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A remark on the global asymptotic stability of a dynamical system modeling two species competition

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1. Introduction

In this paper we shall consider a strongly order preserving semiflow modeling the two species competition in population biology. To motivate the principal result, we consider three examples of competitive dynamical systems in an infinite dimensional space which is a product space. The first example is the model of the unstirred chemostat with equal diffusions [6]

$$(1.1)' \qquad \begin{aligned} S_t &= dS_{xx} - \frac{m_1 S}{a_1 + S} u - \frac{m_2 S}{a_2 + S} v, \\ u_t &= du_{xx} + \frac{m_1 S}{a_1 + S} u, \\ v_t &= dv_{xx} + \frac{m_2 S}{a_2 + S} v, \end{aligned} \qquad 0 < x < 1$$

with initial and boundary conditions of the form

$$\begin{aligned} &S_x(0, t) = -S^{(0)}, && u_x(0, t) = v_x(0, t) = 0, \\ &S_x(1, t) + rS(1, t) = 0, && u_x(1, t) + ru(1, t) = 0, \\ &v_x(1, t) + rv(1, t) = 0, && \\ &u(x, 0) = u_0(x) \ge 0, && v(x, 0) = v_0(x) \ge 0, \\ &S(x, 0) = S_0(x) \ge 0, && 0 < x < 1, \end{aligned}$$

where d, $S^{(0)}$, r, m_i, a_i, $i = 1, 2$ are positive constants.
From [6], it follows that

$$S(x, t) + u(x, t) + v(x, t) = \varphi(x) + 0(e^{-\alpha t}) \text{ as } t \longrightarrow \infty$$

$$\text{for some } \alpha > 0, \text{ where } \varphi(x) = S^{(0)}\left(\frac{1 + r}{r} - x\right),\ 0 < x < 1.$$

From [10] it suffice to study the dynamics on the ω-limit set, i.e., solutions of

$$(1.1)\qquad \begin{aligned} u_t &= du_{xx} + \frac{m_1(\varphi(x)-u-v)}{a_1+(\varphi(x)-u-v)}u, \\ v_t &= dv_{xx} + \frac{m_2(\varphi(x)-u-v)}{a_2+(\varphi(x)-u-v)}v, \end{aligned} \qquad 0 < x < 1$$

$$u_x(0, t) = v_x(0, t) = 0,\ u_x(1, t) + ru(1, t) = 0,$$
$$v_x(1, t) + rv(1, t) = 0,$$
$$u(x, 0) = u_0(x) \geq 0,\ v(x, 0) = v_0(x) \geq 0.$$

The second example is the classical Lotka-Volterrra two-species competition model with diffusions and Dirichlet boundary conditions

$$(1.2)\qquad \begin{aligned} Lu_t &= \Delta u + u(a-u-cv), \\ v_t &= \Delta v + v(d-v-cu), \end{aligned} \qquad \text{in } \Omega \times \mathbb{R}^+$$
$$u = v = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+,$$
$$u(x, 0) = u_0(x) \geq 0, \qquad v(x, 0) = v_0(x) \geq 0,$$

where $a, c, d, e, L > 0$ and Ω is a bounded open set in $\mathbb{R}^m$. The existence and uniqueness of positive solutions for the corresponding elliptic problem of (1.2) was studied by Dancer [1].

The third example is the two-species delayed chemostat model [2],

$$(1.3)'\qquad \begin{aligned} \frac{dS}{dt} &= (S^{(0)} - S(t))D - \frac{m_1 S}{a_1+S}x_1 - \frac{m_2 S}{a_2+S}x_2, \\ \frac{dx_1}{dt} &= -Dx_1 + e^{-D\tau_1}\frac{m_1 S(t-\tau_1)}{a_1+S(t-\tau_1)}x_1(t-\tau_1), \\ \frac{dx_2}{dt} &= -Dx_2 + e^{-D\tau_2}\frac{m_2 S(t-\tau_2)}{a_2+S(t-\tau_2)}x_2(t-\tau_2), \\ S(\theta) &= \varphi(\theta) \geq 0, \qquad -\max(\tau_1, \tau_2) \leq \theta \leq 0, \\ x_1(\theta) &= \varphi_1(\theta) \geq 0, \qquad -\tau_1 \leq \theta \leq 0, \\ x_2(\theta) &= \varphi_2(\theta) \geq 0, \qquad -\tau_2 \leq \theta \leq 0. \end{aligned}$$

Let $z_1(t) = x_1(t+\tau_1)e^{D\tau_1}$, $z_2(t) = x_2(t+\tau_2)e^{D\tau_2}$. Then it follows that

$$S(t) + z_1(t) + z_2(t) = S^{(0)} + O(e^{-Dt}) \quad \text{as} \quad t \longrightarrow \infty.$$

From [10] it suffices to confine the study to the dynamics on the ω-limit set, i.e., to solutions of

$$
(1.3)\qquad
\begin{aligned}
\frac{dz_1}{dt} &= \frac{m_1(S^{(0)} - z_1(t) - z_2(t))}{a_1 + (S^{(0)} - z_1(t) - z_2(t))} e^{-D\tau_1} z_1(t - \tau_1) - Dz_1(t),\\
\frac{dz_2}{dt} &= \frac{m_2(S^{(0)} - z_1(t) - z_2(t))}{a_2 + (S^{(0)} - z_1(t) - z_2(t))} e^{-D\tau_2} z_2(t - \tau_2) - Dz_2(t),\\
z_1(t) &= \varphi_1(\theta) \geq 0, \qquad -\tau_1 \leq \theta \leq 0,\\
z_2(t) &= \varphi_2(\theta) \geq 0, \qquad -\tau_2 \leq \theta \leq 0.
\end{aligned}
$$

For appropriate parameter values these three problems have rest points of the form, $E_0 = (0, 0)$, $\hat{E}_0 = (\hat{u}, 0)$, $\tilde{E} = (0, \tilde{v})$ where E_0 is a repeller and $\hat{E}$, $\tilde{E}$ are globally stable when there are trivial initial conditions for the "other" population. Only nonnegative solutions are meaningful since the variables are populations. For large time t, each of these problems generates a compact, dissipative semidynamical system on the nonnegative cone of a product space $C_1 \times C_2$ where C_i, $i = 1, 2$, is an ordered real, separable Banach space. Not only is the nonnegative cone positively invariant, but sets of the form $C_1 \times \{0\}$ and $\{0\} \times C_2$ are also invariant. These sets represent one population in the absence of the other, and hence carry a semi-dynamical system of "lower" complexity.

In the next section, a generalization which encompass all three examples is discussed, and the basic hypothesis and the main theorem are stated. The main theorem is proved in section 3. The application of the main theorem to the problems (1.1), (1.2) and (1.3) is discussed in section 4.

2. The semi-dynamical system

In this section we set up the general framework which encompass the examples (1.1)–(1.3) and state our main result, Theorem 2.1.

Let C_i, $i = 1, 2$, be ordered, separable Banach spaces with respective partial orderings $\leq_{C_i}$. Let C_i^+ be the nonnegative cone of C_i, i.e.,

$$C_i^+ = \{x_i \in C_i \mid x_i \geq_{C_i} 0\},$$

and let $\mathring{C}_i^+$ be the interior of C_i^+ with respect to the metric on C_i. Then, for any x_i, $y_i \in C_i$, define the order relations $<_{C_i}$ and $\ll_{C_i}$ as follows: $x_i <_{C_i} y_i$ if $y_i - x_i \in C_i^+$ and $y_i - x_i \neq 0$; $x_i \ll_{C_i} y_i$ if $y_i - x_i \in \mathring{C}_i^+$. Next, let K be the cone in $C_1 \times C_2$ defined by

$$K = \{x = (x_1, x_2) \in C_1 \times C_2 \mid x_1 \in C_1^+ \text{ and } -x_2 \in C_2^+\}$$

and let $\mathring{K} = \mathring{C}_1^+ \times \mathring{C}_2^+$. The order relations $\leq_K$, $<_K$, and $\ll_K$ for elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $C_1 \times C_2$ are then defined by: $x \leq_K y$ if

$y - x \in K$; $x <_K y$ if $y - x \in K$ and $y - x \neq 0$; $x \ll_K y$ if $y - x \in \mathring{K}$. Note, for example, that $x \leq_K y$ means that $x_1 \leq_{C_1} y_1$ and $y_2 \leq_{C_2} x_2$.

Following Hirsch [5], we define the semiflow $T(t)$: $C_1^+ \times C_2^+ \to C_1^+ \times C_2^+$, $t \geq 0$, to be monotone (eventually strongly monotone) as follows:

Definition 2.1. T is said to be monotone with respect to K on $C_1^+ \times C_2^+$ if $x, y \in C_1^+ \times C_2^+$ with $x \leq_K y$ implies that $T(t)x \leq_K T(t)y$ for all $t \geq 0$. T is said to be eventually strongly monotone with respect to K on $C_1^+ \times C_2^+$ if $x, y \in C_1^+ \times C_2^+$ with $x <_K y$ implies that $T(t)x \ll_K T(t)y$ for all $t \geq \tau$ for some $\tau \geq 0$.

Concerning T we will assume

(H1) $T(t)$: $C_1^+ \times C_2^+ \to C_1^+ \times C_2^+$ is a compact, C_0 semigroup

(H2) $T(t)$ is monotone and is eventually strongly monotone [8] on the set $C_1^+ \times C_2^+$ and $C_1^+ \times C_2^+ \setminus (\{0\} \times C_2^+\} \cup (C_1^+ \times \{0\})$ respectively.

Remark 2.1. Let $(u_0, v_0) \in C_1^+ \times C_2^+$, $u_0 \neq 0$, $v_0 \neq 0$ and $T(t)(u_0, v_0) = (u(t), v(t))$. Since $\left(\dfrac{u_0}{2}, v_0\right) \leq_K (u_0, v_0)$ and $(u_0, v_0) \leq_K \left(u_0, \dfrac{v_0}{2}\right)$, from (H2) it follows that $T(t)\left(\dfrac{u_0}{2}, v_0\right) \ll T(t)(u_0, v_0)$ and $T(t)(u_0, v_0) \ll_K T(t)\left(u_0, \dfrac{v_0}{2}\right)$ for $t \geq \tau$ for some $\tau > 0$. Hence $(u(t), v(t)) \in \mathring{C}_1^+ \times \mathring{C}_2^+$ for $t \geq \tau$.

Concerning the restrictions on the boundary $C_1^+ \times \{0\}$ and $\{0\} \times C_2^+$, we assume

($\ddot{\mathrm{H}}$1) $C_1^+ \times \{0\}$ is an invariant set under $T(t)$ for all $t > 0$

($\ddot{\mathrm{H}}$2) $\hat{E} = (\hat{u}, 0)$ is a fixed point in $C_1^+ \times \{0\}$

($\ddot{\mathrm{H}}$3) The basin of attraction of $\hat{E}$ in $C_1^+ \times \{0\}$ is $C_1^+ \times \{0\} \setminus E_0$

and

($\tilde{\mathrm{H}}$1) $\{0\} \times C_2^+$ is an invariant set under $T(t)$ for all $t > 0$

($\tilde{\mathrm{H}}$2) $\tilde{E} = (0, \tilde{v})$ is a fixed point in $\{0\} \times C_2^+$

($\tilde{\mathrm{H}}$3) The basin of attraction of $\tilde{E}$ in $\{0\} \times C_2^+$ is $\{0\} \times C_2^+ \setminus E_0$

The hypotheses ($\hat{\mathrm{H}}$1)-($\hat{\mathrm{H}}$3) or ($\tilde{\mathrm{H}}$1)-($\tilde{\mathrm{H}}$3) reflect the fact that each population, given nonzero initial conditions in the appropriate space, will grow to a "fixed" element in its space.

The principal result is:

Theorem 2.1. Suppose that (H1)-(H2), ($\hat{\mathrm{H}}$1)-($\hat{\mathrm{H}}$3) and ($\tilde{\mathrm{H}}$1)-($\tilde{\mathrm{H}}$3) hold. Assume E_0 and $\tilde{E}$ are repellers with respect to $\mathring{C}_1^+ \times \mathring{C}_2^+$. If $\hat{E}$ is a local attractor, then $\hat{E}$ is a global attractor of $\mathring{C}_1^+ \times \mathring{C}_2^+$ provided there are no rest points in $\mathring{C}_1^+ \times \mathring{C}_2^+$.

3. The Proof

Before we prove Theorem 2.1, we establish the following lemma:

LEMMA 3.1. Under the assumptions of Theorem 2.1, $T(t)$ is dissipative and the set

$$A = \{(u, v) \in C_1^+ \times C_2^+ : \tilde{E} \leq (u, v) \leq \hat{E}\}$$

is positively invariant and attracts all orbits.

PROOF. A simple comparison argument shows that T is dissipative on $C_1^+ \times C_2^+$. From Remark 2.1 we may consider (u_0, v_0) in $\mathring{C}_1^+ \times \mathring{C}_2^+$. Let $T(t)(u_0, v_0) = (u(t), v(t))$. From (H2) we have $T(t)(u_0, v_0) \leq_K T(t)(u_0, 0)$ and from ($\hat{\mathrm{H}}$1)-($\hat{\mathrm{H}}$3) $\lim_{t\to\infty} T(t)(u_0, 0) = (\hat{u}, 0)$. Similarly $(0, v_0) \in \{0\} \times \mathring{C}_2^+$ and $(0, v_0) \leq (u_0, v_0)$. From (H2) and ($\tilde{\mathrm{H}}$1)-($\tilde{\mathrm{H}}$3), we have $T(t)(0, v_0) \leq_K T(t)(u_0, v_0)$ and $\lim_{t\to\infty} T(t)(0, v_0) = (0, \tilde{v})$. Hence the orbit of (u_0, v_0) tends to the bounded set $A = \{(u, v) \in C_1^+ \times C_2^+ : \tilde{E} \leq (u, v) \leq \hat{E}\}$. The positive invariance of A under the flow $T(t)$ follows from (H2). Since $T(t)\tilde{E} \leq_K T(t)(u, v) \leq_K T(t)\hat{E}$ for all $t > 0$, $(u, v) \in A$, then $\tilde{E} \leq_K T(t)(u, v) \leq_K \hat{E}$, or $T(t)(u, v) \in A$ for all $t > 0$, $(u, v) \in A$.

REMARK 3.2. The set A contains a global attractor [3, p. 40].

PROOF OF THEOREM 2.1.

Let $W^+(\hat{E})$ be the basin of attraction of the rest point $\hat{E}$ and B be the boundary of $W^+(\hat{E})$. From Lemma 3.1 and Remark 2.1, it suffices to show that $(u, v) \in W^+(\hat{E})$ for any (u, v) in $\mathring{C}_1^+ \times \mathring{C}_2^+$ with $u \ll_{C_1} \hat{u}$, $v \ll_{C_2} \tilde{v}$. The proof is established by contradiction. Suppose there exists $z = (z_1, z_2) \in \mathring{C}_1^+ \times \mathring{C}_2^+$, $z_1 \ll_{C_1} \hat{u}$, $z_2 \ll_{C_2} \tilde{v}$, $z \notin W^+(\hat{E})$. Obviously $z \gg \tilde{E}$. From [5], Theorem 5.2, there are at most countable number of nonconvergent points of a totally ordered set. We note that, in order to apply Hirsch's result, C_i, $i = 1, 2$ are assumed to be separable. From the assumptions that $\tilde{E}$, E_0 are repellers with respect to the interior of the cone and there is no rest point in the interior, every convergent point must be in $W^+(\hat{E})$. Choose $s > 0$ such that $(1 - s)\tilde{E} + sz \in W^+(\hat{E})$. From (H2) and the fact that $(1 - s)\tilde{E} + sz \ll z \ll \hat{E}$, it follows that $\lim_{t\to\infty} T(t)z = \hat{E}$. Thus, $z \in W^+(\hat{E})$, which is the desired contradiction.

4. Applications

In this section we shall apply Theorem 2.1 to the problem (1.1)–(1.3) and obtain new results for each. First we consider the reduced problem (1.1) from the unstirred chemostant with equal diffusions.

Let $T(t)$ be the operator defined by the flow generated by (1.1).

$$T(t)\colon C_1^+ \times C_2^+ \longrightarrow C_1^+ \times C_2^+,$$

$$C_1 = C_2 = C([0, 1], \mathbb{R})$$

$$T(t)(u_0, v_0) = (u(\cdot\ , t), v(\cdot\ , t))$$

From [3], the operator $T(t)$ is compact. Consider the following case in [6]

$$m_1 > d\lambda_0,\ m_2 > d\mu_0 \tag{4.1}$$

where $\lambda_0 > 0$, $\mu_0 > 0$ are the first eigenvalues of

$$\psi'' + \lambda\left(\frac{\varphi(x)}{a_1 + \varphi(x)}\right)\psi = 0$$

$$\psi'(0) = 0,\ \psi'(1) + r\psi(1) = 0$$

and

$$\psi'' + \mu\left(\frac{\varphi(x)}{a_2 + \varphi(x)}\right)\psi = 0$$

$$\psi'(0) = 0,\ \psi'(1) + r\psi(1) = 0$$

respectively.

Under the conditions (4.1), it follows (Theorem 3.2 [6]) that the equilibrium $\hat{E} = (\hat{u}, 0)$ and $\tilde{E} = (0, \tilde{v})$ attract each point in $\mathring{C}_1^+ \times \{0\}$ and $\{0\} \times \mathring{C}_2^+$ respectively. Thus ($\hat{\mathrm{H}}$1)-($\hat{\mathrm{H}}$3) and ($\tilde{\mathrm{H}}$1)-($\tilde{\mathrm{H}}$3) hold. The strongly order preserving of $T(t)$ follows directly from the competitive properties of the reaction terms in (1.1) and the maximum principle [7]. Thus (H1)-(H2) hold. If $\hat{E}$ is asymptotically stable and $\tilde{E}$ is unstable (i.e. $m_2 < m_2^*$ and $m_1 > m_1^*$ in [6]), then, from the proof of Theorem 4.1 in [6], it follows that $\tilde{E}$ is a repeller with respect to the interior $\mathring{C}_1^+ \times \mathring{C}_2^+$. Since E_0 is obviously a repeller, $\hat{E}$ is globally asymptotically stable if and only if there is no interior equilibrium in $\mathring{C}_1^+ \times \mathring{C}_2^+$.

We conjecture there is no interior equilibrium provided $\hat{E}$ is locally asymptotically stable and $\tilde{E}$ is unstable. In the following we provide a sufficient condition for the nonexistence of the positive solutions for the steady state problem of (1.1),

$$du'' + \frac{m_1(\varphi(x) - u - v)}{a_1 + (\varphi(x) - u - v)}u = 0, \tag{4.2}$$

$$
\begin{aligned}
&dv'' + \frac{m_2(\varphi(x) - u - v)}{a_2 + (\varphi(x) - u - v)} v = 0, \\
&u'(0) = v'(0) = 0, \\
&u'(1) + ru(1) = 0, \ v'(1) + rv(1) = 0,
\end{aligned}
$$

subject to the condition

$$
S(x) = \varphi(x) - u(x) - v(x) > 0, \qquad \text{for } 0 \le x \le 1. \tag{4.3}
$$

Let $(u(x), v(x))$ be a positive solution of (4.2), (4.3) and

$$
w_1 = \frac{u'}{u}, \ w_2 = \frac{v'}{v}.
$$

From (4.2), (4.3), we have

$$
\begin{aligned}
&w_1' + w_1^2 = -\frac{1}{d}\frac{m_1 S}{a_1 + S}, \qquad 0 < x < 1, \\
&w_1(0) = 0, \ w_1(1) = -r,
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
&w_2' + w_2^2 = -\frac{1}{d}\frac{m_2 S}{a_2 + S}, \qquad 0 < x < 1, \\
&w_2(0) = 0, \ w_2(1) = -r.
\end{aligned} \tag{4.5}
$$

If

$$
\frac{m_1 S}{a_1 + S} > \frac{m_2 S}{a_2 + S} \qquad \text{for all } S > 0, \tag{4.6}
$$

then (4.4), (4.5) and the standard comparison theorem lead to $w_1(x) < w_2(x)$ for $0 < x \le 1$. Then $w_1(1) < w_2(1)$ is a desired contradiction. Next we need to verify that, under the assumption (4.6), $\hat{E}$ is locally asymptotically stable and $\tilde{E}$ is locally unstable. From [6], $\hat{E}$ is locally asymptotically stable if and only if $\hat{\lambda}(m_2) < 0$ where

$$
\hat{\lambda}(m_2) = \sup_{q>0} \left(\frac{-\mathrm{drq}^2(1) - d\int_0^1 q^2(x)dx + \int_0^1 \frac{m_2 \hat{S}(x)}{a_2 + \hat{S}(x)} q^2(x)dx}{\int_0^1 q^2(x)dx} \right) \tag{4.7}
$$

and $\hat{S}(x) = \varphi(x) - \hat{u}(x)$.

From the assumption (4.6) and (4.7), one has

$$(4.8) \qquad \hat{\lambda}(m_2) < \sup_{q>0} \left(\frac{-\operatorname{drq}^2(1) - d\int_0^1 q^2(x)dx + \int_0^1 \frac{m_1\hat{S}(x)}{a_1+\hat{S}(x)} q^2(x)dx}{\int_0^1 q^2(x)dx} \right)$$

Since the right hand side of (4.8) is the largest eigenvalue of

$$dq'' + \frac{m_1\hat{S}(x)}{a_1+\hat{S}(x)} q = \lambda q,$$

$$q'(0)=0,\ q'(1)+rq(1)=0,$$

the fact that it is zero follows directly from the identity

$$d\hat{u}'' + \frac{m_1\hat{S}(x)}{a_1+\hat{S}(x)} \hat{u} = 0.$$

From [6] $\tilde{E}$ is unstable if and only if $\tilde{\lambda}(m_1) > 0$. $\tilde{\lambda}(m_1) > 0$ follows by a similar argument.

We note that Theorem 3.7 of [6], we showed that under the assumption (4.6), $v(x, t)$ decays to zero exponentially and $u(x, t)$ converges to $\hat{u}(x)$ as $t \to \infty$. The result is consistent with the present one and, indeed, it is a special case.

Next we consider the classical Lotka-Volterra two-species competition model with diffusions and Dirichlet boundary conditions (1.2).
Let $T(t)$ be the operator defined by the flow generated by (1.2)

$$T(t)\colon C_1^+ \times C_2^+ \longrightarrow C_1^+ \times C_2^+$$

$$T(t)(u_0, v_0) = (u(\cdot\,, t), v(\cdot\,, t))$$

where $C_1 = C_2 = C_0(\Omega) = \{f \in C(\Omega) \colon f|_{\partial\Omega} = 0\}$ is a separable space. As in example (1.1), the operator $T(t)$ is compact and strongly order preserving. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. From [1], if $a > \lambda_1$ then there exists a unique nontrivial nonnegative steady state $\hat{u}$ of

$$Lu_t = \Delta u + u(a-u) \qquad \text{in } \Omega \times \mathbb{R}^+,$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+,$$

and $\hat{u}$ attracts each point in $C_0^+(\Omega)$ [3]. Similarly if $d > \lambda_1$, then there exists a unique nontirivial nonnegative steady state $\tilde{v}$ of

$$v_t = \Delta v + v(d-v) \qquad \text{in } \Omega \times \mathbb{R}^+,$$

$$v = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+,$$

and $\tilde{v}$ attracts each point in $C_0^+(\Omega)$. Hence ($\hat{\mathrm{H}}$1)-($\hat{\mathrm{H}}$3) and ($\tilde{\mathrm{H}}$1)-($\tilde{\mathrm{H}}$3) hold. It is proved in [1] that there exists $\hat{e} > 0$ such that $(\hat{u}, 0)$ is locally stable if $e > \hat{e}$ and is unstable if $e < \hat{e}$. Similarly there exists $\tilde{c} > 0$ such that $(0, \tilde{v})$ is locally asymptotically stable if $c > \tilde{c}$ and is unstable if $c < \tilde{c}$. Consider the case $(\hat{u}, 0)$ is locally asymptotically stable and $(0, \tilde{v})$ is unstable, i.e., $e > \hat{e}$, $0 < c < \tilde{c}$. It can be verified that $(0, \tilde{v})$ is a repeller with respect to the interior of $C_0^+(\Omega) \times C_0^+(\Omega)$. Then Theorem 2.1 says that $(\hat{u}, 0)$ is globally stable iff there is no positive solution of the elliptic problem.

$$(4.9) \qquad \begin{aligned} &-\Delta u = u(a - u - cv) \\ &-\Delta v = v(d - v - eu) \end{aligned} \quad \text{in } \Omega \qquad\qquad u = v = 0 \quad \text{on } \partial\Omega$$

It is interesting to note that in Theorem 2 (ii) of [1] Dancer shows for almost all (a, d) in $(\lambda_1, \infty) \times (\lambda_1, \infty)$ there exist $c > 0$ and $e > 0$ such that $(\hat{u}, 0)$ is locally asymptotically stable and $(0, \tilde{v})$ is unstable, and there exists a positive solution for (4.9). Thus the global stability of $(\tilde{u}, 0)$, in general, does not hold and Theorem 2.1 is of limited applicability.

The third application of Theorem 2.1 is the study of the reduced problem (1.3) of the two-species delayed chemostat model (1.3)′,
Let $T(t)$ be the operator defined by the flow generated by (1.3)

$$T(t)\colon C_1^+ \times C_2^+ \longrightarrow C_1^+ \times C_2^+,$$
$$T(t)(\varphi_1, \varphi_2) = ((z_1)_t, (z_2)_t),$$

where for $i = 1, 2$

$$C_i = C([-\tau_i, 0], \mathbb{R})$$
$$(z_i)_t(\theta) = z_i(t + \theta), \qquad -\tau_i \le \kappa \le 0.$$

From [4], the operator $T(t)$ is compact for $t > \max(\tau_1, \tau_2)$.
Let $\lambda_i > 0$, $i = 1, 2$ satisfy

$$\frac{m_i \lambda_i}{a_i + \lambda_i} = D e^{D\tau_i}.$$

The basic assumption for the two-species delayed chemostat equation is

$$(4.10) \qquad 0 < \lambda_1 < \lambda_2 < S^{(0)}.$$

From [2], under the assumption (4.10) the equilibrium $(E)_1 = (\lambda_1, x_1^*, 0)$ of (1.3)′, $\lambda_1 + x_1^* = S^{(0)}$, is a local attractor and the equilibrium $(E)_2 = (\lambda_2, 0, x_2^*)$ of (1.3)′, $\lambda_2 + x_2^* = S^{(0)}$, is a repeller with respect to the interior of the positive cone $C_0^+ \times C_1^+ \times C_2^+$ where $C_0 = C([-\tau, 0], \mathbb{R})$, $\tau = \max(\tau_1, \tau_2)$. For the reduced system (1.3), the equilibrium $(x_1^*, 0)$ is locally stable and the equilibrium $(0, x_2^*)$ is a repeller with respect to the interior of the cone $C_1^+ \times C_2^+$. Since the flow generated by (1.3) is eventually strongly monotone ([8] Section 4) with repect to the partial order $\leq$ where

$$\begin{aligned}
&(\varphi_1, \varphi_2) \leq (\psi_1, \psi_2) \text{ if and only if} \\
&\varphi_1(\theta) \leq \psi_1(\theta) \qquad \text{for all} \quad -\tau_1 \leq \theta \leq 0, \\
&\varphi_2(\theta) \geq \psi_2(\theta) \qquad \text{for all} \quad -\tau_2 \leq \theta \leq 0,
\end{aligned}$$

[9], the assumption (H1)-(H2) hold. From [1] and (4.10), $(x_1^*, 0)$ and $(0, x_2^*)$ attract each point in $C_1^+ \times \{0\}$ and $\{0\} \times C_2^+$ respectively. Hence $(\hat{\mathrm{H}}1)$-$(\hat{\mathrm{H}}3)$ and $(\tilde{\mathrm{H}}1)$-$(\tilde{\mathrm{H}}3)$ hold. Since there exists no positive equilibrium, from Theorem 2.1 it follows that $(x_1^*, 0)$ is a globalattractor in $\mathring{C}_1^+ \times \mathring{C}_2^+$ or $(E)_1$ is a global attractor in $\mathring{C}_0^+ \times \mathring{C}_1^+ \times \mathring{C}_2^+$ for system (1.3).

References

[1] E. N. Dancer, On the existence and uniqueness of positive solutions for competing species models with diffusion, Tran. A.M.S., **36** (1991), 829–859.

[2] S. Ellermeyer, Competition in the chemostat: global asymptotic behavior of a model with delayed response in growth, SIAM J. Applied Math, to appear

[3] J. Hale, Asymptotic Behavior of Dissipative System, American Mathematical Society, Providence, 1988.

[4] J. Hale, Theory of Functional Differential Equation, Springer-Verlag, Berlin, 1977.

[5] M. W. Hirsch, The dynamical system approach to differential equations, Bull. A.M.S., **11** (1984), 1–64.

[6] S. B. Hsu and P. Waltman, On a system of reaction-diffusion equations arising from competition in an unstirred chemostat, SIAM J. Applied Math, **53** (1993), 1026–1044.

[7] H. Matano, Strong comparison principle in nonlinear parabolic equations, Nonlinear Parabolic Equations: Qualitative Properties of Solutions (Boccardo, L.; Tesei, A; eds), p. 148–155 Pitman Reseach Notes in Mathematics 149, 1987.

[8] H. Smith, Monotone semiflows generated by functional differential equations, J. Diff. Eqns., **66** (1987), 420–442.

[9] H. Smith and H. Thieme, Strongly order preserving semi-flows generated by functional differential equations, J. Diff. Eqns., **93** (1991), 332–363.

[10] H. R. Thieme, Convergence results and a Poincare Bendixon trichotomy for asymptotically autonomous differential equations, J. Math. Biology, **30** (1992), 755–763.

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