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Note on singular semilinear elliptic equations

Yasuhiro FURUSHO

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1. Introduction

In this note we study the existence of positive entire solutions for the singular semilinear elliptic equation

$$-\Delta u + c(x)u = p(x)u^{-\gamma}, \qquad x \in \boldsymbol{R}^N,\ N \geq 3,\ \gamma > 0 \tag{1}$$

under the the hypothesis

(H) c and p are locally Hölder continuous functions in $\boldsymbol{R}^N$ with exponent θ, $0 < \theta < 1$, and $c(x) \geq 0$ in $\boldsymbol{R}^N$.

An entire solution of (1) is defined to be a function $u \in C^{2+\theta}_{\mathrm{loc}}(\boldsymbol{R}^N)$ satisfying (1) pointwise in $\boldsymbol{R}^N$.

For the equation (1) with $c(x) \equiv 0$, i.e.,

$$-\Delta u = p(x)u^{-\gamma}, \qquad x \in \boldsymbol{R}^N,\ N \geq 3, \tag{2}$$

Kusano and Swanson [9] proved the existence of a positive entire solution u such that $|x|^{N-2}u(x)$ is bounded above and below as $|x| \to \infty$ under the assumptions that $0 < \gamma < 1$, $p(x) > 0$ in $\boldsymbol{R}^N$ and

$$\int^{\infty} t^{N-1+\gamma(N-2)} p^*(t)\, dt < \infty, \tag{3}$$

where $p^*(t) = \max_{|x|=t} p(x)$. This result was extented afterwards by Dalmasso [2] to cover the case $\gamma \geq 1$.

On the other hand, for the equation (1) with negative γ, it is known that if $-1 < \gamma < 0$, and $p(x)$ satisfies $p(x) > 0$, $\not\equiv 0$ in $\boldsymbol{R}^N$ and

$$\int^{\infty} t p^*(t)\, dt < \infty, \tag{4}$$

then there exists a positive entire solution decaying to 0 at infinity (see e.g. [4], [6], [7] and [10]). However, as far as we are aware, no such result is obtained for the singular type equation (1) under the condition (4).

Our first result, Theorem 1 below, concerns the existence of positive entire solutions of (1) which have uniform positive limits at infnity. In Theorem 2, we show that there exists a decaying entire solution of (1) under the condition

(4). Finally, Theorem 3 gives an extension of the results of Kusano and Swanson [9] and Dalmasso [2] stated above. Our proof of Theorem 3 is simpler than that of Dalmasso [2].

For other closely related papers to this note we refer to the papers [3–5, 8, 11]. Among them, Fukagai [5] has studied the existence and asymptotic behavior at infinity of positive entire solutions of (1) with $c(x) \equiv m^2 > 0$, where m is a constant.

2. Statement of theorems

THEOREM 1. *Assume that* (H) *holds. If*

$$\int^{\infty} tc^*(t)\,dt < \infty \quad \text{and} \quad \int^{\infty} tp^*(t)\,dt < \infty, \tag{5}$$

where $c^*(t) = \max_{|x|=t} c(x)$ *and* $p^*(t) = \max_{|x|=t} |p(x)|$, *then there exist infinitely many positive entire solutions* u *of* (1) *such that*

$$\lim_{|x|\to\infty} u(x) = \xi \tag{6}$$

for some constants $\xi > 0$.

THEOREM 2. *Assume that* (H) *holds and that* $p(x) \geq 0$, $\not\equiv 0$, *in* $\boldsymbol{R}^N$. *Then, condition* (4) *is sufficient for* (1) *to have a positive entire solution* u *tending uniformly to* 0 *as* $|x| \to \infty$.

THEOREM 3. *Assume that* (H) *holds* (*with* $c(x) \equiv 0$) *and* $p(x) \geq 0$, $\not\equiv 0$ *in* $\boldsymbol{R}^N$. *Then, condition* (3) *is sufficient for* (2) *to have a positive entire solution* u *such that*

$$k^{-1}|x|^{2-N} \leq u(x) \leq k|x|^{2-N}, \qquad |x| \geq 1, \tag{7}$$

for some constant $k > 1$.

REMARK 1. Theorem 3 has been proved by Kusano and Swanson [9] and Dalmasso [2] under the condition that $k_0 p^*(|x|) \leq p(x) \leq p^*(|x|)$ in $\boldsymbol{R}^N$ for some $0 < k_0 \leq 1$. We note that this condition is not assumed in Theorem 3.

3. Proof of theorems

The proofs of Theorems 1–3 are based on the following supersolution-subsolution method by Akô and Kusano [1].

THEOREM 0. *Assume that* (H) *holds. If there exist functions* V *and* $W \in C^{\theta}_{\mathrm{loc}}(\boldsymbol{R}^N)$ *such that*

$$-\Delta V(x) + c(x)V(x) \le p(x)V(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N, \tag{8}$$

$$-\Delta W(x) + c(x)W(x) \ge p(x)W(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N, \tag{9}$$

$$0 < V(x) \le W(x), \qquad x \in \boldsymbol{R}^N, \tag{10}$$

then (1) *has an entire solution* u *satisfying* $V(x) \le u(x) \le W(x)$ *in* $\boldsymbol{R}^N$.

A function V(resp. W) satisfying (8) (resp. (9)) is called a subsolution (resp. a supersolution) of (1).

PROOF OF THEOREM 1. Consider the linear elliptic equations

$$-\Delta v + c(x)v = -|p(x)|, \qquad x \in \boldsymbol{R}^N, \tag{11}$$

$$-\Delta w + c(x)w = |p(x)|, \qquad x \in \boldsymbol{R}^N. \tag{12}$$

By (5) and [7; Theorem 2.2] there exist positive solutions v and w in $C^{2+\theta}_{\mathrm{loc}}(\boldsymbol{R}^N)$ of (11) and (12), respectively, such that

$$\lim_{|x| \to \infty} v(x) = \lim_{|x| \to \infty} w(x) = \tilde{\xi} \tag{13}$$

for some constant $\tilde{\xi} > 0$. Furthermore, the maximum principle combined with (11)–(13) implies the relation

$$0 < v(x) \le w(x), \qquad x \in \boldsymbol{R}^N. \tag{14}$$

For any fixed $\kappa \ge (\inf_{x \in \boldsymbol{R}^N} v(x))^{-\gamma/(1+\gamma)}$, put $V(x) = \kappa v(x)$ and $W(x) = \kappa w(x)$ for $x \in \boldsymbol{R}^N$. Then, the functions V and W are a subsolution and a supersolution of (1), respectively, and satisfy (10). In fact,

$$\begin{aligned} -\Delta V(x) + c(x)V(x) &= -\kappa|p(x)| \\ &= -\kappa V(x)^{\gamma}|p(x)|V(x)^{-\gamma} \le -|p(x)|V(x)^{-\gamma} \\ &\le p(x)V(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N. \end{aligned}$$

A similar argument holds for W. Therefore, the existence of a solution u of (1) lying between V and W follows from Theorem 0. Furthermore, by (13) $u(x)$ tends to $\kappa\tilde{\xi}$ as $|x| \to \infty$. Since κ can be taken arbitrarily as above, equation (1) has an infinitude of entire solutions satisfying (6). This completes the proof.

PROOF OF THEOREM 2. Take a positive function $\tilde{p}^* \in C^{\theta}_{\mathrm{loc}}[0, \infty)$ such that

(15) $$\tilde{p}^*(t) \geq p^*(t),\ t > 0, \quad \text{and} \quad \int^{\infty} t\tilde{p}^*(t)\,dt < \infty.$$

Suggested by the proof of Theorem 10 in Fukagai [5], we define a function y by

(16) $$y(t) = \left(\frac{N-2}{1+\gamma}\int_t^{\infty} t\tilde{p}^*(t)\,dt\right)^{1/(1+\gamma)}, \qquad t \geq 0.$$

Then, y satisfies $y(t) > 0$ for $t \geq 0$, $\lim_{t\to\infty} y(t) = 0$ and

(17) $$y'(t) = -\frac{1}{N-2}\, t\tilde{p}^*(t)y(t)^{-\gamma}, \qquad t > 0,$$

where $' = d/dt$. Integrating (17) from t to ∞, we obtain

(18) $$y(t) = \frac{1}{N-2}\int_t^{\infty} s\tilde{p}^*(s)y(s)^{-\gamma}\,ds, \qquad t \geq 0.$$

Using this y, we define a function z by

(19)
$$\begin{cases} z(t) = y(0) & \text{for } t = 0, \\ z(t) = \dfrac{t^{2-N}}{N-2}\displaystyle\int_0^t s^{N-1}\tilde{p}^*(s)y(s)^{-\gamma}\,ds + \dfrac{1}{N-2}\displaystyle\int_t^{\infty} s\tilde{p}^*(s)y(s)^{-\gamma}\,ds & \text{for } t > 0. \end{cases}$$

Then, z is a solution of the boundary value problem

(20) $$z''(t) + \frac{N-1}{t}z'(t) = -\tilde{p}^*(t)y(t)^{-\gamma}, \qquad t > 0,$$

(21) $$z'(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} z(t) = 0.$$

The relation (20) and $z'(0) = 0$ follow from (19). Integrating the first term in (19) by parts and using (17), we obtain

$$z(t) = (N-2)t^{2-N}\int_0^t s^{N-3}y(s)\,ds, \qquad t > 0$$

which implies that $\lim_{t\to\infty} z(t) = 0$. The relation (20) means that the function $W(x) = z(|x|)$ satisfies

$$-\Delta W(x) = \tilde{p}^*(|x|)y(|x|)^{-\gamma}, \qquad x \in \boldsymbol{R}^N.$$

Since $c(x) \geq 0$ and $z(t) \geq y(t)$ by (18) and (19), we see that

$$(22)\qquad -\Delta W(x) + c(x)W(x) \geq \tilde{p}^*(|x|)\,W(x)^{-\gamma} \geq p(x)\,W(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N,$$

which means that W is a supersolition of (1).

To construct a subsolution, we take a function $v \in C^{2+\theta}_{\text{loc}}(\boldsymbol{R}^N)$ such that

$$(23)\qquad \begin{cases} -\Delta v(x) + c(x)v(x) = p(x), & x \in \boldsymbol{R}^N, \\ v(x) > 0,\ x \in \boldsymbol{R}^N \text{ and } \lim_{|x| \to \infty} v(x) = 0. \end{cases}$$

The existence of such a v is guaranteed by (4) and [7; Theorem 2.2]. Let

$$\kappa = \min\{(\sup_{x \in \boldsymbol{R}^N} v(x))^{-\gamma/(1+\gamma)},\ (\sup_{x \in \boldsymbol{R}^N} W(x))^{-\gamma}\},$$

and define $V(x) = \kappa v(x)$ for $x \in \boldsymbol{R}^N$. Then, V is a subsolution of (1), since

$$(24)\quad -\Delta V(x) + c(x)V(x) = \kappa p(x) = \kappa p(x)\,V(x)^{-\gamma}\,V(x)^{\gamma} \leq p(x)\,V(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N.$$

Furthermore, from (22), (24) and the inequalities

$$\kappa p(x) \leq (\sup_{x \in \boldsymbol{R}^N} W(x))^{-\gamma} p(x) \leq p(x)\,W(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N,$$

it follows that

$$\begin{cases} -\Delta(W(x) - V(x)) + c(x)(W(x) - V(x)) \geq 0, & x \in \boldsymbol{R}^N, \\ \lim_{|x| \to \infty} (W(x) - V(x)) = 0. \end{cases}$$

Applying the maximum principle to $W - V$, we see that $V(x) \leq W(x)$ in $\boldsymbol{R}^N$. Consequently, the assertion of Theorem 2 follows from Theorem 0. This completes the proof.

PROOF OF THEOREM 3. Since $\int^\infty t^{N-1} p^*(t)\,dt < \infty$ by (3), from [6; Corollary 3.1] there exists a unique positive function $v \in C^{2+\theta}_{\text{loc}}(\boldsymbol{R}^N)$ such that

$$(25)\qquad \begin{cases} -\Delta v(x) = p(x), & x \in \boldsymbol{R}^N \\ k_1^{-1}|x|^{2-N} \leq v(x) \leq k_1|x|^{2-N}, & |x| \geq 1, \text{ for some } k_1 > 1. \end{cases}$$

Then, as in the proof of Theorem 2, the function $V(x) = \kappa v(x)$ with $\kappa = (\sup_{x \in \boldsymbol{R}^N} v(x))^{-\gamma/(1+\gamma)}$ is a subsolution of (2).

To construct a supersolution of (2), let us solve the linear equation

$$(26)\qquad -\Delta w = p(x)\,V(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N.$$

By (3) and (25), the function $g^*(t) = \max_{|x|=t}\{p(x)\,V(x)^{-\gamma}\}$ satisfies

$$\int^\infty t^{N-1} g^*(t)\,dt \leq \text{constant} \int^\infty t^{N-1+\gamma(N-2)} p^*(t)\,dt < \infty.$$

Hence, by [6; Corollary 3.1] there exists a positive solution $w \in C_{\mathrm{loc}}^{2+\theta}(\boldsymbol{R}^N)$ of (26) such that

$$k_2^{-1}|x|^{2-N} \leq w(x) \leq k_2|x|^{2-N}, \qquad |x| \geq 1 \tag{27}$$

for some $k_2 > 1$. Choosing a constant $\mu \geq 1$ large enough so that $\mu w(x) \geq V(x)$ in $\boldsymbol{R}^N$, we see that the function $W = \mu w$ satisfies

$$-\Delta W(x) = \mu p(x) V(x)^{-\gamma} \geq p(x) V(x)^{-\gamma} \geq p(x) W(x)^{-\gamma}, \qquad x \in \boldsymbol{R}^N. \tag{28}$$

Therefore, by Theorem 0 there exists a solution u of (2) such that $0 < V(x) \leq u(x) \leq W(x)$ in $\boldsymbol{R}^N$. This solution obviously satisfies the relation (7) by (25) and (27). Thus the proof is completed.

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Department of Mathematics,
Faculty of Science and Engineering,
Saga University