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Component-wise convergence of quasilinearization method for nonlinear boundary value problems

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1. Introduction

The boundary value problem

$$x' = f(t, x), \quad t \in J = [a, b] \tag{1.1}$$

$$F[x] = 0, \tag{1.2}$$

where x and $f(t, x)$ are n dimensional vectors and F is an operator from $C(J)$ into R^n, $C(J)$ is the space of all real n vector functions continuous on J, has been the subject of many recent investigations [1, 2, 5, 8, 13–15, 20]. These, in particular include the existence, uniqueness of the solutions and the convergence of the Picard's and Approximate Picard's iterative schemes. The purpose of this paper is to provide a priori sufficient conditions so that the Quasilinear and Approximate Quasilinear iterative methods for (1.1), (1.2) converge to its unique solution. Since the introduction of the term Quasilinearization by Bellman and Kalaba [7] in the year 1965, also see [4, 6, 9], quasilinear methods have been extensively used to construct the solutions of nonlinear boundary value problems. Therefore, the obtained results in this paper are of theoretical as well as of computational importance. To make the analysis widely applicable all the results are proved component-wise. The significance of such a study for systems is now well recognized from the fact that it enlarges the domain of existence and uniqueness of solutions, weakens the convergence conditions and provides sharper error estimates, e.g., see [1–3, 5, 10–12, 16–20].

2. Notations and Assumptions

Throughout, we shall consider the inequalities between two vectors in R^n component-wise whereas between two $n \times n$ matrices element-wise. The following well known properties of matrices will be used frequently without further mention.

1. For any square matrix A, $\lim_{m \to \infty} A^m = 0$ if and only if $\rho(A) < 1$, where $\rho(A)$

denotes the spectral radius of A.

2. For any square matrix A, $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{m=0}^{\infty} A^m$ if $\rho(A) < 1$, where I denotes the unit matrix. Also, if $A \geq 0$, then $(I - A)^{-1}$ exists and is nonnegative if and only if $\rho(A) < 1$.
3. If $0 \leq B \leq A$ and $\rho(A) < 1$, then $\rho(B) < 1$.
4. If $A \geq 0$ then $\rho(3A) = 3\rho(A) < 1$ if and only if $\rho(2A(I - A)^{-1}) < 1$.
5. (Toeplitz Lemma). For a given square matrix $A \geq 0$ with $\rho(A) < 1$ and a sequence of vectors $\{d^m\}$, we define the sequence $\{s^m\}$, where $s^m = \sum_{i=0}^{m} A^{m-i} d^i$; $m = 0, 1, \ldots$. Then, $\lim_{m \to \infty} s^m = 0$ if and only if $d^m \to 0$.

For $x(t) = (x_1(t), \ldots, x_n(t)) \in C(J)$ we shall denote by

$$|x(t)| = (|x_1(t)|, \ldots, |x_n(t)|) \quad \text{and} \quad \|x\| = (\sup_{t \in J} |x_1(t)|, \ldots, \sup_{t \in J} |x_n(t)|).$$

The space $C(J)$ equiped with this norm is a 'generalized normed space' [6]. If $x \in R^n$, then obviously $x \in C(J)$ is a constant function, and hence $|x| = \|x\| = (|x_1|, \ldots, |x_n|)$. The same notations will be used for the $n \times n$ matrix valued functions.

Consider the linear differential system

$$x' = A(t)x + \phi(t), \qquad t \in J \tag{2.1}$$

together with

$$L[x] = \ell, \tag{2.2}$$

where $A(t)$ is an $n \times n$ continuous matrix on J, $\phi(t)$ is an $n \times 1$ continuous vector on J, L is a linear operator mapping $C(J)$ into R^n, i.e., $\ell \in R^n$.

In what follows, we shall denote by $Y(t)$ as the fundamental matrix solution of the homogeneous system

$$y' = A(t)y \tag{2.3}$$

such that $Y(a) = I$. Let $G = L[Y(t)]$ represent the $n \times n$ matrix whose column vectors are $L[y^i(t)]$, $1 \leq i \leq n$ where $y^i(t)$ is the i-th column vector of $Y(t)$.

LEMMA 2.1. [15] If the matrix G is nonsingular then the boundary value problem (2.1), (2.2) has a unique solution $x(t)$ and it can be represented as

$$x(t) = H^1[\phi(t)] + H^2[\ell], \tag{2.4}$$

where H^1 is the linear operator mapping $C(J)$ into $C^{(1)}(J)$ such that

$$H^1[\phi(t)] = Y(t)\int_a^t Y^{-1}(s)\phi(s)\,ds - Y(t)G^{-1}L\left[Y(t)\int_a^t Y^{-1}(s)\phi(s)\,ds\right]$$

and H^2 is the linear operator mapping R^n into $C^{(1)}(J)$ such that

$$H^2[\ell] = Y(t)G^{-1}\ell. \tag{2.5}$$

DEFINITION 2.1. A function $\bar{x}(t) \in C^{(1)}(J)$ is called an approximate solution of (1.1), (1.2) if there exist δ^1 and δ^2 nonnegative vectors such that $\|\bar{x}'(t) - f(t, \bar{x}(t))\| \le \delta^1$ and $\|F[\bar{x}]\| \le \delta^2$, i.e., there exist a function $\eta(t)$ and a constant vector ℓ^1 such that $\bar{x}'(t) = f(t, \bar{x}(t)) + \eta(t)$ and $F[\bar{x}] = \ell^1$ with $|\eta(t)| \le \delta^1$ and $|\ell^1| \le \delta^2$.

With respect to the boundary value problem (1.1), (1.2) each result we shall prove will require some of the following assumptions.

(c_1) There exists an approximate solution $\bar{x}(t)$.

(c_2) There exists an $n \times n$ continuous matrix $A(t)$, $t \in J$ and a linear operator L mapping $C(J)$ into R^n such that if $Y(t)$ is the fundamental matrix solution of (2.3), then $G = L[Y(t)]$ is nonsingular.

Since the problem (1.1), (1.2) is the same as

$$x' = A(t)x + f(t, x) - A(t)x$$
$$L[x] = L[x] - F[x]$$

if the condition (c_2) is satisfied then from Lemma 2.1 it follows that (1.1), (1.2) is equivalent to the equation

$$x(t) = H^1[f(t, x(t)) - A(t)x(t)] + H^2[L[x] - F[x]]. \tag{2.6}$$

Similarly, if (c_1) and (c_2) are satisfied then

$$\bar{x}(t) = H^1[f(t, \bar{x}(t)) + \eta(t) - A(t)\bar{x}(t)] + H^2[L[\bar{x}] - F[\bar{x}] + \ell^1]. \tag{2.7}$$

(c_3) There exist nonnegative matrices M^1 and M^2 such that $\|H^1\| \le M^1$ and $\|H^2\| \le M^2$.

(c_4) The function $f(t, x)$ is continuously differentiable with respect to x in $J \times R^n$ and $f_x(t, x)$ represents the Jacobian matrix of $f(t, x)$ with respect to x; $F[x]$ is continuously Fréchet differentiable in $C(J)$ and $F_x[x]$ denotes the Fréchet derivative of $F[x]$, which is a linear operator maps $C(J)$ into R^n.

(c_5) There exist nonnegative matrices M^3 and M^4, and a positive vector r such that for all $t \in J$ and $x \in \bar{S}(\bar{x}, r) = \{x(t) \in C(J): \|x - \bar{x}\| \le r\}$, $\|f_x(t, x) - A(t)\| \le M^3$ and $\|F_x[x] - L\| \le M^4$.

(c_6) For all $t \in J$ and $x, y \in \bar{S}(\bar{x}, r)$

$$\|f(t, x) - f(t, y) - f_x(t, y)(x - y)\| \le P \cdot \|x - y\| \cdot \|x - y\|,$$

where $P = (p_{ij\ell})$ is a symmetric tensor of the third order with nonnegative components. (Obviously, if f is twice continuously differentiable with respect to x for all $(t, x) \in J \times \bar{S}(\bar{x}, r)$ and all the second derivatives $\dfrac{\partial^2 f_i}{\partial x_j \partial x_\ell}$ are bounded there, then this condition is satisfied, with

$$P_{ij\ell} = \frac{1}{2} \sup_{(t,x) \in J \times S(\bar{x}, r)} \left| \frac{\partial^2 f_i}{\partial x_j \partial x_\ell} \right|.$$

(c$_7$) For all $x, y \in \bar{S}(\bar{x}, r)$

$$\| F[x] - F[y] - F_x[y][x - y] \| \le Q \cdot \| x - y \| \cdot \| x - y \|,$$

where $Q = (q_{ij\ell})$ is a symmetric tensor of the third order with nonnegative components.

3. Quasilinearization method

For the boundary value problem (1.1), (1.2) quasilinear method leads to the construction of the sequence $\{x^m(t)\}$ generated by the iterative scheme

$$\begin{aligned} &(x^{m+1})'(t) = f(t, x^m(t)) + f_x(t, x^m(t))(x^{m+1}(t) - x^m(t)) \\ &F[x^m] + F_x[x^m][x^{m+1} - x^m] = 0;\ m = 0, 1, \ldots, \end{aligned} \tag{3.1}$$

where $x^0(t) = \bar{x}(t)$. In the following result we shall provide sufficient conditions so that this sequence $\{x^m(t)\}$ indeed exists and converges to the unique solution of (1.1), (1.2).

THEOREM 3.1. With respect to the boundary value problem (1.1), (1.2) we assume that the conditions (c$_1$)–(c$_5$) are satisfied. Further, let $\rho(3K) < 1$, where $K = M^1 M^3 + M^2 M^4$ and $r^1 = (I - 3K)^{-1}(M^1 \delta^1 + M^2 \delta^2) \le r$. Then, the following hold

1. the sequence $\{x^m(t)\}$ obtained by (3.1) remains in $\bar{S}(\bar{x}, r^1)$
2. the sequence $\{x^m(t)\}$ converges to the unique solution $x^*(t)$ of (1.1), (1.2)
3. a bound on the error involving the matrix $K^* = 2K(I - K)^{-1}$ is given by

$$\| x^m - x^* \| \le (K^*)^m (I - K^*)^{-1} \| x^1 - x^0 \| \tag{3.2}$$

$$\le (K^*)^m (I - 3K)^{-1} (M^1 \delta^1 + M^2 \delta^2). \tag{3.3}$$

PROOF. First we shall show that $\{x^m(t)\} \subseteq \bar{S}(\bar{x}, r^1)$. For this, on $C(J)$ we define an implicit operator T as follows

$$\begin{aligned} Tx(t) = {} & H^1[f(t, x(t)) + f_x(t, x(t))(Tx(t) - x(t)) - A(t) Tx(t)] \\ & + H^2[L[Tx] - F[x] - F_x[x][Tx - x]] \end{aligned} \tag{3.4}$$

whose form is patterned similar to that of (2.6) for (3.1).

Since $\bar{x}(t)\in\bar{S}(\bar{x}, r^1)$, it suffices to show that if $x(t)\in\bar{S}(\bar{x}, r^1)$, then $Tx(t)\in\bar{S}(\bar{x}, r^1)$. For this, let $x(t)\in\bar{S}(\bar{x}, r^1)$, then from (2.7) and (3.4), we have

$$\begin{aligned}
Tx(t)-\bar{x}(t) &= H^1[f(t, x(t))+f_x(t, x(t))(Tx(t)-x(t))-A(t)Tx(t) \\
&\quad -f(t, \bar{x}(t))-\eta(t)+A(t)\bar{x}(t)] \\
&\quad +H^2[L[Tx]-F[x]-F_x[x][Tx-x] \\
&\quad -L[\bar{x}]+F[\bar{x}]-\ell^1] \\
&= H^1[f(t, x(t))-f(t, \bar{x}(t))-A(t)(x(t)-\bar{x}(t)) \\
&\quad +(f_x(t, x(t))-A(t))(Tx(t)-x(t))-\eta(t)] \\
&\quad +H^2[-(F[x]-F[\bar{x}]-L[x-\bar{x}]) \\
&\quad -(F_x[x]-L)[Tx-x]-\ell^1] \\
&= H^1\left[\int_0^1[f_x(t, \bar{x}(t)+\theta_1(x(t)-\bar{x}(t)))-A(t)](x(t)-\bar{x}(t))\,d\theta_1\right. \\
&\quad \left.+(f_x(t, x(t))-A(t))(Tx(t)-x(t))-\eta(t)\right] \\
&\quad +H^2\left[-\int_0^1[F_x[\bar{x}+\theta_2(x-\bar{x})]-L][x-\bar{x}]\,d\theta_2\right. \\
&\quad \left.-(F_x[x]-L)[Tx-x]-\ell^1\right]
\end{aligned} \tag{3.5}$$

and hence it follows that

$$\begin{aligned}
\|Tx-\bar{x}\| &\le M^1[M^3\|x-\bar{x}\|+M^3\|Tx-x\|+\delta^1] \\
&\quad +M^2[M^4\|x-\bar{x}\|+M^4\|Tx-x\|+\delta^2] \\
&\le M^1[2M^3\|x-\bar{x}\|+M^3\|Tx-\bar{x}\|+\delta^1] \\
&\quad +M^2[2M^4\|x-\bar{x}\|+M^4\|Tx-\bar{x}\|+\delta^2] \\
&= 2K\|x-\bar{x}\|+K\|Tx-\bar{x}\|+(M^1\delta^1+M^2\delta^2),
\end{aligned}$$

which is the same as

$$\|Tx-\bar{x}\|\le 2K(I-K)^{-1}\|x-\bar{x}\|+(I-K)^{-1}(M^1\delta^1+M^2\delta^2).$$

Thus, we find that

$$\begin{aligned}
\|Tx-\bar{x}\| &\le 2K(I-K)^{-1}(I-3K)^{-1}(M^1\delta^1+M^2\delta^2) \\
&\quad +(I-K)^{-1}(M^1\delta^1+M^2\delta^2)
\end{aligned}$$

$$= (I - K)^{-1}[2K(I - 3K)^{-1} + I](M^1\delta^1 + M^2\delta^2)$$
$$= (I - K)^{-1}(I - K)(I - 3K)^{-1}(M^1\delta^1 + M^2\delta^2)$$
$$= (I - 3K)^{-1}(M^1\delta^1 + M^2\delta^2).$$

Therefore, $\| Tx - \bar{x} \| \leq r^1$ follows from the definition of r^1.

Next, we shall show the convergence of the sequence $\{x^m(t)\}$. From (3.1), in view of (3.4), we have

$$\begin{aligned} x^{m+1}(t) - x^m(t) &= H^1[f(t, x^m(t)) + f_x(t, x^m(t))(x^{m+1}(t) - x^m(t)) \\ &\quad - A(t)x^{m+1}(t) - f(t, x^{m-1}(t)) - f_x(t, x^{m-1}(t))(x^m(t) \\ &\quad - x^{m-1}(t)) + A(t)x^m(t)] \\ &\quad + H^2[L[x^{m+1}] - F[x^m] - F_x[x^m][x^{m+1} - x^m] \\ &\quad - L[x^m] + F[x^{m-1}] + F_x[x^{m-1}][x^m - x^{m-1}]] \\ &= H^1[f(t, x^m(t)) - f(t, x^{m-1}(t)) - A(t)(x^m(t) - x^{m-1}(t)) \\ &\quad + (f_x(t, x^m(t)) - A(t))(x^{m+1}(t) - x^m(t)) \\ &\quad - (f_x(t, x^{m-1}(t)) - A(t))(x^m(t) - x^{m-1}(t))] \\ &\quad + H^2[-(F[x^m] - F[x^{m-1}] - L[x^m - x^{m-1}]) \\ &\quad - (F_x[x^m] - L)[x^{m+1} - x^m] \\ &\quad + (F_x[x^{m-1}] - L)[x^m - x^{m-1}]] \end{aligned} \tag{3.6}$$

and hence, as earlier in (3.5), it follows that

$$\begin{aligned} \|x^{m+1} - x^m\| &\leq M^1[M^3\|x^m - x^{m-1}\| + M^3\|x^{m+1} - x^m\| + M^3\|x^m - x^{m-1}\|] \\ &\quad + M^2[M^4\|x^m - x^{m-1}\| + M^4\|x^{m+1} - x^m\| + M^4\|x^m - x^{m-1}\|] \\ &= 2K\|x^m - x^{m-1}\| + K\|x^{m+1} - x^m\|, \end{aligned}$$

which is the same as

$$\|x^{m+1} - x^m\| \leq 2K(I - K)^{-1}\|x^m - x^{m-1}\|.$$

Thus, by an easy induction, we get

$$\|x^{m+1} - x^m\| \leq (2K(I - K)^{-1})^m\|x^1 - x^0\|. \tag{3.7}$$

However, since $3\rho(K) < 1$ implies that $\rho(2K(I - K)^{-1}) < 1$, from (3.7) it is clear that $\{x^m(t)\}$ is a Cauchy sequence, and therefore converges to some $x^*(t) \in \bar{S}(\bar{x}, r^1)$. This $x^*(t)$ is the unique solution of (1.1), (1.2) can easily be verified.

The error bound (3.2) follows from (3.7) and the triangle inequality

$$\begin{aligned}
\|x^{m+p}-x^m\| &\le \sum_{i=0}^{p-1} \|x^{m+p-i}-x^{m+p-i-1}\| \\
&\le \sum_{i=0}^{p-1} (K^*)^{m+p-i-1}\|x^1-\bar{x}\| \\
&= (K^*)^m \sum_{i=0}^{p-1} (K^*)^i \|x^1-\bar{x}\| \\
&\le (K^*)^m (I-K^*)^{-1}\|x^1-\bar{x}\|
\end{aligned}$$

and now taking the limit as $p \to \infty$.

Finally, from (3.5) we have

$$\begin{aligned}
x^1(t)-\bar{x}(t) = H^1[(f_x(t,\bar{x}(t)) - A(t))(x^1(t)-\bar{x}(t)) - \eta(t)] \\
+ H^2[-(F_x[\bar{x}]-L)[x^1-\bar{x}] - \ell^1]
\end{aligned}$$

and hence

$$\|x^1-\bar{x}\| \le M^1[M^3\|x^1-\bar{x}\| + \delta^1] + M^2[M^4\|x^1-\bar{x}\| + \delta^2],$$

which is the same as

$$\|x^1-\bar{x}\| \le (I-K)^{-1}(M^1\delta^1 + M^2\delta^2). \tag{3.8}$$

Using this inequality in (3.2) and the fact that $(I-K^*)^{-1}(I-K)^{-1} = (I-3K)^{-1}$, the required estimate (3.3) follows.

THEOREM 3.2. Let in addition to the hypotheses of Theorem 3.1, suppose the conditions (c_6) and (c_7) be satisfied. Then, the following holds

$$\|x^{m+1}-x^m\| \le H\cdot\|x^m-x^{m-1}\|\cdot\|x^m-x^{m-1}\|, \tag{3.9}$$

where $H=(I-K)^{-1}(M^1P+M^2Q)$ is a tensor of the third order with nonnegative components.

PROOF. From (3.6), we have

$$\begin{aligned}
x^{m+1}(t)-x^m(t) = H^1[&f(t,x^m(t)) - f(t,x^{m-1}(t)) - f_x(t,x^{m-1}(t))(x^m(t) \\
&- x^{m-1}(t)) + (f_x(t,x^m(t)) - A(t))(x^{m+1}(t)-x^m(t))] \\
&+ H^2[-(F[x^m] - F[x^{m-1}] - F_x[x^{m-1}][x^m-x^{m-1}]) \\
&-(F_x[x^m]-L)[x^{m+1}-x^m]].
\end{aligned}$$

Thus, on using the given hypotheses and the fact that $\{x^m\} \subseteq \bar{S}(\bar{x}, r^1)$, we obtain

$$\|x^{m+1}-x^m\| \le M^1[P\cdot\|x^m-x^{m-1}\|\cdot\|x^m-x^{m-1}\| + M^3\|x^{m+1}-x^m\|]$$

$$
\begin{aligned}
&+ M^2[Q \cdot \|x^m - x^{m-1}\| \cdot \|x^m - x^{m-1}\| + M^4\|x^{m+1} - x^m\|] \\
&= (M^1 P + M^2 Q) \cdot \|x^m - x^{m-1}\| \cdot \|x^m - x^{m-1}\| \\
&+ K\|x^{m+1} - x^m\|,
\end{aligned}
$$

which is the same as (3.9).

REMARK 3.1. In view of (3.7) and (3.8) the inequality (3.9) can be written as

$$
\begin{aligned}
\|x^{m+1} - x^m\| \le H \cdot [(K^*)^{m-1}(I - K)^{-1}(M^1\delta^1 + M^2\delta^2)] \cdot \\
[(K^*)^{m-1}(I - K)^{-1}(M^1\delta^1 + M^2\delta^2)]. \qquad (3.10)
\end{aligned}
$$

4. Approximate quasilinearization method

In Theorem 3.1 the conclusion 2 ensures that the sequence $\{x^m(t)\}$ generated by the quasilinear method (3.1) converges to the unique solution $x^*(t)$ of the problem (1.1), (1.2). However, in practical evaluation this sequence is approximated by the computed sequence, say, $\{y^m(t)\}$. To find $y^{m+1}(t)$ the function f is approximated by f^m, and the operator F by F^m. Therefore, the computed sequence $\{y^m(t)\}$ satisfies the recurrence relation

$$
\begin{aligned}
y^{m+1}(t) = H^1[f^m(t, y^m(t)) + f_x^m(t, y^m(t))(y^{m+1}(t) - y^m(t)) \\
- A(t)y^{m+1}(t)] + H^2[L[y^{m+1}] - F^m[y^m] \\
- F_x^m[y^m][y^{m+1} - y^m]] \\
y^0(t) = x^0(t) = \bar{x}(t);\ m = 0, 1, \ldots.
\end{aligned} \qquad (4.1)
$$

With respect to f^m and F^m, we shall assume that the following conditions are satisfied.

(d_1) For all $(t, x) \in J \times \bar{S}(\bar{x}, r)$ the function $f^m(t, x)$ is continuously differentiable with respect to x, and $\|f_x^m(t, x) - A(t)\| \le M^3$. Also, for all $t \in J$ and $y^m(t)$ obtained from (4.1) the inequality

$$
|f(t, y^m(t)) - f^m(t, y^m(t))| \le M^5|f(t, y^m(t))| \qquad (4.2)
$$

holds, where M^5 is an $n \times n$ nonnegative matrix with $\rho(M^5) < 1$.

(d_2) For all $x \in \bar{S}(\bar{x}, r)$, $F^m[x]$ is continuously Fréchet differentiable and $\|F_x^m[x] - L\| \le M^4$. Also, for $y^m(t)$ obtained from (4.1) the inequality

$$
\|F[y^m] - F^m[y^m]\| \le M^6\|F[y^m]\| \qquad (4.3)
$$

holds, where M^6 is an $n \times n$ nonnegative matrix with $\rho(M^6) < 1$.

Inequalities (4.2) and (4.3) correspond to the relative error in approximating

f and F by f^m and F^m. Further, since $\rho(M^5)$, $\rho(M^6) < 1$ these inequalities provide that

$$|f(t, y^m(t))| \le (I - M^5)^{-1} |f^m(t, y^m(t))| \tag{4.4}$$

and

$$\| F[y^m] \| \le (I - M^6)^{-1} \| F^m[y^m] \|. \tag{4.5}$$

THEOREM 4.1. With respect to the boundary value problem (1.1), (1.2) we assume that the conditions (c_1)–(c_5), (d_1) and (d_2) are satisfied. Further, let $\rho(\hat{K}) < 1$, where

$$\hat{K} = (I - K)^{-1}[2K + M^1 M^5(M^3 + \|A(t)\|) + M^2 M^6(M^4 + \|L\|)]$$

and

$$r^2 = (I - \hat{K})^{-1}(I - K)^{-1}[M^1\delta^1 + M^2\delta^2 + M^1 M^5(I - M^5)^{-1}\|f^0(t, \bar{x}(t))\| + M^2 M^6(I - M^6)^{-1}\|F^0[\bar{x}]\|] \le r.$$

Then, the following hold

1. all the conclusions $1 - 3$ of Theorem 3.1 hold
2. the sequence $\{y^m(t)\}$ obtained from (4.1) remains in $\bar{S}(\bar{x}, r^2)$
3. the sequence $\{y^m(t)\}$ converges to $x^*(t)$ the solution of (1.1), (1.2) if and only if $\lim_{m\to\infty} p^m = 0$, where

$$p^m = \| y^{m+1}(t) - H^1[f(t, y^m(t)) + f_x(t, y^m(t))(y^{m+1}(t) - y^m(t)) - A(t)y^{m+1}(t)] - H^2[L[y^{m+1}] - F[y^m] - F_x[y^m][y^{m+1} - y^m]] \| \tag{4.6}$$

and, also

$$\|x^* - y^{m+1}\| \le (I - K)^{-1}[M^1 M^5(I - M^5)^{-1}\|f^m(t, y^m(t))\| + M^2 M^6(I - M^6)^{-1}\|F^m[y^m]\| + 2K\|y^{m+1} - y^m\|]. \tag{4.7}$$

PROOF. Since $\hat{K} \ge 2(I - K)^{-1}K$, $\rho(\hat{K}) < 1$ implies $\rho(2K(I - K)^{-1}) < 1$, which in turn implies $\rho(3K) < 1$, and obviously

$$r^2 \ge [I - 2K(I - K)^{-1}]^{-1}(I - K)^{-1}(M^1\delta^1 + M^2\delta^2) = (I - 3K)^{-1}(M^1\delta^1 + M^2\delta^2) = r^1$$

the conditions of Theorem 3.1 are satisfied and conclusion 1 follows.

To prove 2, we note that $\bar{x}(t) \in \bar{S}(\bar{x}, r^2)$, and from (2.7) and (4.1) we find

$$y^1(t) - \bar{x}(t) = H^1[f^0(t, \bar{x}(t)) - f(t, \bar{x}(t)) + (f_x^0(t, \bar{x}(t)) - A(t))(y^1(t) - \bar{x}(t)) - \eta(t)] + H^2[-(F^0[\bar{x}] - F[\bar{x}]) - (F_x^0[\bar{x}] - L)[y^1 - \bar{x}] - \ell^1]$$

and hence

$$\|y^1 - \bar{x}\| \leq M^1[M^5\|f(t, \bar{x}(t))\| + M^3\|y^1 - \bar{x}\| + \delta^1] + M^2[M^6\|F[\bar{x}]\| + M^4\|y^1 - \bar{x}\| + \delta^2],$$

which in view of (4.4) and (4.5) implies that

$$\begin{aligned}\|y^1 - \bar{x}\| &\leq (I - K)^{-1}[M^1\delta^1 + M^2\delta^2 + M^1M^5(I - M^5)^{-1}\|f^0(t, \bar{x}(t))\| \\ &\quad + M^2M^6(I - M^6)^{-1}\|F^0[\bar{x}]\|] \\ &\leq r^2.\end{aligned}$$

Now we assume that $y^m(t) \in \bar{S}(\bar{x}, r^2)$ and will show that $y^{m+1}(t) \in \bar{S}(\bar{x}, r^2)$. From (2.7) and (4.1), we have

$$\begin{aligned}y^{m+1}(t) - \bar{x}(t) = H^1[&f^m(t, y^m(t)) - f(t, y^m(t)) \\ &+ f(t, y^m(t)) - f(t, \bar{x}(t)) - A(t)(y^m(t) - \bar{x}(t)) \\ &+ (f_x^m(t, y^m(t)) - A(t))(y^{m+1}(t) - \bar{x}(t)) \\ &- (f_x^m(t, y^m(t)) - A(t))(y^m(t) - \bar{x}(t)) - \eta(t)] \\ &+ H^2[-(F^m[y^m] - F[y^m]) \\ &- (F[y^m] - F[\bar{x}] - L[y^m - \bar{x}]) \\ &- (F_x^m[y^m] - L)[y^{m+1} - \bar{x}] \\ &+ (F_x^m[y^m] - L)[y^m - \bar{x}] - \ell^1]\end{aligned}$$

and hence

$$\begin{aligned}\|y^{m+1} - \bar{x}\| \leq M^1[&M^5\|f(t, y^m(t))\| + M^3\|y^m - \bar{x}\| + M^3\|y^{m+1} - \bar{x}\| \\ &+ M^3\|y^m - \bar{x}\| + \delta^1] + M^2[M^6\|F[y^m]\| \\ &+ M^4\|y^m - \bar{x}\| + M^4\|y^{m+1} - \bar{x}\| + M^4\|y^m - \bar{x}\| + \delta^2].\end{aligned}$$

However, since

$$\begin{aligned}|f(t, y^m(t))| &\leq |f(t, y^m(t)) - f(t, \bar{x}(t)) - A(t)(y^m(t) - \bar{x}(t))| \\ &\quad + |f(t, \bar{x}(t))| + |A(t)||y^m(t) - \bar{x}(t)| \\ &\leq M^3\|y^m - \bar{x}\| + (I - M^5)^{-1}\|f^0(t, \bar{x}(t))\| + \|A(t)\|\,\|y^m - \bar{x}\|\end{aligned}$$

and, similarly

$$\|F[y^m]\| \leq M^4\|y^m - \bar{x}\| + (I - M^6)^{-1}\|F^0[\bar{x}]\| + \|L\|\,\|y^m - \bar{x}\|$$

it follows that

$$
\begin{aligned}
(I-K)\|y^{m+1}-\bar{x}\| &\leq M^1[M^5(M^3+\|A(t)\|)+2M^3]\|y^m-\bar{x}\| \\
&\quad + M^1[M^5(I-M^5)^{-1}\|f^0(t,\bar{x}(t))\|+\delta^1] \\
&\quad + M^2[M^6(M^4+\|L\|)+2M^4]\|y^m-\bar{x}\| \\
&\quad + M^2[M^6(I-M^6)^{-1}\|F^0[\bar{x}]\|+\delta^2] \\
&\leq (I-K)\hat{K}r^2+(I-K)(I-\hat{K})r^2 \\
&= (I-K)r^2.
\end{aligned}
$$

Thus, $\|y^{m+1}-\bar{x}\| \leq r^2$ and this completes the proof of conclusion 2.

Next we shall prove 3. From the definitions of $x^{m+1}(t)$ and $y^{m+1}(t)$ in (3.1) and (4.1), we have

$$
\begin{aligned}
&x^{m+1}(t)-y^{m+1}(t) \\
&\quad = -y^{m+1}(t)+H^1[f(t,y^m(t))+f_x(t,y^m(t))(y^{m+1}(t)-y^m(t)) \\
&\qquad - A(t)y^{m+1}(t)]+H^2[L[y^{m+1}]-F[y^m]-F_x[y^m][y^{m+1}-y^m]] \\
&\qquad + H^1[f(t,x^m(t))-f(t,y^m(t))-A(t)(x^m(t)-y^m(t)) \\
&\qquad + (f_x(t,x^m(t))-A(t))(x^{m+1}(t)-x^m(t)) \\
&\qquad - (f_x(t,y^m(t))-A(t))(y^{m+1}(t)-y^m(t))] \\
&\qquad + H^2[-(F[x^m]-F[y^m]-L[x^m-y^m]) \\
&\qquad - (F_x[x^m]-L)[x^{m+1}-x^m] \\
&\qquad + (F_x[y^m]-L)[y^{m+1}-y^m]]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|x^{m+1}-y^{m+1}\| \\
&\quad \leq p^m+M^1[M^3\|x^m-y^m\|+M^3\|x^{m+1}-x^m\|+M^3\|y^{m+1}-y^m\|] \\
&\qquad + M^2[M^4\|x^m-y^m\|+M^4\|x^{m+1}-x^m\|+M^4\|y^{m+1}-y^m\|] \\
&\quad \leq p^m+K\|x^m-y^m\|+K\|x^{m+1}-x^m\| \\
&\qquad + K(\|y^{m+1}-x^{m+1}\|+\|x^{m+1}-x^m\|+\|x^m-y^m\|),
\end{aligned}
$$

which implies that

$$
\|x^{m+1}-y^{m+1}\| \leq (I-K)^{-1}p^m+K^*\|x^m-y^m\|+K^*\|x^{m+1}-x^m\|.
$$

Thus, from (3.7) we find that

$$
\|x^{m+1}-y^{m+1}\| \leq [(I-K)^{-1}p^m+(K^*)^{m+1}\|x^1-x^0\|]+K^*\|x^m-y^m\|.
$$

Using the fact that $\|x^0-y^0\|=0$, the above inequality gives

$$\|x^{m+1}-y^{m+1}\| \le \sum_{i=0}^{m}(K^*)^{m-i}[(I-K)^{-1}p^i+(K^*)^{i+1}\|x^1-x^0\|].$$

Thus, from the triangle inequality, we have

$$\begin{aligned}\|x^*-y^{m+1}\| \le \sum_{i=0}^{m}(K^*)^{m-i}(I-K)^{-1}p^i+(m+1)(K^*)^{m+1}\|x^1-x^0\| \\ +\|x^*-x^{m+1}\|.\end{aligned} \tag{4.8}$$

In (4.8) Theorem 3.1 ensures that $\lim_{m\to\infty}\|x^*-x^{m+1}\|=0$, and since $\rho(K^*)<1$, $\lim_{m\to\infty}[(m+1)(K^*)^{m+1}\|x^1-x^0\|]=0$. Thus, the condition $\lim_{m\to\infty}p^m=0$ is necessary and sufficient for the convergence of the sequence $\{y^m(t)\}$ to $x^*(t)$ follows from the Toeplitz lemma.

Finally, we shall prove (4.7). For this, we have

$$\begin{aligned}x^*(t)-y^{m+1}(t)=H^1[&f(t,x^*(t))-f(t,y^m(t))-A(t)(x^*(t)-y^m(t))\\&+f(t,y^m(t))-f^m(t,y^m(t))\\&-(f_x^m(t,y^m(t))-A(t))(y^{m+1}(t)-y^m(t))]\\&+H^2[-(F[x^*]-F[y^m]-L[x^*-y^m])\\&-(F[y^m]-F^m[y^m])\\&+(F_x^m[y^m]-L)[y^{m+1}-y^m]]\end{aligned}$$

and hence

$$\begin{aligned}\|x^*-y^{m+1}\| &\le M^1[M^3\|x^*-y^m\|+M^5(I-M^5)^{-1}\|f^m(t,y^m(t))\|\\&\quad+M^3\|y^{m+1}-y^m\|]+M^2[M^4\|x^*-y^m\|\\&\quad+M^6(I-M^6)^{-1}\|F^m[y^m]\|+M^4\|y^{m+1}-y^m\|]\\&\le K\|x^*-y^{m+1}\|+M^1M^5(I-M^5)^{-1}\|f^m(t,y^m(t))\|\\&\quad+M^2M^6(I-M^6)^{-1}\|F^m[y^m]\|+2K\|y^{m+1}-y^m\|,\end{aligned}$$

which is the same as (4.7).

In our next result we shall need the following conditions.

$(d_1)'$ Condition (d_1) holds with inequality (4.2) replaced by

$$|f(t,y^m(t))-f^m(t,y^m(t))|\le r^3, \tag{4.9}$$

where r^3 is a nonnegative vector.

$(d_2)'$ Condition (d_2) holds with inequality (4.3) replaced by

$$\|F[y^m]-F^m[y^m]\|\le r^4, \tag{4.10}$$

where r^4 is a nonnegative vector.

Inequalities (4.9) and (4.10) correspond to the absolute error in approximating f and F by f^m and F^m.

THEOREM 4.2. With respect to the boundary value problem (1.1), (1.2) we assume that the conditions (c_1)–(c_5), $(d_1)'$ and $(d_2)'$ are satisfied. Further, let $\rho(3K) < 1$, and

$$r^5 = (I - K^*)^{-1}(I - K)^{-1}(M^1(r^3 + \delta^1) + M^2(r^4 + \delta^2)) \leq r.$$

Then, the following hold

1. all the conclusions 1–3 of Theorem 3.1 hold
2. the sequence $\{y^m(t)\}$ obtained from (4.1) remains in $\bar{S}(\bar{x}, r^5)$
3. the condition $\lim_{m \to \infty} p^m = 0$ is necessary and sufficient for the convergence of $\{y^m(t)\}$ to the solution $x^*(t)$ of (1.1), (1.2) where p^m are defined in (4.6); and

$$\| x^* - y^{m+1} \| \leq (I - K)^{-1}(M^1 r^3 + M^2 r^4 + 2K \| y^{m+1} - y^m \|).$$

PROOF. The proof is contained in Theorem 4.1.

5. An example

The following example illustrates the importance of our results.

EXAMPLE 5.1. The boundary value problem

$$\begin{aligned} &u'' + u + (u - t)^3 = t + 0.1 \\ &u(-1) = -0.9, \ u(1) = 1.1 \end{aligned} \tag{5.1}$$

is due to Urabe [14], and it has also appeared in [1, 8, 20].

In system form (5.1) is the same as

$$\begin{aligned} &x_1' = x_2 \\ &x_2' = -x_1 - (x_1 - t)^3 + t + 0.1 \\ &x_1(-1) = -0.9, \ x_1(1) = 1.1. \end{aligned} \tag{5.2}$$

For (5.2) we choose $\bar{x}(t) = (t + 0.1, 1)^T$ so that $\delta^1 = [0, 10^{-3}]^T$ and $\delta^2 = [0, 0]^T$. Next, we take

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } L[x(t)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(-1) \\ x_2(-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$$

so that

$$Y(t) = \begin{bmatrix} \cos(t+1) & \sin(t+1) \\ -\sin(t+1) & \cos(t+1) \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ \cos 2 & \sin 2 \end{bmatrix}$$

and

$$H^1[\phi(t)] = \int_{-1}^{1} H(t, s)\phi(s)\, ds,$$

where

$\sin 2\, H(t, s)$

$$= \begin{cases} \begin{bmatrix} \sin(1-t)\cos(1+s) & -\sin(1-t)\sin(1+s) \\ -\cos(1-t)\cos(1+s) & \cos(1-t)\sin(1+s) \end{bmatrix}, & -1 \le s \le t \le 1 \\ \begin{bmatrix} -\sin(1+t)\cos(1-s) & -\sin(1+t)\sin(1-s) \\ -\cos(1+t)\cos(1-s) & -\cos(1+t)\sin(1-s) \end{bmatrix}, & -1 \le t \le s \le 1. \end{cases}$$

Thus, it is easy to compute

$$\|H^1\| \le \left\| \int_{-1}^{1} |H(t, s)|\, ds \right\| \le \frac{1}{\sin 2} \begin{bmatrix} 2 & 2\sin 1 - \sin 2 \\ 2 & 2\sin^2 1 \end{bmatrix}$$

$$\le \begin{bmatrix} 2.2001 & 0.8657 \\ 2.2001 & 1.5822 \end{bmatrix} = M^1;$$

$$\|H^2\| = \|Y(t)G^{-1}\| \le \frac{1}{\sin 2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \le 1.10005 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = M^2,$$

and for (t, x), such that $t \in [-1, 1]$, $x \in \bar{S}(\bar{x}, r)$

$$\|f_x(t, x) - A(t)\| = \left\| \begin{bmatrix} 0 & 0 \\ 3(x_1(t) - t)^2 & 0 \end{bmatrix} \right\| \le \begin{bmatrix} 0 & 0 \\ 3(0.1 + r_1)^2 & 0 \end{bmatrix} = M^3;$$

$$M^4 = 0.$$

Therefore, we find that

$$K = M^1 M^3 = \begin{bmatrix} 2.5971 & 0 \\ 4.7466 & 0 \end{bmatrix} (0.1 + r_1)^2.$$

Hence, the conditions of Theorem 3.1 are satisfied provided

$$\theta = 3 \times 2.5971\,(0.1 + r_1)^2 < 1 \tag{5.3}$$

and

$$\frac{0.8657 \times 10^{-3}}{1 - \theta} \le r_1 \tag{5.4}$$

also

$$10^{-3}\left[1.5822 + \frac{4.11\,(0.1 + r_1)^2}{1-\theta}\right] \le r_2. \tag{5.5}$$

Inequality (5.3) implies that $0 \le r_1 \le 0.2582572$, whereas (5.4) gives $0.94035 \times 10^{-3} \le r_1 \le 0.257654925$.

Thus, for $r_1 = 0.1$ both the inequalities (5.3) and (5.4) are satisfied. Further, for this value of r_1, from (5.3) and (5.5) we get $\theta = 0.311652$ and $0.001822 \le r_2$.

Therefore, in conclusion the sequence $\{x^m(t)\}$ generated by (3.1) for the problem (5.2) has the following properties.

1. $\{x^m(t)\} \subseteq \bar{S}(\bar{x}, r) = \{(x_1, x_2)^T : |x_1 - (t + 0.1)| \le 0.1,\ |x_2 - 1| \le 0.001822\}$,
2. $x^m(t) \to x^*(t)$, which is the solution of (5.2),
3. the error estimate (3.3) reduces to

$$|x_1^m(t) - x_1^*(t)| \le (0.23186)^m (1.2577 \times 10^{-3})$$
$$|x_2^m(t) - x_2^*(t)| \le (0.23186)^m (4.2013 \times 10^{-3}).$$

Further, with this choice of $r_1 = 0.1$, we have

$$P = \frac{1}{2}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1.2 & 0 & 0 & 0 \end{array}\right], \quad Q = 0.$$

Thus, it follows that

$$H = \frac{1}{2}\left[\begin{array}{cc|cc} 1.1593 & 0 & 0 & 0 \\ 2.1188 & 0 & 0 & 0 \end{array}\right].$$

Hence, the error bound (3.9) gives

$$|x_1^{m+1} - x_1^m| \le (0.57965)\,|x_1^m - x_1^{m-1}|^2 \tag{5.6}$$

$$|x_2^{m+1} - x_2^m| \le (1.0594)\,|x_1^m - x_1^{m-1}|^2. \tag{5.7}$$

Since from (3.8), we have

$$|x_1^1 - x_1^0| \le 0.9661 \times 10^{-3} \tag{5.8}$$

$$|x_2^1 - x_2^0| \le 1.7656 \times 10^{-3} \tag{5.9}$$

inequality (5.6) easily determines

$$\begin{aligned} |x_1^{m+1} - x_1^m| &\le (0.57965)|x_1^1 - x_1^0|)^{2^m}(1.7252) \\ &\le (0.56 \times 10^{-3})^{2^m}(1.7252). \end{aligned} \tag{5.10}$$

Finally, using this estimate in (5.7), we obtain

$$|x_2^{m+1} - x_2^m| \leq (0.56 \times 10^{-3})^{2^m}(3.1532). \tag{5.11}$$

We also note that for this particular example the error bound (3.10) reduces to

$$\|x^{m+1} - x^m\| \leq (0.23186)^{2m-2} \times 10^{-6}(0.55, 0.99)^T. \tag{5.12}$$

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