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A local Crank-Nicolson method for solving the heat equation

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1. Introduction

A wide range of computations for n-dimensional heat equation $\frac{\partial u}{\partial t} = \alpha \sum_{l=1}^{n} \frac{\partial^2 u}{\partial x_l^2}$ have been extensively investigated today [1], [3], [5], [8], because of their importance in applied sciences. Although the explicit method is computationally simple, it has one serious drawback: The time step δt should be taken to be very small because the process is stable only for $\alpha \sum_{l=1}^{n} \frac{\delta t}{(\delta x_l)^2} \leq \frac{1}{2}$, where δx_l is step size on the space variable. The Crank-Nicolson method has widly been used since it reduces the total volume of calculation and is valid for all small finite value of $r_{x_l} = \frac{\delta t}{(\delta x_l)^2}$. It is however, necessary to solve a large linear system. In this paper, based on the representation of the Trotter product [6], we shall propose a new technique, which does not yield a large linear system by using a splitting of coefficient matrix that is obtained by applying the usual centered difference to the partial differential term in space of the above equation. The proposed method has an explicit form and unconditionally stable. Furthermore, we find that it is superior to the Crank-Nicolson method as is illustrated by numerical examples.

2. The formulation of the local Crank-Nicolson method for one-dimensional problem with the Dirichlet boundary conditions

Let us first consider the following heat equation of (2.1):

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0, 1),\ t \geq 0, \tag{2.1}$$

with the initial and the boundary conditions:

$$u(x, 0) = f(x), \qquad x \in (0, 1), \tag{2.2}$$

$$u(0, t) = 0, \qquad t \geq 0, \tag{2.3}$$

$$u(1, t) = 0\,, \qquad t \geq 0\,, \tag{2.4}$$

where α is the constant thermal conductivity, $u(x, t)$ the non-dimensional temperature at point (x, t), and $f(x)$ a known function. We apply the usual centered difference to the partial differential term in space in (2.1) to obtain the following equations [5]:

$$\frac{dV(t)}{dt} = \frac{\alpha}{h^2} A V(t)\,, \tag{2.5}$$

where $V(t) = [v(x_1, t), v(x_2, t), \dots, v(x_{N-1}, t)]^T$, $x_i = ih$, $i = 1, \dots, N-1$, $h = 1/N$. For small h, we could find that $v(x_i, t)$ is an approximation to $u(x_i, t)$ and A is the following tridiagonal matrix:

$$A = \begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}. \tag{2.6}$$

Without loss of generality, we may set $\alpha = 1$. Then the solution of (2.5) is given by

$$V(t) = \exp\left(\frac{t}{h^2} A\right) V(0)\,, \tag{2.7}$$

where $V(0) = [f(x_1), f(x_2), \dots, f(x_{N-1})]^T$. Here the exponential function of the matrix A is

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.8}$$

If the matrix A is bounded, the right hand side of (2.8) converges. Equation (2.7) may be written in a stepwise fashion as

$$V(t_m + k) = \exp\left(\frac{k}{h^2} A\right) V(t_m)\,, \tag{2.9}$$

where $t_m = mk$ $(m = 1, \dots, N)$ and k is a convenient time step. Let $V(t_m) = (v_{1,m}, v_{2,m}, \dots, v_{N-1,m})^T$, where $v_{i,m}$ is an approximation to $u(x_i, t_m)$. The well-known explicit, the implicit and the Crank-Nicolson methods are expressed as the following forms, respectively:

$$V(t_{m+1}) = \left(I + \frac{k}{h^2} A\right) V(t_m)\,, \tag{2.10}$$

$$V(t_{m+1})=\left(I-\frac{k}{h^2}A\right)^{-1}V(t_m) \tag{2.11}$$

and

$$V(t_{m+1})=\left(I-\frac{k}{2h^2}A\right)^{-1}\left(I+\frac{k}{2h^2}A\right)V(t_m)\,. \tag{2.12}$$

From (2.9), (2.10), (2.11) and (2.12), we have following approximate equations.

$$\exp\left(\frac{k}{h^2}A\right)=I+\frac{k}{h^2}A+O\left(\left(\frac{k}{h^2}\right)^2\right), \tag{2.13}$$

$$\exp\left(\frac{k}{h^2}A\right)=\left(I-\frac{k}{h^2}A\right)^{-1}+O\left(\left(\frac{k}{h^2}\right)^2\right) \tag{2.14}$$

and

$$\exp\left(\frac{k}{h^2}A\right)=\left(I-\frac{k}{2h^2}A\right)^{-1}\left(I+\frac{k}{2h^2}A\right)+O\left(\left(\frac{k}{h^2}\right)^3\right). \tag{2.15}$$

In this paper we shall propose a method named a local Crank-Nicolson method. In the following, we consider the concept of a splitting theory of coefficient matrix A and give an approximation of $\exp\left(\frac{t}{h^2}A\right)$. We first show the following lemma for the exponential function of the matrix.

LEMMA 1. *Suppose that the coefficient matrix in* (2.5) *has the form expressed as*

$$A=\sum_{i=1}^{S}A_i\,.$$

If the matrix A_i is bounded for any i, then we have the Trotter product form of the semi-group such that

$$\exp\left(\frac{t}{h^2}A\right)=\lim_{\sigma\to\infty}\left\{\prod_{i=1}^{S}\exp\left(\frac{tA_i}{\sigma h^2}\right)\right\}^{\sigma}, \tag{2.16}$$

where S and σ are some positive integers.

By lemma 1, for any $\sigma=\xi$, (2.16) can be approximated in the following form

$$\exp\left(\frac{k}{h^2}A\right)\approx\left\{\prod_{i=1}^{S}\exp\left(\frac{kA_i}{\xi h^2}\right)\right\}^{\xi}. \tag{2.17}$$

We decompose A in (2.6) into $S=(N-1)$-part splitting in the following block form:

$$A_1 = \begin{bmatrix} -2 & 1 & 0 & & & 0 \\ 1 & -1 & 0 & & & \\ 0 & 0 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & 0 \end{bmatrix},$$

$$A_i = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & -1 & 1 & \\ & & 1 & -1 & \\ & & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}, \quad (2 \le i \le N-2) \tag{2.18}$$

$$A_{N-1} = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & 0 & 0 \\ & & & 0 & -1 & 1 \\ 0 & & & 0 & 1 & -2 \end{bmatrix},$$

where each matrix A_i is a negative semi definite matrix. Let us consider the approximation of the operator $\exp\left(\frac{k}{\xi h^2} A_i\right)$ by using the Crank-Nicolson method (2.15). For each i the following relation holds:

$$\exp\left(\frac{k}{\xi h^2} A_i\right) \approx \left(I - \frac{k}{2\xi h^2} A_i\right)^{-1} \left(I + \frac{k}{2\xi h^2} A_i\right). \tag{2.19}$$

Substituting (2.19) into (2.17) we obtain

$$\exp\left(\frac{k}{h^2} A\right) \approx \left\{\prod_{i=1}^{N-1} \left[\left(I - \frac{k}{2\xi h^2} A_i\right)^{-1} \left(I + \frac{k}{2\xi h^2} A_i\right)\right]\right\}^{\xi}. \tag{2.20}$$

Hence we have from (2.9) and (2.20)

$$V_1(t_{m+1}) = \left\{\prod_{i=1}^{N-1} \left[\left(I - \frac{k}{2\xi h^2} A_i\right)^{-1} \left(I + \frac{k}{2\xi h^2} A_i\right)\right]\right\}^{\xi} V(t_m), \tag{2.21}$$

which is an approximation of $u(x_i, t_m)$. If we replace A_i by $B_i = A_{N-i}$ then we obtain the following form:

$$V_2(t_{m+1}) = \left\{\prod_{i=1}^{N-1} \left[\left(I - \frac{k}{2\xi h^2} B_i\right)^{-1} \left(I + \frac{k}{2\xi h^2} B_i\right)\right]\right\}^{\xi} V(t_m). \tag{2.22}$$

We now put

$$\tilde{V}(t_{m+1}) = \frac{1}{2}\left\{\left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}A_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}A_i\right)\right]^{\xi} + \left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}B_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}B_i\right)\right]^{\xi}\right\} \tag{2.23}$$

and employ $\tilde{V}(t_{m+1})$ as a numerical solution of (2.5). This scheme is called the local Crank-Nicolson scheme.

LEMMA 2. *The local Crank-Nicolson method have the second-order approximation in time.*

PROOF. By the expansion formula, we have

$$\exp\left(\frac{k}{h^2}A\right) = \sum_{n=0}^{\infty}\left(\frac{k}{h^2}A\right)^n. \tag{2.24}$$

The equation on right hand side of (2.17) is rewritten as

$$\left\{\prod_{i=1}^{N-1}\exp\left(\frac{kA_i}{\xi h^2}\right)\right\}^{\xi} = I + \frac{k}{h^2}A + \cdots. \tag{2.25}$$

By the comparison of (2.24) and (2.25), it turns out that (2.21) is an approximation of order k in time.

If we replace A_i by $B_i = A_{N-i}$ then we obtain

$$\left\{\prod_{i=1}^{N-1}\exp\left(\frac{kB_i}{\xi h^2}\right)\right\}^{\xi} = I + \frac{k}{h^2}A + \cdots. \tag{2.26}$$

By taking the arithmetic mean of (2.25) and (2.26) we have

$$\frac{1}{2}\left\{\left[\prod_{i=1}^{N-1}\exp\left(\frac{kA_i}{\xi h^2}\right)\right]^{\xi} + \left[\prod_{i=1}^{N-1}\exp\left(\frac{kB_i}{\xi h^2}\right)\right]^{\xi}\right\} = I + \frac{k}{h^2}A + \frac{1}{2!}\left(\frac{k}{h^2}A\right)^2 + \cdots. \tag{2.27}$$

We could find that scheme (2.23) approximates the solution u of (2.5) with the second order accuracy in t.

It is not difficult to check that the coefficient matrix of (2.21) is the transposed matrix of the coefficient matrix of (2.22). The matrix $[I - (k/(2\xi h^2))A_i]$ has the simple form:

$$\begin{bmatrix} I_{i-1} & & \\ & R_i & \\ & & I_{N-i-2} \end{bmatrix}, \tag{2.28}$$

where I_i is the unit matrix of order i and R_i is given by

$$R_i = I_2 - \frac{r}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Hence, the inversion of $[I - (k/(2h^2))A_i]$ can be writen as

$$[I - (k/(2h^2))A_i]^{-1} = \begin{bmatrix} I_{i-1} & & \\ & R_i^{-1} & \\ & & I_{N-i-2} \end{bmatrix} \tag{2.29}$$

for each i. Here the inversion of R_i can be written as

$$R_i^{-1} = \frac{1}{1+r} I_2 - \frac{r}{2(1+r)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, the inversion of $[I - (k/(2\xi h^2))A_i]$ can be computed quite easily, which leads to the explicit description of our method. It means that it is unnecessary to solve a large linear system to carry out the method. This is an important matter in computation processes.

3. Stability, consistency and convergence

In this section, we discuss the stability, consistency and convergence of the difference scheme (2.23).

THEOREM 1. *Suppose that the matrix A is of the form $A = \sum_{i=1}^{N-1} A_i$, where each A_i is negative semi definite ($i = 1, 2, \ldots, N-1$). Then the local Crank-Nicolson method is unconditionally stable.*

PROOF. Let λ_i be an eigenvalue of the matrix A_i and η_i be an eigenvalue of the matrix on right hand side of (2.19). Then we know $\lambda_i \leq 0$ and

$$\eta_i = \frac{1 + (k/(2\xi h^2))\lambda_i}{1 - (k/(2\xi h^2))\lambda_i}. \tag{3.1}$$

Since it is easily shown that $|\eta_i| \leq 1$, the following relation holds:

$$\left(\prod_{i=1}^{N-1} |\eta_i|\right)^{\xi} \leq 1 .$$

Therefore, this scheme is an unconditionally stable [4].

In the next place, we examine the consistency condition of this scheme.

THEOREM 2. *The local Crank-Nicolson method satisfies the consistency condition.*

PROOF. Using the initial function f, the solution u of (2.1)–(2.4) at time t can be expressed as

$$u(t) = E(t)f, \tag{3.2}$$

where $E(t)$ is a solution operator of (2.1)–(2.4). By the property of the semigroup [7],

$$u(t+k) = E(k)u(t), \tag{3.3}$$

we know that the truncation error T from (2.23) is

$$T = u(t+k) - \frac{1}{2}\left\{\left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}A_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}A_i\right)\right]^{\xi} + \left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}B_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}B_i\right)\right]^{\xi}\right\}u(t_m).$$

Since $E(k) \to I$ as $k \to 0$, we find that, as $k \to 0$

$$\frac{1}{2}\left\{\left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}A_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}A_i\right)\right]^{\xi} + \left[\prod_{i=1}^{N-1}\left(I - \frac{k}{2\xi h^2}B_i\right)^{-1}\left(I + \frac{k}{2\xi h^2}B_i\right)\right]^{\xi}\right\} \to I.$$

So we obtain $T \to 0$. It means that the proposed scheme satisfies the consistency condition.

From the above results, by using the Lax equivalence theorem it is easily shown that the proposed scheme has the convergence property.

THEOREM 3. *Given a properly posed linear initial-value problem and a linear finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.*

4. The formulation of the local Crank-Nicolson method for two-dimensional problem

Let us consider the following two-dimensional initial-boundary value problem for

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad \text{in } \Omega \times (0, T] \tag{4.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^2. The initial and the boundary conditions

$$u(x, y, 0) = f(x, y) \qquad \text{on } \Omega\,, \tag{4.2}$$

$$u(x, y, t) = 0 \qquad \text{on } \partial\Omega\,, \tag{4.3}$$

are imposed, respectively. Over the domain Ω we set an orthogonal finite difference lattice in which lattice points are numbered with the natural ordering. Applying the centered difference approximation in space variables by using the lattice to the equation (4.1) we have the system of the differential equation

$$\frac{dV(t)}{dt} = \frac{1}{h^2} A V(t)\,, \tag{4.4}$$

where the matrix A is a block tridiagonal matrix since we take the natural ordering. For the equation (4.4) the local Crank-Nicolson method can be formulated by using the block decomposition. In this case it is necessary to solve sublinear systems induced from the decomposition. The order of the each sublinear system is small comparing with the order of the matrix A. For simplicity, we shall show the concrete formulation to the problem defined on the rectangler domain. Set $\Delta x = \Delta y = h = 1/N$, $x_i = ih$ $(i = 1, 2, \dots, N-1)$, $y_j = jh\,(j = 1, 2, \dots, N-1)$, and $V(t) = [v(x_1, y_1, t), v(x_1, y_2, t), \dots, v(x_N, y_N, t)]^T$ and $v(x_i, y_j, t)$ denotes an approximate solution for $u(x_i, y_j, t)$. In this case, the matrix A is an $N^2 \times N^2$ block tridiagonal matrix which is given as

$$A = \begin{bmatrix} H & I & & & 0 \\ I & H & I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ 0 & & & I & H \end{bmatrix} \tag{4.5}$$

where I is an $N \times N$ unit matrix and H is an $N \times N$ tridiagonal matrix such as

$$H = \begin{bmatrix} -4 & 1 & & & 0 \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -4 \end{bmatrix}. \tag{4.6}$$

If we give an initial vector $V(0) = [v(x_1, y_1, 0), v(x_1, y_2, 0), \dots, v(x_N, y_N, 0)]^T$ then the solution of the equation (4.4) at the time t is given as follows:

$$V(t) = \exp\left(\frac{t}{h^2} A\right) V(0)\,. \tag{4.7}$$

By applying the formulation for the one-dimensional problem, we are able to formulate the method for (4.7). In order to obtain the scheme we consider a decomposition of the matrix A, as follows:

$$A_1 = \begin{bmatrix} H & I & & & 0 \\ I & H/2 & & & \\ & & \ddots & & \\ & & & \ddots & 0 \\ 0 & & & 0 & 0 \end{bmatrix},$$

$$A_i = \begin{bmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & H/2 & I & & \\ & & I & H/2 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix}, \qquad (2 \le i \le N-2) \tag{4.8}$$

$$A_{N-1} = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & \ddots & & & \\ & & \ddots & & \\ & & & H/2 & I \\ 0 & & & I & H \end{bmatrix}.$$

By using the same manner for the one-dimensional scheme, we obtain the following iteration scheme:

$$V(t_{m+1}) = \left\{ \prod_{i=1}^{N-1} \left[\left(I - \frac{k}{2\xi h^2} A_i \right)^{-1} \left(I + \frac{k}{2\xi h^2} A_i \right) \right] \right\}^{\xi} V(t_m) . \tag{4.9}$$

Then the solutions of (4.9) are obtained by using the L-U decomposition:

$$\begin{aligned} \left(I - \frac{k}{2\xi h^2} A_{N-1} \right) V_1 &= \left(I + \frac{k}{2\xi h^2} A_{N-1} \right) V_0 , \\ \left(I - \frac{k}{2\xi h^2} A_{N-2} \right) V_2 &= \left(I + \frac{k}{2\xi h^2} A_{N-2} \right) V_1 , \\ &\cdots\cdots\cdots\cdots \\ \left(I - \frac{k}{2\xi h^2} A_1 \right) V_{N-1} &= \left(I + \frac{k}{2\xi h^2} A_1 \right) V_{N-2} , \end{aligned} \tag{4.10}$$

where V_i is an N^2 vector and $V_i = V(t_i)$. The stability and convergence for the present scheme are satisfied as same as the scheme for one-dimensional problem.

5. Numerical test

In order to test the feasibility and efficiency of our proposed algorithm we carried out several numerical tests. Firstly, we solved the equation (2.1) imposed the initial condition

$$f(x) = 100 \sin(\pi x) \qquad \text{for } x \in (0, 1),$$

and the boundary conditions (2.3) and (2.4). In this case, the exact solution is given as

$$u(x, t) = 100 \exp(-\pi^2 t) \sin(\pi x).$$

The result is shown in Fig. 1 and Table 1, at $t = 0.5$ with $N = 20$, $k = 0.05$, $\xi = 20$. Secondly, we solved the model problem [2]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0, 2), \quad t \geq 0,$$

$$u(x, 0) = 1.0, \qquad u(0, t) = u(2, t) = 0,$$

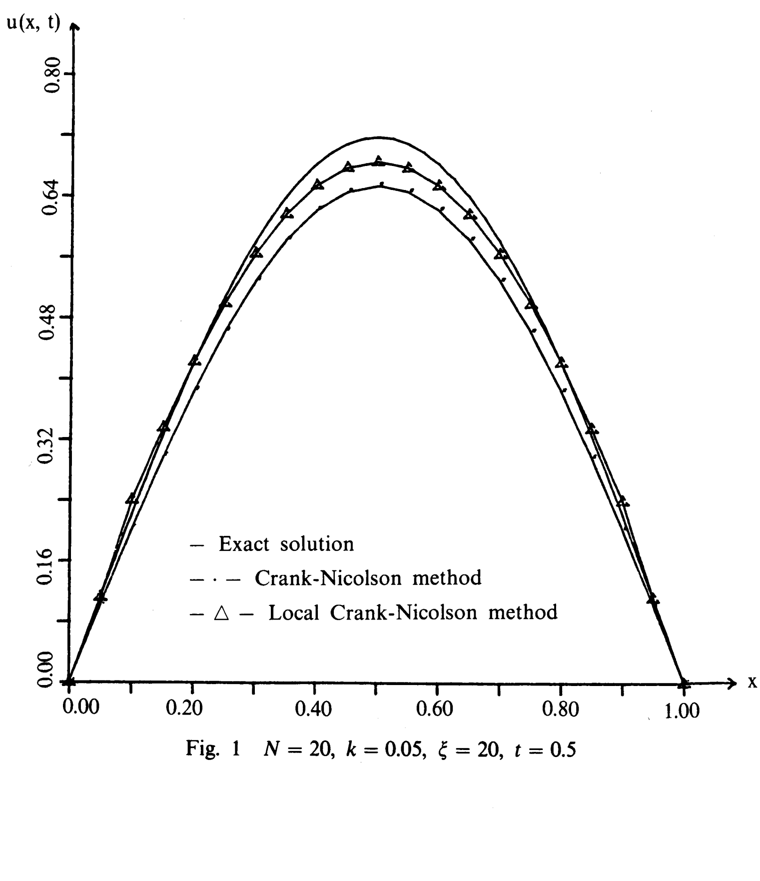

Fig. 1 $N = 20$, $k = 0.05$, $\xi = 20$, $t = 0.5$

Table 1. Comparisons of the exact solution with the numerical solutions for $N = 20$, $k = 0.05$, $\xi = 20$.

Method	$x_{10} = 0.5$, $t = 0.5$	Error
Exact solution	0.71918809	
Crank-Nicolson	0.65520500	0.08896573
Local Crank-Nicolson	0.68657660	0.04534487

which has a theoretical solution

$$u(x, t) = \sum_{i=1}^{\infty} [1 - (-1)^n] \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \exp\left(\frac{-n^2\pi^2 t}{4}\right).$$

The numerical results for $h = 0.05$ and $k = 0.1$ are depicted in Fig. 2. We have tabulated relative errors in Table 2. From Fig. 1, 2 and Table 1, 2 we concluded that local Crank-Nicolson method gives better approximations than the Crank-Nicolson method.

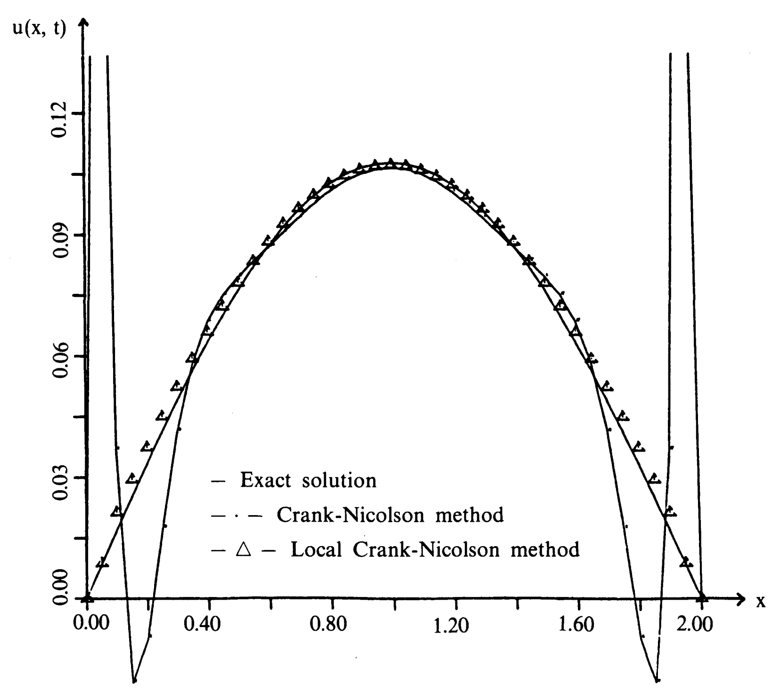

Fig. 2 $N = 40$, $k = 0.1$, $\xi = 40$, $t = 1.0$

Table 2. Comparisons of the theoretical solution with the numerical solutions for $N = 40$, $k = 0.1$, $\xi = 40$.

Method	$x_{20} = 1.0$, $t = 1.0$	Error
Theoretical solution	0.1079770	
Crank-Nicolson	0.1067305	0.0116789
Local Crank-Nicolson	0.1075128	0.0043176

Finally, we solved the equation (4.1) in the rectangle domain $\Omega = \{(x, y) \in R^2 | 0 < x < 1, 0 < y < 1\}$ with the initial condition

$$f(x, y) = 100 \sin(\pi x) \sin(\pi y) \qquad \text{for } x,\ y \in (0, 1),$$

and the boundary conditions (4.3). In this case, the exact solution is easily given as

$$u(x, y, t) = 100 \exp(-2\pi^2 t) \sin(\pi x) \sin(\pi y)$$

The result is shown in Table 3, at $t = 0.1$ with $N = 10$, $k = 0.01$, $\xi = 10$

From Table 3 we concluded that local Crank-Nicolson method gives better approximation than the original Crank-Nicolson one.

Table 3. The comparison of the exact solution with the numerical solutions for $N = 10$, $k = 0.01$, $\xi = 10$.

Method	$x_5 = y_5 = 0.5$, $t = 1.0$	Error
Exact solution	13.891119	
Crank-Nicolson	14.095628	0.0147223
Local Crank-Nicolson	14.028540	0.009827

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