HIROSHIMA MATH. J.
24 (1994), 423–434

Oscillatory properties of systems of neutral differential equations

Dedicated to Professor Takaŝi Kusano on his sixtieth birthday

Anatoli F. IVANOV and Pavol MARUŠIAK

(Received November 16, 1992)

Abstract. We study oscillatory properties of solutions and existence of nonoscillatory solutions with a power growth at the infinity for the system of differential equations of neutral type

$$\frac{d^{n_i}}{dt^{n_i}}[x_i(t) - a_i(t)x_i(h_i(t))] = p_i(t)f_i(x_{3-i}(g_i(t))), \qquad n_i \in \mathbb{N},\ i = 1, 2.$$

1. Introduction

In this paper we consider systems of neutral differential equations of the form

$$\frac{d^{n_i}}{dt^{n_i}}[x_i(t) - a_i(t)x_i(h_i(t))] = p_i(t)f_i(x_{3-i}(g_i(t))), \qquad n_i \in \mathbb{N},\ i = 1, 2. \tag{S}$$

The following conditions are assumed to hold without further mention:

(a) $a_i, h_i, g_i, p_i\colon \mathbb{R}^+ \to \mathbb{R}$, $f_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are continuous functions;
(b) $h_i(t) \le t$ for $t \in \mathbb{R}^+$, $\lim_{t\to\infty} h_i(t) = \infty$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2$;
(c) $zf_i(z) > 0$ for $z \ne 0$.

We put

$$x_i(t) - a_i(t)x_i(h_i(t)) = u_i(t), \qquad i = 1, 2.$$

For $t_0 \ge 0$ denote

$$t_1 = \min\{\inf_{t\ge t_0} h_i(t),\ \inf_{t\ge t_0} g_i(t),\ i = 1, 2\}.$$

A vector function $X = (x_1, x_2)$ is defined to be a solution of system (S) if there exists a $t_0 \ge 0$ such that X is continuous on $[t_1, \infty)$, u_i is n_i times

The first author (A.F.I.) has been supported by the Alexander von Humboldt Stiftung during his stay at the Mathematisches Institut der Ludwig-Maximilians-Universität, München. He is grateful to both institutions for their support and hospitality.

continuously differentiable in $[t_0, \infty)$, $i = 1, 2$, and X satisfies system (S) on $[t_0, \infty)$.

Solution $X = (x_1, x_2)$ is called a *proper solution* if $\sup\{|x_1(t)| + |x_2(t)|, t \geq T\} > 0$ for any $T \geq 0$.

A proper solution X of (S) is defined to be *nonoscillatory* if there exists a $t_0 \geq 0$ such that every its component is different from zero for all $t \geq t_0$. A proper solution X is defined as *oscillatory* otherwise.

Present paper consists of two parts. In the first part we prove the existence of nonoscillatory solutions of system (S) with a polynomial growth at the infinity. An additional assumption is made that

$$a_i(t) = \lambda_i = \text{const}, \quad t - h_i(t) = \tau_i = \text{const} > 0.$$

A solution $X = (x_1, x_2)$ is said to have a polynomial growth at the infinity if there exist nonnegative numbers β_1, β_2 such that

$$\lim_{t\to\infty} \frac{x_1(t)}{t^{\beta_1}} = \text{const} \neq 0, \quad \lim_{t\to\infty} \frac{x_2(t)}{t^{\beta_2}} = \text{const} \neq 0.$$

Existence of nonoscillatory solutions of scalar equations or systems of equations of neutral type with certain asymptotic properties, including the polynomial growth solutions, has been studied in, e.g. [1–5, 7, 8].

The second part deals with the oscillatory behavior of solutions of (S). Sufficient conditions for the oscillation of all solutions of linear systems have been obtained by V. N. Shevelo et. al. [8], Győri and Ladas [1].

The results of this paper generalize those obtained in a previous paper by the authors [3].

2. Existence of nonoscillatory solutions

Throughout this section we assume the following additional to (a)-(c) conditions to hold:

$$a_i(t) = \lambda_i = \text{const} \neq \pm 1, \; h_i(t) = t - \tau_i, \; \tau_i > 0, \; i = 1, 2; \tag{2.1}$$

$$\limsup_{t\to\infty} \frac{g_i(t)}{t} \leq \sigma_i < \infty, \; \sigma_i > 0, \; i = 1, 2; \tag{2.2}$$

$|f_i(z)| \leq \delta_i |z|^{\alpha_i}$ for large $|z|$ and some positive constants $\alpha_i, \delta_i, i = 1, 2.$ (2.3)

2.1. Auxiliary transformations and Lemmas

Let $C[T, \infty)$ be the Fréchet space of continuous functions in $[T, \infty)$ with the topologyof the uniform convergence on compact subintervals.

A. Let $|\lambda| > 1$ and let $C_{\lambda,\tau}[T, \infty)$ stand for a subset of $C[T, \infty)$ consisiting of all functions $u(t)$ such that the series $\sum_{k=1}^{\infty} \lambda^{-k} u(t+k\tau)$ are uniformly convergent on every compact subinterval of $[T, \infty)$.

Define the operator $\Psi_{T,\lambda,\tau}$: $C_{\lambda,\tau}[T, \infty) \to C[T-\tau, \infty)$ as follows:

$$\Psi_{T,\lambda,\tau} u(t) = \sum_{k=1}^{\infty} \lambda^{-k} u(t+k\tau), \qquad t \geq T-\tau. \tag{2.4}$$

B. Let $|\lambda| < 1$. Define the operator $\Phi_{T,\lambda,\tau}$: $C[T, \infty) \to C[T-\tau, \infty)$ as follows:

$$\Phi_{T,\lambda,\tau} u(t) = \sum_{k=0}^{n(t)-1} |\lambda|^k u(t-k\tau) + \frac{|\lambda|^{n(t)} u(T)}{1-|\lambda|}, \qquad t \geq T,$$

$$\Phi_{T,\lambda,\tau} u(t) = \frac{u(T)}{1-|\lambda|}, \qquad T-\tau \leq t \leq T, \tag{2.5}$$

where $n(t)$ stands for the smallest integer such that $t - n(t)\tau \leq T$.

LEMMA 2.1. *If* $u \in C_{\lambda,\tau}[T, \infty)$ *then* $x = \Psi_{T,\lambda,\tau} u$ *satisfies the difference equation*

$$x(t) - \lambda x(t-\tau) = -u(t), \qquad t \geq T.$$

LEMMA 2.2. *If* $u \in C[T, \infty)$ *then* $x = \Phi_{T,\lambda,\tau} u$ *satisfies the difference equation*

$$x(t) - \lambda x(t-\tau) = u(t), \qquad t \geq T.$$

Proof of Lemmas follows immediately from (2.4) and (2.5) respectively.

LEMMA 2.3 ([4, Lemma 1.3]). *Let* $u \in C[T, \infty)$ *be positive and nonincreasing. Then for every constant* $\rho \in (0, 1)$ *there exist positive constants* c_1, c_2 *and* c_3 *depending on* λ *and* τ *only and such that*

$$\Phi_{T,\lambda,\tau} u(t) \leq c_1 u(\rho t) + c_2 u(T) \lambda^{(1-\rho)t/\tau} + c_3 u(T) \lambda^{(t-T)/tau}. \tag{2.6}$$

COROLLARY 2.1. *Let* $u(t)$ *be as in Lemma* 2.3 *and* m *be a nonnegative real number. If*

$$\int_T^{\infty} t^m u(t)dt < \infty$$

then

$$\int_T^{\infty} t^m \Phi_{T,\lambda,\tau} u(t)dt < \infty.$$

Proof of Corollary 2.1 follows from (2.6).

2.2. Solutions with a power growth

THEOREM 2.1. *Let the assumptions* (2.1)–(2.3) *hold and* $l_i \in \{0, 1, \ldots, n_i - 1\}$, $i = 1, 2$, *be given. Suppose there exist continuous nonincreasing functions* $q_i\colon [t_0, \infty) \to \mathbb{R}$, $i = 1, 2$, *such that* $|p_i(t) \le q_i(t)$ *for* $t \ge t_0$ *and*

$$\int_{t_0}^{\infty} t^{n_i - l_i - 1 + \alpha_i l_{3-i}} q_i(t)dt < \infty, \qquad i = 1, 2.$$

Then for arbitrary (b_1, b_2) $(b_1 b_2 > 0)$ *system* (S) *has a nonoscillatory solution* (x_1, x_2) *with the property*

$$\lim_{t \to \infty} \frac{x_i(t)}{t^{l_i}} = b_i, \qquad i = 1, 2.$$

PROOF. A). Let $|\lambda_i| > 1$, $i = 1, 2$. Set $T_* = \min\{\inf_{t \ge T} g_i(t),\ T - \tau_i,\ i = 1, 2\}$. For $x_i(t) \in C[T_*, \infty)$, $i = 1, 2$, define the mapping $F(X) = F((x_1, x_2)) = (F_1 X, F_2 X)$ given by

$$F_i X(t) = c_i + \frac{d_i(t-T)^{l_i}}{l_i!} + (-1)^{n_i - l_i - 1} \int_T^{\infty} \frac{(t-s)^{l_i - 1}}{(l_i - 1)!}$$
$$\times \int_s^{\infty} \frac{(r-s)^{n_i - l_i - 1}}{(n_i - l_i - 1)!} \Psi_{T, \lambda_i, \tau_i}(p_i(r) f_i(x_{3-i}(g_i(r)))\, dr ds, \qquad t \ge T,$$
$$F_i X(t) = c_i, \ T_* \le t \le T, \ i = 1, 2, \tag{2.8}$$

where $c_i \neq 0$, $d_i > 0$, $i = 1, 2$.

Assume that the mapping F has a fixed point $X^0 = (x_1^0, x_2^0)$, i.e. $X^0 = (x_1^0, x_2^0) = (F_1 X^0(t), F_2 X^0(t))$. Differentiation of (2.8) shows that

$$(x_i^0(t))^{(n_i)} = -\Psi_{T, \lambda_i, \tau_i}(p_i(t) f_i(x_{3-i}^0(g_i(t))), \qquad i = 1, 2,$$

and therefore, in view of Lemma 2.1, $(x_1^0(t), x_2^0(t))$ is a solution of system (S). From (2.8) it follows that $\lim_{t \to \infty} x_i^{(l_i)}(t) = d_i$, implying $\lim_{t \to \infty} x_i(t)/t^{l_i} = \text{const} > 0$, $i = 1, 2$.

Define subsets $W_i \subset C[T_*, \infty)$, $i = 1, 2$, as follows:

$$W_i = \left\{ w \in C[T, \infty) \colon c_i \le w(t) \le c_i + \frac{2d_i(t-T)^{l_i}}{l_i!}, \ t \ge T, \text{ and} \right.$$
$$\left. w(t) = c_i, \ T_* \le t \le T \right\}, \qquad i = 1, 2.$$

It is easy to see that W_i, $i = 1, 2$, are convex subsets of $C[T_*, \infty)$. We shall show next that for arbitrary constants $d_i > 0$, $c_i \neq 0$, $i = 1, 2$, T can be chosen in such a way that F maps $W_1 \times W_2$ into itself. The standard arguments show that $F(W_1 \times W_2)$ is relatively compact in the topology of $C[T_*, \infty) \times C[T_*, \infty)$. Therefore, the Schauder-Tychonoff theorem is applied to derive the existence of a fixed point of F.

Take $x_i \in W_i$, $i = 1, 2$. Then in view of (2.2), (2.3) we obtain

$$|\Psi_{T,\lambda_i,\tau_i}(p_i(t) f_i(x_{3-i}^0(g_i(t)))| \leq \sum_{k=1}^{\infty} |\lambda_i|^{-k} q_i(t + k\tau_i) f_i(k_{3-i}(g_i(t + k\tau_i)))$$

$$\leq q_i(t) \sum_{k=1}^{\infty} |\lambda_i|^{-k} f_i(k_{3-i}(g_i(t + k\tau_i))) \leq \delta_i q_i(t) \sum_{k=q}^{\infty} |\lambda_i|^{-k} g_i^{\alpha_i l_{3-i}}(t + k\tau_i)$$

$$\leq \delta_i \sigma_i^{\alpha_i l_{3-i}} q_i(t) \sum_{k=1}^{\infty} |\lambda_i|^{-k} (t + k\tau_i)^{\alpha_i l_{3-i}} \leq \gamma_i q_i(t) t^{\alpha_i l_{3-i}}, \qquad i = 1, 2,$$

where

$$\gamma_i = \delta_i \sigma_i^{\alpha_i l_{3-i}} \sum_{k=1}^{\infty} |\alpha_i|^{-k} (1 + k)^{\lambda_i l_{3-i}}, \qquad i = 1, 2.$$

In view of (2.7) for every $d_i' > 0$, $i = 1, 2$, there exists a $T > 0$ such that

$$\left|(-1)^{n_i - l_i - 1} \int_T^{\infty} \frac{(t - s)^{l_i - 1}}{(l_i - 1)!} \int_s^{\infty} \frac{(r - s)^{n_i - l_i - 1}}{(n_i - l_i - 1)!} \Psi_{T,\lambda_i,\lambda_i,\tau_i}(p_i(r) f_i(x_{3-i}(g_i(r)))dr\,ds\right|$$

$$\leq \gamma_i \int_T^{\infty} \frac{(t - s)^{l_i - 1}}{(l_i - 1)!} ds \int_T^{\infty} \frac{(r - s)^{n_i - l_i - 1}}{(n_i - l_i - 1)!} r^{\alpha_i l_{i-3}} q_i(r) dr$$

$$\leq \gamma_i \int_T^{\infty} \frac{(t - s)^{l_i - 1}}{(l_i - 1)!} ds \int_T^{\infty} r^{n_i - l_i - 1 + \alpha_i l_{3-i}} q_i(r) dr \leq d_i'(t - T)^{l_i}, \; t \geq T, \; i = 1, 2.$$

It follows that $F_i X_i \subset W_i (i = 1, 2)$ provided $x_i \in W_i (i = 1, 2)$. This shows that $F(W_1 \times W_2) \subset W_1 \times W_2$. Therefore, F has a fixed point.

B). Case $|\lambda_i| < 1$, $i = 1, 2$. Let T_* be defined as above and $x_i \in C[T_*, \infty)$, $i = 1, 2$. Define the mapping $F(X) = F((x_1, x_2)) = (F_1 X, F_2 X)$ as follows:

$$F_i(X(t) = c_i + \frac{d_i(t - T)^{l_i}}{l_i!} + (-1)^{n_i - l_i} \int_T^{\infty} \frac{(t - s)^{l_i - 1}}{(l_i - 1)!}$$

$$\times \int_s^{\infty} \frac{(r - s)^{n_i - l_i - 1}}{(n_i - l_i - 1)!} \Phi_{T,\lambda_i,\tau_i}(p_i(r) f_i(x_{3-i}(g_i(r)))\,dr\,ds, \qquad t \geq T, \tag{2.9}$$

$$F_i X(t) = c_i, \; T_* \leq t \leq T, \; i = 1, 2,$$

where $c_i \neq 0$, $d_i > 0$, $i = 1, 2$. If the mapping F has a fixed point (x_1^0, x_2^0)

then differentiation of (2.9) shows that

$$(x_i^0(t))^{(n_i)} = \Phi_{T,\lambda_i,\tau_i}(p_i(t)f_i(x_{3-i}^0(g_i(t)))), \qquad i = 1, 2,$$

and Lemma 2.1 shows that $(x_1^0(t), x_2^0(t))$ is a solution of (S). From equation (2.9) it follows that $\lim_{t\to\infty} x_i(t)/t^{l_i} = \text{const} > 0$, $i = 1, 2$.

Let subsets W_i be given as above. Take $x_i \in W_i$, $i = 1, 2$. Then in view of (2.2), (2.3), (2.7) and Corrolary 2.1 we obtain

$$\left|\int_s^\infty \frac{(r-s)^{n_i-l_i-1}}{(n_i-l_i-1)!}\Phi_{T,\lambda_i,\tau_i}(p_i(r)f_i(x_{3-i}(g_i(r)))dr\right|$$
$$\leq \gamma_i \int_s^\infty \frac{(r-s)^{n_i-l_i-1}}{(n_i-l_i-1)!}\Phi_{T,\lambda_i,\tau_i}(q_i(r)|x_{3-i}^{\alpha_i}(g_i(r))|)dr\,|$$
$$\leq \gamma_i \int_T^\infty r^{n_i-l_i-1+\alpha_i l_{3-i}}\Phi_{T,\lambda_i,\tau_i}(g_i(r))dr < \infty, \qquad i = 1, 2.$$

The last inequality shows that for arbitrary $d_i' > 0$, $i = 1, 2$, there exists a $T > 0$ such that for all $s \geq T$

$$\left|\int_s^\infty \frac{(r-s)^{n_i-l_i-1}}{(n_i-l_i-1)!}\Phi_{T,\lambda_i,\tau_i}(p_i(r)f_i(x_{3-i}(g_i(r)))dr\right| \leq d_i', \qquad i = 1, 2.$$

As for the case A) this implies $F(W_1 \times W_2) \subset W_1 \times W_2$ provided $x_i \in W_i$ and $T > 0$ is sufficiently large. As before, F is continuous and $F(W_1 \times W_2)$ is relatively compact in the topology $C[T_*, \infty) \times C[T_*, \infty)$. Therefore, F has a fixed point, which gives a nonoscillatory solution of (S). This completes the proof of the theorem.

3. Oscillation criteria

In addition to the assumptions (a)-(c) we suppose

$$|a_i(t)| \leq \lambda_i < 1,\ a_i(t)a_i(h_i(t)) \geq 0, \qquad i = 1, 2; \tag{3.1}$$

$$p_1(t) = \bar{p}_1(t),\ p_2(t) = \sigma\bar{p}_2(t),\ \bar{p}_2(t) > 0, \qquad t \geq 0,\ \sigma \in \{-1, +1\}; \tag{3.2}$$

For every $d > 0$ there exist $\delta_i > 0$, $i = 1, 2$,

$$\text{such that } \inf\{|f_i(z)| : |z| > d\} > \delta_i, \qquad i = 1, 2. \tag{3.3}$$

REMARK 3.1. System (S) whose coefficients p_i, $i = 1, 2$, satisfy (3.2) will be denoted (S_σ).

In the sequel the following Lemma (called Kiguradze's Lemma) will be used.

LEMMA 3.1 ([6]). *Let* $z \in C^m[t_0, \infty)$ *be such that* $z(t) \neq 0$, $vz(t) \cdot z^m(t) \geq 0$ *for* $t \geq t_0$, $v \in \{-1, +1\}$. *Then there exists an integer* $l \in \{0, 1, \dots, m\}$ *with* $v(-1)^{l+m} = 1$ *and* $T_0 \geq t_0$ *such that for* $t \geq T_0$ *one has*

$$z(t)z^{(k)}(t) > 0 \qquad \text{for } k = 0, 1, \dots, l,$$
$$(-1)^{l+k} z(t)z^{(k)}(t) \geq 0 \qquad \text{for } k = l+1, \dots, m.$$

COROLLARY 3.1. *If under assumptions of Lemma* 3.1 $\lim_{t\to\infty} z(t) = 0$ *then* $z^{(k)}(t)$ $(k = 0, 1, \dots, m)$ *tend monotonically to zero as* $t \to \infty$.

Denote

$$\gamma(h) = \sup\{s\colon h_i(s) \leq t,\ g_i(t) \leq t,\ i = 1, 2,\ t \geq 0\}.$$

Let (x_1, x_2) be a nonoscillatory solution of system (S_σ). Let (3.1)–(3.3) and (1.1) hold. Then in view of Lemma 3.1 from (S_σ) we get for all sufficiently large t either

$$x_i(t)u_i(t) > 0, \qquad i \in \{1, 2\}, \tag{3.4}$$

or

$$x_i(t)u_i(t) < 0, \qquad i \in \{1, 2\}. \tag{3.5}$$

Denote by N^+ (respectively N^-) the set of components of all nonoscillatory solutions (x_1, x_2) of system (S_σ) such that (3.4) (respectively (3.5)) is satisfied.

LEMMA 3.2. *Let* $x_i \in N^+$, $i \in \{1, 2\}$, *and* (1.1), (3.1) *hold.*

(a) *If* $x_i(t)u_i'(t) > 0$ *for* $t \geq t_0$, *then there exist constants* b_i *and* $T_0 > t_0$ *such that* $x_i b_i > 0$ *and*

$$|x_i(t)| \geq |u_i(t)|(1 - \lambda_i) \geq |b_i|(1 - \lambda_i) > 0, \qquad i = 1, 2,\ t \geq T_0. \tag{3.6}$$

(b) *Let* $x_i(t)u_i'(t) < 0$ *and* $|u_i(t)| \geq \delta > 0$ *for* $t \geq t_0$. *Then there exist* $\varepsilon > 0$ *and* $T_1 \geq t_0$ *such that*

$$x_i(t) \geq \varepsilon_1 |u_i(t)| \geq \varepsilon_1 \delta = \varepsilon > 0 \qquad \text{for } t \geq T_1, \qquad i = 1, 2. \tag{3.7}$$

PROOF. (a) Without loss of generality we assume that $x_1(t) > 0$, $u_1(t) > 0$ and $u_1'(t) > 0$ for $t \geq t_0$. The last two inequalities imply that there exist b_1 and $t_1 \geq t_0$ such that $u_1 \geq b_1$ for $t \geq t_1$. Then (1.1) together with (3.1) gives $x_1(t) \geq u_1(t) + a_1(t)u_1(h_1(t)) \geq u_1(t)(1 - \lambda_1) \geq b_1(1 - \lambda_1) > 0$ for $t \geq \gamma(\gamma(t_1)) = T_0$.

(b) Let $x_1(t) > 0$, $u_1'(t) < 0$ and $u_1(t) \geq \delta > 0$ for $t \geq t_0$. Choose δ_1: $1 < \delta_1 \leq 1/\lambda_1$. Then there exists $t_2 \geq t_1$ such that $0 < \delta \leq u_1(t) \leq u_1(h_1(t)) \leq \delta\delta_1$. The last inequality, in view of (1.1) and(3.1), implies $x_1(t) \geq u_1(t) + a_1(t)u_1(h_1(t)) \geq u_1(t)(1 - \lambda_1\delta_1) \geq \delta(1 - \lambda_1\delta_1) = \varepsilon > 0$ for $t \geq \gamma(\gamma(t_2)) = T_1$.

LEMMA 3.3. *Let* $x_i \in N^-$, $i = 1, 2$, *and* (1.1), (3.1) *hold. Then* $\lim_{t\to\infty} u_i(t) = 0$, $\lim_{t\to-\infty} x_i(t) = 0$, $i = 1, 2$.

PROOF. Without loss of generality we assume that $x_1(t) > 0$, $u_1(t) < 0$ for $t \geq t_0$. Then (1.1) in view of (3.1) implies $0 < x_1(t) < a_1(t)x_1(h_1(t)) \leq x_1(h_1(t))$ for $t \geq t_0$. Function $x_1(t)$ is nonincreasing and therefore, $\lim_{t\to\infty} x_1(t) = c \geq 0$. Then with regard to $0 < x_1(t) < \lambda_1 x_1(h_1(t))$ we have $c \leq \lambda_1 c$, a contradiction to $0 < \lambda_1 < 1$. Therefore $c = 0$. Then (1.1) together with (3.1) implies $\lim_{t\to\infty} u_2(t) = 0$.

REMARK 3.4. The case $x_i \in N^-$ can occur only if $a_i(t) > 0$ and $\nu(-1)^{n_i} = 1$, $i = 1, 2$.

LEMMA 3.4. *Let* a_i, $\bar{p}_i$ *and* f_i, $i = 1, 2$, *satisfy the assumptions* (3.1)–(3.3) *and let*

$$w_i(t) = z(t) - a_i(t)z(h_i(t)), \qquad i = 1, 2,$$

where $z(t)$ *is a solution of the equation*

$$\nu w^{(m)}(t) = \bar{p}_i(t) f_i(z(g_i(t))), \qquad i = 1, 2,\ t \geq t_0, \tag{E}$$

where $m \in \mathbb{N}$, $\nu \in \{-1, +1\}$. *If* $z(t) \geq d > 0$ $(z(t) \leq d < 0)$ *for* $t \geq t_0$ *and*

$$\int_{t_0}^{\infty} p_i(t)dt = \infty, \qquad i = 1, 2, \tag{3.8}$$

then

$$\lim_{t\to\infty} w_i^{(k)}(t) = \nu\infty \ \ (\lim_{t\to\infty} w_i^{(k)}(t) = -\nu\infty), \quad k = 0, 1, \dots, m-1,\ i = 1, 2. \tag{3.9}$$

PROOF. Assume $z(t) \geq d > 0$. Then with regard to (3.3) there exist $\delta > 0$ and $T_1 \geq t_0$ such that $f_i(z(g_i(t))) \geq \delta$ for $t \geq T_1$, $i = 1, 2$. Integrating equation (E) from T_1 to t and using the last inequality we obtain

$$\nu[w_i^{(m-1)}(t) - w_i^{(m-1)}(T_1)] \geq \delta \int_T^t p_i(s)ds, \qquad i = 1, 2.$$

This last inequality together with (3.8) implies (3.9).

THEOREM 3.1. *Let* $\sigma = -1$ *and let the assumptions* (3.1)–(3.3) *and* (3.8) *hold. Then every proper solution* (x_1, x_2) *of* (S_{-1}) *is either oscillaltory or* $u_i^{(k_i)}(t)$ $(k_i = 0, 1, \dots, n_i,\ i = 1, 2)$ *tend monotonically to zero as* $t \to \infty$.

PROOF. Let $\sigma = -1$ and let (x_1, x_2) be a nonoscillatory solution of (S_{-1}) defined in $[t_0, \infty)$.

A). Assume first that $x_i \in N^+$ and $x_i(t) > 0$ for $t \geq t_0$, $i = 1, 2$. (The proofs for cases $x_i(t) < 0$, $t \geq t_0$, $i = 1, 2$, and $x_1(t)x_2(t) < 0$ are similar.) Then in view of (3.1)–(3.3) we obtain from system (S): $u_i(t) > 0$, $i = 1, 2$, $u_1^{(n_1)}(t) > 0$, $u_2^{(n_2)}(t) < 0$ for $t \geq t_1 := \gamma(t_0)$.

(i) Let n_1 be odd and n_2 be either odd or even. Then by Lemma 3.1 there exists $t_2 \geq t_1$ such that $u_1' > 0$ for $t \geq t_2$. With regard to Lemma 3.2 there exist $a_1 > 0$ and $t_3 \geq t_2$ such that $x_1(t) \geq a_1$ for $t \geq t_3$. Using Lemma 3.4 we obtain $\lim_{t \to \infty} u_2(t) = -\infty$, which contradicts $u_2(t) > 0$ for $t \geq t_2$.

(ii) Let n_i, $i = 1, 2$, be even. By Lemma 3.1 there exists $t_2 \geq t_0$ such that $u_2' > 0$ for $t \geq t_2$. Using Lemma 3.2 and Lemma 3.4 we get $\lim_{t \to \infty} u_2(t) = \infty$. Then, in view of Lemma 3.1, $u_1' > 0$ for large t. Then we can proceed the same way as for the case (i) to get a contradiction.

(iii) Let n_1 be even and n_2 be odd. Then by Lemma 3.1 either $u_i'(t) > 0$ or $u_i'(t) < 0$ for $t \geq t_2$. If $u_1'(t) > 0$ we have the case (i), and if $u_2'(t) > 0$ we have the case (ii), which lead to the same contradictions.

Let $u_i'(t) < 0$, $i = 1, 2$, for $t \geq t_2$. Then since $u_i(t) > 0$, $i = 1, 2$, for $t \geq t_0$ there exist $\lim_{t \to \infty} u_i(t) = a_i$, $i = 1, 2$. Assume $a_i > 0$. Then by Lemma 3.2 and Lemma 3.4 we get $\lim_{t \to \infty} u_1(t) = \infty$, $\lim_{t \to \infty} u_2(t) = -\infty$, which contradicts the fact that $u_i(t)$, $i = 1, 2$, are bounded. Therefore $a_i = 0$, $i = 1, 2$ and in view of Corollary 3.1 (P_1) holds ($n + m$ is odd for this case).

B). Let $x_i(t) \in N^-$, $i = 1, 2$ ($a_i(t) > 0$, $i = 1, 2$, for $t \geq t_0$). Assume $x_i(t) < 0$ for $t \geq t_0$, $i = 1, 2$ (the proofs for the cases $x_i < 0$, $i = 1, 2$, and $x_1(t)x_2(t) < 0$ for large t are similar). From system (S) with regard to (3.1)–(3.2) we obtain $u_i(t) < 0$, $i = 1, 2$, $u_1^{(n_1)}(t) > 0$, $u_2^{(n_2)}(t) < 0$ for $t \geq t_1 := \gamma(t_0)$. Therefore, in view of Remark 3.2 and Lemma 3.3, n_1 is odd, n_2 is even and $\lim_{t \to \infty} u_i(t) = 0$, $i = 1, 2$. Then by Corollary 3.1 (P_1) holds ($n_1 + n_2$ is odd).

C). Let $x_1 \in N^+$, $x_2 \in N^-$ ($a_2(t) > 0$ for $t \geq t_0$). Assume $x_i(t) > 0$, $i = 1, 2$, for $t \geq t_0$. (The proofs in the cases $x_i(t) < 0$, $i = 1, 2$, and $x_1(t)x_2(t) > 0$ are analogous.)

From system (S) with regard to (3.1)–(3.3) we obtain $u_1(t) > 0$, $u_1^{(n_1)}(t) > 0$, $u_2(t) < 0$, $u_2^{(n_2)}(t) < 0$ for $t \geq t_1 = \gamma(t_0)$. Using Lemma 3.3 and Remark 3.1 we get $\lim_{t \to \infty} u_2(t) = 0$, where n_2 is even. By virtue of Lemma 3.1 there exists $t_2 \geq t_1$ such that for n_1 odd $u_2'(t) > 0$ and for n_1 even either $u_2'(t) > 0$ or $u_2'(t) < 0$ for $t \geq t_2$. If $u_2'(t) > 0$ or $u_2'(t) < 0$ and $\lim_{t \to \infty} u_1(t) = a_1 > 0$ then by Lemmas 3.2 and 3.4 we get $\lim_{t \to \infty} u_2(t) = -\infty$ which is a contradiction to $\lim_{t \to \infty} u_2(t) = 0$. Therefore, $\lim_{t \to \infty} u_1(t) = 0$. Using Corollary 3.1 we conclude that (P_1) holds ($n_1 + n_2$ is even).

The proof of Theorem 3.1 is complete.

THEOREM 3.2. *Let* $\sigma = 1$ *and let the assumptions* (3.1)–(3.3) *and* (3.8) *hold. Then every proper solution* (x_1, x_2) *of system* (S) *is either oscillatory or*

(P_1) *holds or*

$$\lim_{t\to\infty} u_i^{(k_i)}(t) = (\operatorname{sgn} u_i)\,\infty, \qquad k_i = 0, 1, \dots, n_i - 1,\ i = 1, 2. \tag{P_2}$$

holds.

PROOF. Let $\sigma = 1$ and (x_1, x_2) be a nonoscillatory solution of (S) in $[t_0, \infty)$.

A). Assume first that $x_i \in N^+$, $i = 1, 2$.

(I) Let $x_i(t) > 0$, $i = 1, 2$, for $t \geq t_0$ (the proof for the case $x_i(t) < 0$, $i = 1, 2$, is similar). Then from system (S) in view of (3.1)–(3.2) we obtain $u_i(t) > 0$, $u_i^{(n_i)}(t) > 0$, $i = 1, 2$, for $t \geq t_1 = \gamma(t_0)$.

(i) Let n_1 be odd and n_2 be either odd or even. Then by Lemma 3.1 there exists $t_2 \geq t_1$ such that $u_1'(t) > 0$ for $t \geq t_2$. Using Lemma 3.2 and then Lemma 3.4 we get $\lim_{t\to\infty} u_2^{(k_2)}(t) = \infty$, $k_2 = 0, 1, \dots, n_2 - 1$. The last relation with regard to Lemmas 3.2 and 3.4 implies $\lim_{t\to\infty} u_1^{(k_1)}(t) = \infty$, $k_1 = 0, 1, \dots, n_1 - 1$. Therefore, ($P_2$) holds.

(ii) Let n_2 be even and n_1 be either odd or even. By Lemma 3.1 $u_1'(t) > 0$ or $u_1'(t) < 0$ for $t \geq t_1$. If $u_1'(t) > 0$, then we proceed as in the case (i) and conclude that (P_2) holds. Therefore, the case $u_1'(t) < 0$ for $t \geq t_0$ is impossible.

(II) Let $x_1(t) > 0$, $x_2(t) < 0$ for $t \geq t_0$ (the proof of the case $x_1(t) < 0$, $x_2(t) > 0$ for $t \geq t_0$ is similar). Then from system (S) in view of (3.1)–(3.2) we get $u_1(t) > 0$, $u_1^{(n_1)}(t) < 0$, $u_2(t) < 0$, $u_2^{(n_2)}(t) > 0$ for $t \geq t_1 = \gamma(t_0)$.

(i) Let n_2 be even and n_1 be either odd or even. By Lemma 3.1 there exists $t_2 \geq t_1$ such that $u_2'(t) < 0$ for $t \geq t_2$. Using Lemma 3.3 and then Lemma 3.4 we get $\lim_{t\to\infty} u_1(t) = -\infty$, which contradicts $u_1(t) > 0$ for $t \geq t_0$.

(ii) Let n_1 be even and n_2 be odd. By Lemma 3.1, $u_1'(t) > 0$ for $t \geq t_2 \geq t_1$. Using Lemma 3.2 and then Lemma 3.4 we get $\lim_{t\to\infty} u_2(t) = \infty$, which contradicts $u_2(t) < 0$ for $t \geq t_0$.

(iii) Let n_i, $i = 1, 2$, be odd. Then by Lemma 3.1 either $u_i'(t) > 0$ or $u_i'(t) < 0$, $i = 1, 2$, for $t \geq t_2 \geq t_1$. If $u_1'(t) > 0$ or $u_1'(t) < 0$ and $\lim_{t\to\infty} u_1(t) = a_1 > 0$ then using Lemma 3.2 and Lemma 3.4 we obtain $\lim_{t\to\infty} u_2(t) = \infty$, which contradicts $u_2'(t) < 0$, $t \geq t_2$. Therefore, $\lim_{t\to\infty} u_2(t) = 0$. If $u_2'(t) < 0$ or $u_2'(t) > 0$ and $\lim_{t\to\infty} u_2(t) = a_2 < 0$, then with regard to Lemmas 3.2 and 3.4 we have $\lim_{t\to\infty} u_1(t) = -\infty$, which contradicts $u_1(t) > 0$, $t \geq t_1$. Therefore, $\lim_{t\to\infty} u_2(t) = 0$. Thus, in view of Corollary 3.1 (P_1) holds.

B). Assume $x_i \in N^-$ ($a_i(t) > 0$ for $t \geq t_0$, $i = 1, 2$.)

Let $x_1(t) > 0$, $x_2(t) < 0$ for $t \geq t_0$ (the cases $x_1(t) < 0, x_2(t) > 0$ and $x_1 x_2 > 0$ are similar). Then (S) together with (3.1)–(3.3) implies $u_1(t) < 0$, $u_1^{(n_1)}(t) < 0$, $u_2(t) > 0$, $u_2^{(n_2)}(t) > 0$ for $t \geq t_1 = \gamma(t_0)$. Therefore, in view of Lemma 3.3 and Remark 3.2 $\lim_{t\to\infty} u_i(t) = 0$, $i = 1, 2$ and n_i, $i = 1, 2$, are even. Then by Corollary 3.1 (P_1) holds ($n_1 + n_2$ is even).

C). We suppose that $x_1 \in N^+$, $x_2 \in N^-$ ($a_2(t) > 0$ for $t \geq t_0$). (The case $x_1 \in N^-$, $x_2 \in N^+$ is treated similarly.) Let $x_1(t) > 0$, $x_2(t) < 0$ for $t \geq t_0$ (the proof in the cases $x_1(t) < 0$, $x_2(t) > 0$ and $x_1(t)x_2(t) > 0$ is similar). Then from system (S) with regards to (3.1)–(3.3) we get $u_i(t) > 0$, $i = 1, 2$, $u_1^{(n_1)}(t) < 0$ and $u_2^{(n_2)}(t) > 0$ for $t \geq t_1 = \gamma(t_0)$. By virtue of Lemma 3.3 and Remark 3.2 we have $\lim_{t\to\infty} u_2(t) = 0$ and n_2 is even. By Lemma 3.1 there exists $t_2 \geq t_1$ such that for n_1 even $u_1'(t) > 0$ and for n_1 odd either $u_1'(t) > 0$ or $u_1'(t) < 0$ for $t \geq t_2$. If $u_1'(t) > 0$ or $u_1'(t) < 0$ and $\lim_{t\to\infty} u_1(t) = a_1 > 0$, then by Lemmas 3.2 and 3.4 we conclude that $\lim_{t\to\infty} u_2(t) = \infty$, which contradicts $\lim_{t\to\infty} u_2(t) = 0$. Therefore, $\lim_{t\to\infty} u_1(t) = 0$. Then, in view of Corollary 3.1, (P_1) holds ($n_1 + n_2$ is odd).

This completes the proof of Theorem 3.2.

We suppose next that for $a_i(t)$ ($i = 1, 2$) one of the following conditions hold

$$\begin{aligned}&a_i(t) > 0, \text{ or } a_i(t) \leq 0, \text{ or}\\ &a_i(t) \leq 0 \text{ and } a_{3-i}(t) \text{ exchanges sign, } i = 1, 2,\\ &\text{or } a_i(t)(i = 1, 2) \text{ exchanges sign.}\end{aligned} \tag{3.10}$$

THEOREM 3.3. *Let the assumptions of Theorem* 3.1 *and* (3.10) *be fulfilled. Then every proper solution* (x_1, x_2) *of system* (S_{-1}) *is oscillatory for* $n_1 + n_2$ *even and is either oscillatory or satisfies* (P_1) *for* $n_1 + n_2$ *odd.*

THEOREM 3.4. *Let the assumptions of Theorem* 3.2 *and* (3.10) *be fulfilled. Then every proper solution* (x_1, x_2) *of system* (S_1) *is either oscillatory or* (P_2) *holds for* $n_1 + n_2$ *odd, or either* (P_1) *or* (P_2) *holds for* $n_1 + n_2$ *even.*

PROOF OF THEOREMS. If the assumptions of Theorem 3.3 and 3.4 hold then the case C) cannot occur. Therefore, proper solutions of (S_{-1}) (respectively (S_1)) cannot have property (P_1) when $n_1 + n_2$ is even (respectively $n_1 + n_2$ is odd). This fact together with the proof of Theorems 3.1 and 3.2 yields conclusions of Theorems 3.3 and 3.4.

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Institute of Mathematics
Ukrainian Academy of Sciences
Kiev, Ukraine
and
Katedra Matematiky
Vysokej Školy Dopravy a Spojov
Žilina, ČSFR