2.1}$$

Similarly we can see that ∇^* satisfies these properties. Thus ∇ and ∇^* are well-defined connections and have the following relation, see Eguchi [2].

PROPOSITION 1. *Let* $\bar{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$. *Then* $\bar{\nabla}$ *is the Riemannian connection with respect to* g.

PROOF. By definition,

$$Xg(Y, Z) = -\rho(XY|Z) - \rho(Y|XZ) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z).$$

This implies

$$Xg(Y, Z) = \frac{1}{2}X\{g(Y, Z) + g(Z, Y)\} = g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}_X Z),$$

which shows that $\bar{\nabla}$ is metric. Next we see that

$$g(\nabla_X Y - \nabla_Y X, Z) = -\rho(XY - YX|Z) = g([X, Y], Z)$$

and

$$g(\nabla^*_X Y - \nabla^*_Y X, Z) = -\rho(Z|XY - YX) = g([X, Y], Z)$$

for all $Z \in \mathfrak{X}(M)$, which implies that both ∇ and ∇^* are torsion-free and hence $\bar{\nabla}$ is. □

If ρ is symmetric, then $\bar{\nabla} = \nabla = \nabla^*$. This case reduces to the Riemannian geometry. A typical example of a contrast function is asymmetric as ρ_1 defined in Introduction. Hence we pay attention to a tensor on M,

$$T(X, Y, Z) = g(\nabla_X Y - \nabla_X^* Y, Z).$$

The tensor T is symmetric because

$$\begin{aligned} T(X, Y, Z) - T(Y, X, Z) &= g(\nabla_X Y - \nabla_Y X - (\nabla_X^* Y - \nabla_Y^* X), Z) \\ &= g([X, Y] - [X, Y], Z) = 0 \end{aligned}$$

and

$$T(X, Y, Z) - T(X, Z, Y) = X\{g(Y, Z) - g(Z, Y)\} = 0.$$

Thus the triple (M, g, T) becomes a statistical manifold according to the terminology by Lauritzen [4].

Nagaoka and Amari [6] introduced a dualistic structure on such a triple (M, g, T), see also Chapter 3 in Amari [1] for extensive discussions. The identity

$$[X, Y]g(Z, W) = XYg(Z, W) - YXg(Z, W)$$

leads to

$$g(R(X, Y)Z, W) = g(Z, R^*(Y, X)W),$$

where R and R^* are the Riemannian curvature tensors associated with ∇ and ∇^*, that is,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

and

$$R^*(X, Y) = \nabla_X^* \nabla_Y^* - \nabla_Y^* \nabla_X^* - \nabla_{[X, Y]}^*.$$

Thus it is seen that M is R-free if and only if it is R^*-free. Further, when M is R-free and R^*-free, the corresponding dual affine coordinates (x^i) and (x_i^*) to ∇ and ∇^*, that is

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = 0, \ \nabla_{\partial/\partial x_i^*}^* \frac{\partial}{\partial x_j^*} = 0 \quad \text{and} \quad g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_j^*}\right) = \delta_i{}^j$$

are connected with the Legendre transformation $\sum_i x^i x_i^* = \psi(x) + \varphi(x^*)$. Here both ψ and φ are convex-conjugate and are called the potential functions. It is shown that

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2}{\partial x^i \partial x^j} \psi, \quad g\left(\frac{\partial}{\partial x_i^*}, \frac{\partial}{\partial x_i^*}\right) = \frac{\partial^2}{\partial x_i^* \partial x_j^*} \varphi. \tag{2.2}$$

Thus the notion of a locally Euclidean space can be extended to a dualistic version.

We now define a pair (D, D^*) of differential operators $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by the conditions

$$g(D_{X,Y}Z, W) = -\rho(XYZ|W) \quad \text{and} \quad g(D^*_{X,Y}Z, W) = -\rho(W|XYZ),$$

which should be satisfied for all $W \in \mathfrak{X}(M)$.

PROPOSITION 2. *The operator D satisfies the following conditions*:

(1) *The mapping* $(X, Y, Z) \longrightarrow D_{X,Y}Z$ *is trilinear.*

(2) $D_{fX,Y}Z = fD_{X,Y}Z,$

(3) $D_{X,fY}Z = fD_{X,Y}Z + Xf\nabla_Y Z$

and

(4) $D_{X,Y}fZ = fD_{X,Y}Z + Xf\nabla_Y Z + Yf\nabla_X Z + X(Yf)Z$

for all $f \in \mathfrak{F}(M)$.

PROOF. By definition, (1) is clear. The Leipnitzs law yields that

$$g(D_{fX,Y}Z, W) = -\rho(fXYZ|W) = g(fD_{X,Y}Z, W),$$

$$g(D_{X,fY}Z, W) = -\rho(fXYZ + (Xf)YZ|W) = g(fD_{X,Y}Z + Xf\nabla_Y Z, W)$$

and

$$\begin{aligned} g(D_{X,Y}fZ, W) &= -\rho(fXYZ + (Xf)YZ + (Yf)XZ + X(Yf)Z|W) \\ &= g(fD_{X,Y}Z + Xf\nabla_Y Z + Yf\nabla_X Z + X(Yf)Z, W) \end{aligned}$$

for all $W \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$, which conclude (2), (3) and (4). □

Take arbitrarily two local coordinate systems $(\lambda, U, (y^i))$, and $(\mu, V, (z^a))$ with $U \cap V \neq \phi$. Then $D_{\partial/\partial y^i, \partial/\partial y^j}\partial/\partial y^k$ defines the components of D in the coordinates $(\lambda, U, (y^i))$. The natural bases $\{\partial/\partial y^i\}$ and $\{\partial/\partial z^a\}$ on $U \cap V$ are related by

$$\frac{\partial}{\partial z^a} = \frac{\partial y^i}{\partial z^a}\frac{\partial}{\partial y^i}$$

from which it follows that

$$D_{\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}}\frac{\partial}{\partial z^c} = \frac{\partial y^k}{\partial z^c} D_{\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}}\frac{\partial}{\partial y^k} + \frac{\partial^2 y^j}{\partial z^a \partial z^c}\nabla_{\frac{\partial}{\partial z^b}}\frac{\partial}{\partial y^j}$$

$$+\frac{\partial^2 y^j}{\partial z^b \partial z^c}\nabla_{\frac{\partial}{\partial z^a}}\frac{\partial}{\partial y^j}+\frac{\partial^3 y^k}{\partial z^a \partial z^b \partial z^c}\frac{\partial}{\partial y^k} \qquad \text{(from (1) and (4))}$$

$$=\frac{\partial y^i}{\partial z^a}\frac{\partial y^j}{\partial z^b}\frac{\partial y^k}{\partial z^c}D_{\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^k}+\frac{\partial^2 y^j}{\partial z^a \partial z^b}\frac{\partial y^k}{\partial z^c}\nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^k}$$

$$+\frac{\partial^2 y^j}{\partial z^a \partial z^c}\frac{\partial y^k}{\partial z^b}\nabla_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}+\frac{\partial^2 y^j}{\partial z^b \partial z^c}\frac{\partial y^k}{\partial z^a}\nabla_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}+\frac{\partial^3 y^k}{\partial z^a \partial z^b \partial z^c}\frac{\partial}{\partial y^k}$$

(from (1), (2) and (3)), where $\{\partial y^i/\partial z^a\}$ denotes the Jacobi matrix of $\lambda^{-1}(\mu(\cdot))$. Here and hereafter the Einstein convention is used for indices i, j and k. Thus we observe that the set of the conditions (1)–(4) determines the transformation rule of components of D for a change of variables. By a similar argument we see that D^* enjoys also the conditions:

(1)′ The mapping $(X, Y, Z) \longrightarrow D^*_{X,Y}Z$ is trilinear.

(2)′ $D^*_{fX,Y}Z = fD^*_{X,Y}Z$

(3)′ $D^*_{X,fY}Z = fD^*_{X,Y}Z + Xf\nabla^*_Y Z$ and

(4)′ $D^*_{X,Y}fZ = fD^*_{X,Y}Z + Xf\nabla^*_Y Z + Yf\nabla^*_X Z + X(Yf)Z$

for all $f \in \mathfrak{F}(M)$.

We now define

$$B(X, Y) = D_{X,Y} - \nabla_X\nabla_Y \quad \text{and} \quad B^*(X, Y) = D^*_{X,Y} - \nabla^*_X\nabla^*_Y.$$

Then we have that

$$B(fX, Y)Z = B(X, fY)Z = B(X, Y)fZ = fB(X, Y)Z$$

for all $f \in \mathfrak{F}(M)$ since $\nabla_X\nabla_Y$ also satisfies the conditions (1)–(4). Thus both $B(X, Y)$ and $B^*(X, Y)$ are $\mathfrak{F}(M)$-linear and are a kind of curvature-like tensors associated with D and D^*. We now show that the antisymmetric part of B is nothing but the Riemannian curvature tensor.

PROPOSITION 3. $R(X, Y) = B(Y, X) - B(X, Y)$.

PROOF. The result follows from $D_{X,Y}Z - D_{Y,X}Z = \nabla_{[X,Y]}Z$. In fact,

$$g(D_{X,Y}Z - D_{Y,X}Z, W) = -\rho([X, Y]Z \mid W) = g(\nabla_{[X,Y]}Z, W)$$

for all $W \in \mathfrak{X}(M)$. □

By a similar argument, $R^*(X, Y) = B^*(Y, X) - B^*(X, Y)$. Proposition 3 directly implies Bianchi's first and second identities:

$$\mathfrak{S}R(X, Y)Z = 0 \quad \text{and} \quad \mathfrak{S}(\nabla_Z R)(X, Y) = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum on X, Y and Z. The symmetry of B is equivalent to R-freeness. Further, the following identities hold.

PROPOSITION 4. (1) $B(X, Y)Z = B(X, Z)Y$.

(2) $$g(B(X, Y)Z, W) = g(B(W, Y)Z, X).$$

(3) $$g(B^*(Y, X)W, Z) = g(B(X, Y)Z, W).$$

PROOF. We get

$$\begin{aligned} B(X, Y)Z &= D_{X,Y}Z - \nabla_X \nabla_Y Z \\ &= D_{X,Y}Y + \nabla_X[Y, Z] - \nabla_X(\nabla_Z Y + [Y, Z]) = B(X, Z)Y \end{aligned}$$

since

$$D_{X,Y}Z = D_{X,Z}Y + \nabla_X[Y, Z].$$

Hence we obtain (1). We next show (2). By applying X to the definition

$$g(\nabla_Y Z, W) = -\rho(YZ|W)$$

we get

$$g(\nabla_X \nabla_Y Z, W) + g(\nabla_Y Z, \nabla_X^* W) = -\rho(XYZ|W) - \rho(YZ|XW),$$

or

$$g(B(X, Y)Z, W) = g(\nabla_Y Z, \nabla_X^* W) + \rho(YZ|XW). \tag{2.3}$$

From this and the torsion-freeness of ∇^* it follows that

$$\begin{aligned} g(B(W, Y)Z, X) &= g(\nabla_Y Z, [W, X] + \nabla_X^* W) + \rho(YZ|WX) \\ &= g(\nabla_Y Z, \nabla_X^* W) + \rho(YZ|XW) = g(B(X, Y)Z, W), \end{aligned}$$

which concludes (2). The identity

$$Y[g(Z, \nabla_X^* W) + \rho(Z|XW)] = 0$$

leads to

$$g(B^*(Y, X)W, Z) = g(\nabla_Y Z, \nabla_X^* W) + \rho(YZ|XW), \tag{2.4}$$

which concludes (3) because of (2.3). □

Since it follows from (3) in Proposition 3 that

$$g(\{B(X, Y) - B^*(X, Y)\}Z, W) = g(B(X, Y)Z, W) - g(B(Y, X)W, Z),$$

we obtain that $B(X, Y) = B^*(X, Y)$ if and only if

$$g(B(X, Y)Z, W) = g(B(Y, X)W, Z)$$

for all Z and W in $\mathfrak{X}(M)$.

From this we get a kind of symmetry associated with B.

COROLLARY 1. *The forth-order tensor $g(B(X, Y)Z, W)$ or $g(B^*(X, Y)Z, W)$ is symmetric if and only if B is equal to B^* and R vanishes.*

PROOF. The result follows from the above statement and Proposition 4 (1) and (2). □

Now we obtain that the contrast function generates a further dualistic structure over M.

THEOREM 1. *The following statements are equivalent:*

(1) *M is B-free.* (2) *M is B^*-free.*

(3) *There exists a system of coordinates (x^i) satisfying*

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = 0 \qquad (1 \le i, j \le d)$$

and

$$D_{\partial/\partial x^i, \partial/\partial x^j} \frac{\partial}{\partial x^k} = 0 \qquad (1 \le i, j, k \le d). \tag{2.5}$$

(4) *There exists a system of coordinates (x_i^*) satisfying*

$$\nabla^*_{\partial/\partial x_i^*} \frac{\partial}{\partial x_j^*} = 0 \qquad (1 \le i, j \le d)$$

and

$$D^*_{\partial/\partial x_i^*, \partial/\partial x_j^*} \frac{\partial}{\partial x_k^*} = 0 \qquad (1 \le i, j, k \le d). \tag{2.6}$$

PROOF. If follows from (3) in Proposition 4 that (1) is equivalent to (2). Next we assume (1). Then M is R-free on account of Proposition 3. Namely M has ∇-affine coordinates (x^i), which are seen from (1) that

$$D_{\partial/\partial x^i, \partial/\partial x^j} \frac{\partial}{\partial x^k} = 0$$

This implies (3). Conversely if (3) holds, then

$$B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} = 0$$

with respect to the coordinates (x^i), which leads M to be B-free since B is a tensor. Similarly (2) is equivalent to (4). □

In the statements (3) and (4), (2.5) and (2.6) can be exchanged for

$$\rho\left(\frac{\partial}{\partial x^i}\ \frac{\partial}{\partial x^j}\,\middle|\,\frac{\partial}{\partial x^k}\ \frac{\partial}{\partial x^1}\right)=0 \quad\text{and}\quad \rho\left(\frac{\partial}{\partial x_i^*}\ \frac{\partial}{\partial x_j^*}\,\middle|\,\frac{\partial}{\partial x_k^*}\ \frac{\partial}{\partial x_l^*}\right)=0,$$

respectively, on account of (2.3). We assume that M is B-free in this paragraph. From (2.2) it is satisfied that

$$g\left(\frac{\partial}{\partial x^i}, \nabla^*_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}\right)=\frac{\partial^3}{\partial x^i\partial x^j\partial x^k}\psi \tag{2.7}$$

and

$$g\left(\nabla_{\frac{\partial}{\partial x_i^*}}\frac{\partial}{\partial x_j^*}, \frac{\partial}{\partial x_k^*}\right)=\frac{\partial^3}{\partial x_i^*\partial x_j^*\partial x_k^*}\varphi.$$

Further, then

$$g\left(\frac{\partial}{\partial x^i}, D^*_{\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right)=\frac{\partial^4}{\partial x^i\partial x^j\partial x^k\partial x^l}\psi \tag{2.8}$$

since

$$\frac{\partial}{\partial x^j}\left[g\left(\frac{\partial}{\partial x^i}, \nabla^*_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right)-\frac{\partial^3}{\partial x^i\partial x^k\partial x^l}\psi\right]=0$$

yields

$$g\left(\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}, \nabla^*_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right)+g\left(\frac{\partial}{\partial x^i}, \nabla^*_{\frac{\partial}{\partial x^j}}\nabla^*_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right)=\frac{\partial^4}{\partial x^i\partial x^j\partial x^k\partial x^l}\psi.$$

Similarly we obtain that

$$g\left(D_{\frac{\partial}{\partial x_i^*},\frac{\partial}{\partial x_j^*}}\frac{\partial}{\partial x_j^*}, \frac{\partial}{\partial x_l^*}\right)=\frac{\partial^4}{\partial x_i^*\partial x_j^*\partial x_k^*\partial x_l^*}\varphi.$$

If M is R-free, then the divergence function can be introduced as

$$d(x_1, x_2^*)=\psi(x_1)+\varphi(x_2^*)-\sum_{i=1}^{d} x_1{}^i x_{2i}^*,$$

where ψ and φ are potential functions with respect to (x^i) and (x_i^*), respectively. Thus d is a contrast function, see [1]. The contrast function ρ is related with d as follows.

COROLLARY 2. *Assume that M is B-free. Then*

$$\rho(x_1, x_2^*) = d(x_1, x_2^*)$$

by neglecting $O(\|x_1 - x_2\|^5)$.

PROOF. We write $\delta(x_1, x_2^*) = \rho(x_1, x_2^*) - d(x_1, x_2^*)$. It suffices to show that the differential coefficients of $\delta(x_1, x_2^*)$ in x_1 vanish at $x_1 = x_2$ up to the forth-order by Taylor's theorem. By definition we have the following identities: $\rho(XY|\cdot) = g(X, Y)$,

$$\rho(XYZ\,|\cdot) = g(\nabla_Y Z, X) + g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$$

and

$$\rho(XYZ\,|\cdot) = g(D_{X,Y}Z, W) + g(\nabla_X \nabla_Z W, Y) + g(\nabla_Z W, \nabla_X^* Y)$$
$$+ g(\nabla_X \nabla_Y Z, W) + g(\nabla_Y Z, \nabla_X^* W) + g(\nabla_X Z, \nabla_Y^* W) + g(Z, \nabla_X^* \nabla_Y^* W).$$

Hence we have

$$\rho\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\,\middle|\,\cdot\right) = \frac{\partial^3}{\partial x^i \partial x^j \partial x^k}\psi$$

and

$$\rho\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}\,\middle|\,\cdot\right) = \frac{\partial^4}{\partial x^i \partial x^j \partial x^k \partial x^l}\psi$$

from Theorem 1, (2.7) and (2.8). Consequently the function δ is of order $O(\|x_1 - x_2\|^5)$. □

We discuss a deformation of a contrast function. Let a function $\Phi\colon [0, \infty) \to \mathbf{R}$ be monotone increasing such that $\Phi(0) = 0$ and $\Phi'(0) = 1$. As typical examples we can mention

$$\Phi_\alpha(t) = \frac{1}{\alpha}\log(1 + \alpha t), \quad \Psi_\alpha(t) = \frac{1}{\alpha}\tan(\alpha t)$$

or their inverse transformations, where α is a positive constant. Then $\rho_1(x, y) = \Phi(\rho(x, y))$ is also a contrast function. The geometric quantities $(g, \nabla, \nabla^*, D, D^*)$ and $(g_1, \nabla_1, \nabla_1^*, D_1, D_1^*)$ associated with ρ and ρ_1 are connected with

$$(g_1, \nabla_1, \nabla_1^*) = (g, \nabla, \nabla^*), \tag{2.9}$$

$$(D_1)_{X,Y}Z = D_{X,Y}Z + \Phi''(0)\,\mathfrak{S}g(X, Y)Z \tag{2.10}$$

and

$$(D_1^*)_{X,Y}Z = D_{X,Y}^*Z + \Phi''(0)\mathfrak{S}g(X, Y)Z.$$

In particular, the deformation of ρ keeps the equality of B with B^*.

Let $\mathscr{S}$ be a simplex of dimension n. As an alternative contrast function on $\mathscr{S}$ to ρ_1 defined in Introduction, we give

$$\rho_0(\boldsymbol{p}, \boldsymbol{q}) = 4\left(1 - \sum_{i=1}^{n+1} \sqrt{p_i q_i}\right)$$

for $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\mathscr{S}$. It follows from a straightforward calculus that ρ_0 and ρ_1 generate a common metric tensor, say g_0. By taking $\Phi(t) = (\cos^{-1}(1 - t/4))^2$, we know that $\Phi(\rho_0(\boldsymbol{p}, \boldsymbol{q}))$ is the squared arc-length of the geodesic curve connecting $\boldsymbol{p}$ and $\boldsymbol{q}$ with respect to g_0.

Let ρ be a contrast function on M such that ρ is C^∞-differentiable and generates a nontrivial metric tensor g. For every $\delta > 0$, $\rho^{(\delta)}(x, y) = \{\rho(x, y)\}^\delta$ is also a contrast function by definition. However if $\delta < 1$, then $\rho^{(\delta)}(x, y)$ is not differentiable at $x = y$. Alternatively if $\delta > 1$, then the metric tensor by $\rho^{(\delta)}$ is reduced to a zero tensor. Thus we see that if ρ yields a nontrivial metric tensor g, then any power change of ρ becomes nonsense. In effect $\rho(x, y)$ has the same order as the squared arc-length of the geodesic curve connecting x with y with respect to g, which will be shown in the following section.

3. Riemannian case

Let (M, g) be a Riemannian manifold and $\bar{\nabla}$ the Riemannian connection with respect to g. We denote the geodesic curve connecting x with y by $C = \{x_t\colon 1 \le t \le 1\}$, where $x_0 = x$ and $x_1 = y$. Define a contrast function by

$$\rho_0(x, y) = \frac{1}{2}\left(\int_C \sqrt{g_{x_t}(\dot{x}_t, \dot{x}_t)}\, dt\right)^2,$$

where $\dot{x}_t = dx_t/dt$. Since the tangent vectors $\dot{x}_t$'s are parallel to each other along the curve C,

$$\rho_0(x, y) = \frac{1}{2} g_{x_t}(\dot{x}_t, \dot{x}_t)$$

for any $t \in [0, 1]$, in particular $\rho_0(x, y) = g_x(\dot{x}_0, \dot{x}_0)/2$. We now investigate what geometry the function ρ_0 generates. Let $(g_0, \nabla_0, \nabla_0^*, D_0, D_0^*)$ be the geometric quantities associated with ρ_0 according to the formulation discussed in Section 2. The symmetry of ρ_0 yields $\nabla_0 = \nabla_0^*$ and $D_0 = D_0^*$ on M. Further, it will be seen that $g_0 = g$ and $\nabla_0 = \nabla_0^* = \bar{\nabla}$, where $\bar{\nabla}$ is the original Riemannian connection.

THEOREM 2. *$g = g_0$, $\nabla_0 = \nabla_0^* = \bar{\nabla}$ and*

$$(D_0)_{X,Y}Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \frac{1}{3}\{\bar{R}(X, Y)Z + \bar{R}(X, Z)Y\},$$

where $\bar{R}$ denotes the Riemannian curvature with respect to $\bar{\nabla}$.

PROOF. For a sufficiently small $\rho_0(x, y)$ there exists a local chart $(x^1, \dots, x^d, U, \varphi)$ of M such that $x \in U$ and $y \in U$. Then the curve $x_t = (x^1(t), \dots, x^d(t))$ satisfies

$$\frac{d^2}{dt^2} x^i(t) + \sum_{j,k} \Gamma^i_{jk}(x(t)) \frac{d}{dt} x^j(t) \frac{d}{dt} x^k(t) = 0 \tag{3.1}$$

with $(x^i(0)) = x$ and $(x^i(1)) = y$, where Γ^i_{jk}'s denote the Christoffel symbols.

We now express the vector $(dx^i(0)/dt)$ as a polynomial of $y - x$ up to the third order. From (3.1),

$$\begin{aligned}\frac{d^3}{dt^3} x^i(t) = \sum_{j,k,l} &\left(-\frac{\partial}{\partial x^l} \Gamma^i_{jk}(x(t)) + 2\sum_{\alpha} \Gamma^i_{j\alpha}(x(t)) \Gamma^{\alpha}_{kl}(x(t)) \right) \\ &\times \frac{d}{dt} x^j(t) \frac{d}{dt} x^k(t) \frac{d}{dt} x^l(t).\end{aligned}$$

A Taylor expansion leads to

$$\begin{aligned}x^i(t) &= x^i + \frac{d}{dt} x^i(0) t + \frac{d^2}{dt^2} x^i(0) \frac{t^2}{2} + \frac{d^3}{dt^3} x^i(0) \frac{t^3}{6} + O(t^4) \\ &= x^i + t\Delta^i - \frac{t^2}{2} \sum_{j,k} \Gamma^i_{jk}(x) \Delta^j \Delta^k \\ &\quad + \frac{t^3}{6} \sum_{j,k,l} \left(-\frac{\partial}{\partial x^l} \Gamma^i_{jk}(x) + 2\sum_{\alpha} \Gamma^i_{j\alpha}(x) \Gamma^{\alpha}_{kl}(x) \right) \Delta^j \Delta^k \Delta^l + O(t^4)\end{aligned}$$

where $\Delta^i = dx^i(0)/dt$. From $(x^i(1)) = y$, it follows that

$$\begin{aligned}\Delta^i &= (y^i - x^i) + \frac{1}{2} \sum_{j,k} \Gamma^i_{jk}(x)(y^j - x^j)(y^k - x^k) \\ &\quad + \frac{1}{6} \sum_{j,k,l} \left(\frac{\partial}{\partial x^l} \Gamma^i_{jk}(x) + \sum_{\alpha} \Gamma^i_{j\alpha}(x) \Gamma^{\alpha}_{kl}(x) \right)(y^j - x^j)(y^k - x^k)(y^l - x^l) \\ &\quad + O(\|y - x\|^4).\end{aligned} \tag{3.2}$$

Let X, Y, Z and W be vector fields on M. Define a mapping $(X, Y) \to X \cdot Y$ by

$$X \cdot Y = \sum_{i,j} X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

for $X = \sum X^i \partial/\partial x^i$ and $Y = \sum Y^j \partial/\partial x^j$. By definition

$$\bar{\nabla}_X Y = X \cdot Y + \Gamma(X, Y),$$

see Loos [5]. Further,

$$\begin{aligned}\bar{\nabla}_X \bar{\nabla}_Y Z = {} & X \cdot (Y \cdot \Gamma) + (X \cdot \Gamma)(Y, Z) + \Gamma(\Gamma(Y, Z), X) \\ & + \Gamma(X \cdot Y, Z) + \Gamma(X \cdot Z, Y) + \Gamma(Y \cdot Z, X)\end{aligned}$$

and the curvature tensor with respect to $\bar{\nabla}$ is expressed as

$$\begin{aligned}\bar{R}(X, Y)Z = {} & (X \cdot \Gamma)(Y, Z) - (Y \cdot \Gamma)(X, Z) \\ & + \Gamma(\Gamma(Y, Z), X) - \Gamma(\Gamma(X, Z), Y). \end{aligned} \tag{3.3}$$

Note that in the right-hand sides of the above equations each term depends on the local coordinate system, while all the left-hand side is coordinate-free. Writing $U = \sum_i (y^i - x^i)(\partial/\partial x^i)_x$, we can express $\dot{x}_0$ as

$$\dot{x}_0 = U + \frac{1}{2}\Gamma_X(U, U) + \frac{1}{6}\{(U \cdot \Gamma_X)(U, U) + \Gamma_X(\Gamma_X(U, U), U)\} + O(\|U\|^4) \tag{3.4}$$

by inverting the equation (3.2). The following relations are deduced from (3.4):

$$\begin{aligned} & (X_y \cdot \dot{x}_0)_* = X, \ (X_X \cdot \dot{x}_0)_* = -X, \ (\nabla_{X_x} \dot{x}_0)_* = -X, \\ & (\bar{\nabla}_{X_x}(Y_y \cdot \dot{x}_0))_* = 0, \ (X_y \cdot (Y_y \cdot \dot{x}_0))_* = \bar{\nabla}_X Y, \\ & (X_y \cdot (Y_y \cdot (Z_y \cdot \dot{x}_0)))_* = X \cdot (Y \cdot Z) + \Gamma(X \cdot Y, Z) + \Gamma(X \cdot Z, Y) + \Gamma(Y \cdot Z, X), \\ & \qquad + \frac{1}{3}\mathfrak{S}\{(X \cdot \Gamma)(Y, Z) + \Gamma(\Gamma(Y, Z), X)\} \end{aligned}$$

and

$$\begin{aligned}(\bar{\nabla}_{W_x} Y_y \cdot (Z_y \cdot \dot{x}_0))_* = {} & (W \cdot \Gamma)(Y, Z) + \Gamma(\Gamma(Y, Z), W) \\ & - \frac{1}{3}\mathfrak{S}\{W \cdot \Gamma)(Y, Z) + \Gamma(\Gamma(Y, Z), W)\},\end{aligned}$$

where $\mathfrak{S}$ denotes cyclic sum and

$$\begin{aligned} & ((X_1)_x \cdots (X_n)_x (Y_1)_y \cdots (Y_m)_y F(x, y))_* \\ & = ((X_1)_x \cdots (X_n)_x (Y_1)_y \cdots (Y_m)_y F(x, y))_{x=z, y=z}. \end{aligned}$$

Specifically we get

$$(X_y \cdot Y_y \cdot Z_y \cdot \dot{x}_0)_* = \bar{\nabla}_X \bar{\nabla}_Y Z - \frac{1}{3}\{\bar{R}(X, Y)Z + \bar{R}(X, Z)Y\},$$

and

$$(\bar{\nabla}_{W_x} Y_y \cdot Z_y \cdot \dot{x}_0)_* = \frac{1}{3}\{\bar{R}(W, Y)Z + \bar{R}(W, Z)Y\}$$

on account of (3.3).

On the basis of the relations established above, we get

$$g_0(X, Y) = -\rho_0(X|Y) = -(g(\dot{x}_0, \nabla_{X_x} Y_y \cdot \dot{x}_0) + g(\nabla_{X_x}\dot{x}_0, Y_y \cdot \dot{x}_0))_* = g(X, Y)$$

and

$$\begin{aligned} g_0(Z, (\nabla_0^*)_X Y) = -\rho_0(Z|XY) &= -(g(\dot{x}_0, \bar{\nabla}_{Z_x} X_y \cdot Y_y \cdot \dot{x}_0) + g(\bar{\nabla}_{Z_x}\dot{x}_0, X_y \cdot Y_y \cdot \dot{x}_0) \\ &\quad + g(Y_y \cdot \dot{x}_0, \bar{\nabla}_{Z_x} X_y \cdot \dot{x}_0) + g(\bar{\nabla}_{Z_x} Y_y \cdot \dot{x}_0, X_y \cdot \dot{x}_0))_* \\ &= g(Z, \bar{\nabla}_X Y). \end{aligned}$$

by the use of the expression $\rho_0(x, y) = g_x(\dot{x}_0, \dot{x}_0)/2$. In this way the metric g_0 is g and both ∇_0 and ∇_0^* are equal to $\bar{\nabla}$. Next we get

$$\begin{aligned} & g(W, D^*)_{X,Y} Z) = -\rho_0(W|XYZ) \\ &= -(g(\dot{x}_0, \bar{\nabla}_{W_x} X_y \cdot Y_y \cdot Z_y \cdot \dot{x}_0) + g(\bar{\nabla}_{W_x}\dot{x}_0, X_y \cdot Y_y \cdot Z_y \cdot \dot{x}_0) \\ &\quad + g(Z_y \cdot \dot{x}_0, \bar{\nabla}_{W_x} X_y \cdot Y_y \cdot \dot{x}_0) + g(\bar{\nabla}_{W_x} Z_y \cdot \dot{x}_0, X_y \cdot Y_y \cdot \dot{x}_0) \\ &\quad + g(Y_y \cdot \dot{x}_0, \bar{\nabla}_{W_x} X_y \cdot Z_y \cdot \dot{x}_0) + g(\bar{\nabla}_{W_x} Y_y \cdot \dot{x}_0, X_y \cdot Z_y \cdot \dot{x}_0) \\ &\quad + g(X_y \cdot \dot{x}_0, \bar{\nabla}_{W_x} Y_y \cdot Z_y \cdot \dot{x}_0) + g(\bar{\nabla}_{W_x} X_y \cdot \dot{x}_0, Y_y \cdot Z_y \cdot \dot{x}_0))_* \\ &= g(W, \bar{\nabla}_X \bar{\nabla}_Y Z) - \frac{1}{3} g(W, \bar{R}(X, Y)Z + \bar{R}(X, Z)Y) + \frac{1}{3} g(X, \bar{R}(W, Y)Z \\ &\quad + \bar{R}(W, Z)Y) + \frac{1}{3} g(Y, \bar{R}(W, X)Z + \bar{R}(W, Z)X) + \frac{1}{3} g(Z, \bar{R}(W, X)Y \\ &\quad + \bar{R}(W, Y)X). \end{aligned}$$

Consequently we obtain

$$(D_0)_{X,Y} Z = (D_0^*)_{X,Y} Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \frac{1}{3}\{\bar{R}(X, Y)Z + \bar{R}(X, Z)Y\},$$

noting $g(W, \bar{R}(X, Y)Z) + g(\bar{R}(X, Y)W, Z) = 0$ and $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + R(Z, X)Y = 0$. □

Let $\bar{B}(X, Y) = (D_0)_{X,Y} - \nabla_X \nabla_Y$. Then the Bianchi's first identity leads to

$$\{\bar{B}(X, Y) + \bar{B}(Y, X)\}Z = -\bar{B}(Z, X)Y.$$

Further it is easily seen from Proposition 3 that M is $\bar{R}$-free if and only if M is also $\bar{B}$-free.

4. Minimum contrast leaf

As discussed in Section 2, a contrast function ρ on M generates a metric tensor g and differential operators ∇, ∇^*, D and D^*, where B-conjugacy is established in addition to R-conjugacy. Let $\tilde{M}$ be a k-dimensional submanifold of M with the immersion f of $\tilde{M}$ in M. By restricting the domain of ρ as $\tilde{\rho} = \rho|_{\tilde{M}\times\tilde{M}}$, the quantities $(g, \nabla, \nabla, D, D^*)$ induce $(\tilde{g}, \tilde{\nabla}, \tilde{\nabla}, \tilde{D}, \tilde{D}^*)$ over $\tilde{M}$. For example,

$$\tilde{g}(U, V) = -\tilde{\rho}(U \mid V)$$

for U and V of $\mathfrak{X}(\tilde{M})$. Of course by definition $\tilde{g}(U, V) = g(f_* U, f_* V)$. Henceforth we identify U with $f_* U$, so that $\tilde{g}(U, V) = g(U, V)$. Let N_f be the normal bundle of $\tilde{M}$ and $Sec(N_f)$ the space of sections of $\tilde{M}$ into N_f, or the space of normal vector fields. We define a mapping α: $\mathfrak{X}(\tilde{M}) \times \mathfrak{X}(\tilde{M}) \rightarrow Sec(N_f)$ by

$$g(\alpha(U, V), \xi) = -\rho(UV \mid \xi)$$

for all ξ of $Sec(N_f)$. Then α is the second-fundamental tensor with respect to ∇ because α is bilinear and it is decomposed that

$$\nabla_U V = \tilde{\nabla}_U V + \alpha(U, V).$$

Alternatively with respect to ∇^*, the tensor α^* is similarly defined and hence

$$\nabla_U^* V = \tilde{\nabla}_U^* V + \alpha^*(U, V).$$

Next for a fixed ξ of $Sec(N_f)$ the shape operator A_ξ with respect to ∇ and the conjugate A_ξ^* are given by

$$\tilde{g}(A_\xi U, V) = -\rho(U\xi \mid V) \quad \text{and} \quad \tilde{g}(V, A_\xi^* U) = -\rho(V \mid U\xi).$$

Note that

$$\nabla_U \xi = -A_\xi U + \nabla_U^\perp \xi \quad \text{and} \quad \nabla_U^* \xi = -A_\xi^* U + \nabla_U^{*\perp} \xi.$$

Thus (α, α^*) and (A_ξ, A_ξ^*) are related to each other as follows:

Proposition 5. *$\tilde{g}(A_\xi^* U, V) + g(\alpha(U, V), \xi) = 0$ and*

$$\tilde{g}(A_\xi U, V) + g(\alpha^*(U, V), \xi) = 0.$$

PROOF. By definition,

$$Ug(V, \xi)=0 \quad \text{and} \quad Ug(\xi, V)=0$$

or

$$-\rho(UV|\xi)-\rho(V|U\xi)=0, \quad \text{and} \quad -\rho(\xi|UV)-\rho(U\xi|V)=0,$$

which conclude the two identities. □

We define a mapping $\beta: \mathfrak{X}(M)\times\mathfrak{X}(M)\times \mathrm{X}(M)\to Sec(N_f)$ by

$$\beta(U, V, W)=\beta_1(U, V, W)-\nabla_U^{\perp}\alpha(V, W)-\nabla_V^{\perp}\alpha(U, W),$$

where β_1 is defined to satisfy

$$g(\beta_1(U, V, W), \xi)=-\rho(UVW|\xi)$$

for any $\xi\in Sec(N_f)$. It should be noted that β is a tensor field and

$$D_{U,V}W=\tilde{D}_{U,V}W+\beta_1(U, V, W).$$

We call β the third fundamental tensor with respect to D. The conjugate counterpart is written by β^*.

PROPOSITION 6. *Assume that M is B-free. Then we have that*

$$\beta(U, V, W)=\alpha(U, \tilde{\nabla}_V W)-\nabla_V^{\perp}\alpha(U, W)$$

and

$$\beta^*(U, V, W)=\alpha^*(U, \tilde{\nabla}_V^* W)-\nabla_V^{*\perp}\alpha^*(U, W).$$

PROOF. From the assumption it follows that

$$\begin{aligned} g(\beta(U, V, W), \xi)&=g(\nabla_U\nabla_V f_* W, \xi)-g(\nabla_U^{\perp}\alpha(V, W)+\nabla_V^{\perp}\alpha(U, W), \xi)\\ &=g(\nabla_U(\nabla_V W+\alpha(V, W)), \xi)-g(\nabla_U^{\perp}\alpha(V, W)+\nabla_V^{\perp}\alpha(U, W), \xi)\\ &=g(\alpha(U, \tilde{\nabla}_V W)-\nabla_V^{\perp}\alpha(U, W), \xi)\end{aligned}$$

for all ξ of $Sec(N_f)$. This shows the first relation. From Theorem 1, M is also B^*-free, which leads to the second relation by a similar argument as above. The proof is complete. □

Hereafter we assume that for any point x of M there exists a unique point u of $\tilde{M}$ such that u minimizes $\rho(x, v)$ in $v\in\tilde{M}$. Then to each point u of $\tilde{M}$ it can be defined that

$$L_u=\{x\in M: \rho(x, u)=\min_{v\in\tilde{M}}\rho(x, v)\},$$

which we call the minimum contrast leaf at u. By the above assumption L_u

is a submanifold of codimension k transversing to $\tilde{M}$ at u. Thus M is decomposed into a foliation $M = \cup \{L_u : u \in \tilde{M}\}$ and

$$T_u(M) = T_u(\tilde{M}) \oplus T_u(L_u).$$

Now let u be fixed. From the above assumption it follows that

$$U_u \rho(x, u) = 0$$

for all U of $\mathfrak{X}(\tilde{M})$ and x of L_u. Thus we have that $g_u(\xi, U) = 0$ for all ξ of $\mathfrak{X}(L_u)$ and U of $\mathfrak{X}(\tilde{M})$, or equivalently that the tangent space of L_u at $f(u)$ is equal to the normal space of $\tilde{M}$ at u. Further,

$$\rho(\xi_1 \cdots \xi_k | U)(u) = 0 \tag{4.1}$$

for any $k \geq 2$. Hence the second fundamental tensor γ of L_u is defined by the condition

$$g(\gamma(\xi, \zeta), \tilde{U}) = -\rho(\xi\zeta | \tilde{U})$$

for all $\tilde{U}$ of $Sec(N(L_u))$. Next the third fundamental tensor δ of L_u is given by

$$\delta(\xi, \zeta, \eta) = \delta_1(\xi, \zeta, \eta) - \nabla_\xi^\perp \gamma(\zeta, \eta) - \nabla_\zeta^\perp \gamma(\xi, \eta)$$

PROPOSITION 6. *Let L_u be a minimum contrast leaf through u of a subspace $\tilde{M}$. Then the tensors γ and δ for L_u, defined as above, vanish at u.*

PROOF. The result follows from (4.1) with $k = 2, 3$.

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