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Kuramochi boundaries of infinite networks and extremal problems

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§1. Introduction and preliminaries

Discrete potential theory has been developed by several authors, e.g., R. J. Duffin [1] and M. Yamasaki [9] among others, and analogies of various potential theoretic properties of Riemann surfaces have been discussed on infinite networks. For example, the extremal length of a family of paths which tend to the boundary of an infinite network is studied by T. Nakamura and M. Yamasaki [5], the boundary limit of Dirichlet finite functions is investigated by T. Kayano and M. Yamasaki [3] and M. Yamasaki [10] and the extremal problems with respect to the ideal boundary components of an infinite network are discussed by A. Murakami and M. Yamasaki [4].

In this paper, we shall be concerned with the Kuramochi boundaries of infinite networks. In §3, we give some examples of Kuramochi functions on infinite networks and in §4, the corresponding Kuramochi boundaries. In the last two sections, we shall study extremal problems related to the Kuramochi boundary; the relation between the extremal distance and the Dirichlet principle related to the Kuramochi boundary in §5 and the relation between the extremal width and the flow problem with respect to the Kuramochi boundary in §6.

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a positive real function on Y. We assume that the graph $\{X, Y, K\}$ is connected, locally finite and has no self-loop. The quartet $N = \{X, Y, K, r\}$ is called an infinite network. For notation and terminologies concerning infinite networks, we mainly follow [3] and [5].

For a set S denote by $L(S)$ (resp. $L^+(S)$) the set of all real functions (resp. non-negative real functions) on S. For $A \subset X$, by $\varepsilon_A (\in L^+(X))$ we shall mean the characteristic function of A. If $A = \{a\}$, we write ε_a for $\varepsilon_{\{a\}}$. The support of a function f is denoted by Sf.

For $u, v \in L(X)$, we set

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y) u(x), \qquad y \in Y$$

and

$$(1.1)\qquad (u, v) = \sum_{y\in Y} r(y)[du(y)][dv(y)],$$

provided the sum is well-defined. $D(u) = (u, u)$ is called the Dirichlet integral of u. If $D(u) < \infty$, then u is called a Dirichlet finite function.

For $u \in L(X)$, we set

$$\Delta u(x) = \sum_{y\in Y} K(x, y)[du(y)], \qquad x \in X,$$

which is called the Laplacian of u.

LEMMA 1.1 (cf. [8; Lemma 3]). *If $u, v \in L(X)$ and Su (or Sv) is finite, then*

$$\sum_{x\in X} \Delta u(x)v(x) = \sum_{x\in X} u(x)\Delta v(x) = -(u, v).$$

A function $u \in L(X)$ is said to be harmonic (resp. superharmonic) on a subset A of X if $\Delta u(x) = 0$ (resp. $\Delta u(x) \leqq 0$) on A.

§ 2. Dirichlet principle

We take a finite nonempty subset A_0 of X once for all, and set $X' = X - A_0$. Set

$$\mathscr{D} = \mathscr{D}^{A_0} = \{u \in L(X);\ D(u) < \infty \text{ and } u = 0 \text{ on } A_0\}.$$

$\mathscr{D}$ is a Hilbert space with respect to the inner product defined by (1.1).

Let A be a subset of X' and let $\varphi \in L(X)$. We denote by $\mathscr{D}_A(\varphi)$ the class of all functions in $\mathscr{D}$ which take values φ on A:

$$\mathscr{D}_A(\varphi) = \{u \in \mathscr{D};\ u = \varphi \text{ on } A\}.$$

We formulate a Dirichlet principle in N as follows.

THEOREM 2.1. *Let A be a subset of X' and let $\varphi \in L(X)$. If $\mathscr{D}_A(\varphi) \neq \emptyset$, then there exists a unique $h \in \mathscr{D}_A(\varphi)$ which has the minimum Dirichlet integral among the functions of $\mathscr{D}_A(\varphi)$. The function h is harmonic in $X' - A$ and is characterized by*

$$(2.1)\qquad h \in \mathscr{D}_A(\varphi) \text{ and } (u - h, h) = 0 \qquad \text{for all } u \in \mathscr{D}_A(\varphi).$$

For a proof of Theorem 2.1, see for example [8; Theorem 2].

We shall denote the above function h by φ_A.

REMARK 2.1. In case A is finite, $\mathscr{D}_A(\varphi) \neq \emptyset$ for any $\varphi \in L(X)$. If $D(\varphi) < \infty$, then $\mathscr{D}_A(\varphi) \neq \emptyset$ for any subset A of X'.

We give some properties of φ_A.

THEOREM 2.2. *Let A and A' be subsets of X' such that $A \subset A'$ and let $\varphi \in L(X)$. If $\mathscr{D}_A(\varphi) \neq \emptyset$, then*

$$(\varphi_A)_{A'} = \varphi_A.$$

PROOF. Since $(\varphi_A)_{A'} \in \mathscr{D}_A(\varphi)$, $D(\varphi_A) \leqq D((\varphi_A)_{A'})$. On the other hand, $\varphi_A \in \mathscr{D}_{A'}(\varphi_A)$, and hence $D((\varphi_A)_{A'}) \leqq D(\varphi_A)$. By the uniqueness, we have $(\varphi_A)_{A'} = \varphi_A$.

For a subset A of X', put $L_A = \{\varphi \in L(X); \mathscr{D}_A(\varphi) \neq \emptyset\}$. It follows from (2.1) that $\varphi \mapsto \varphi_A$ is a linear mapping from L_A into $\mathscr{D}$.

THEOREM 2.3. *Let $\varphi \in L_A$. Then*

$$\min(0, \inf_A \varphi) \leqq \varphi_A \leqq \max(0, \sup_A \varphi). \tag{2.2}$$

PROOF. Suppose $\alpha = \max(0, \sup_A \varphi) < \infty$. Then $\min(u, \alpha) \in \mathscr{D}_A(\varphi)$ for any $u \in \mathscr{D}_A(\varphi)$ and $D(\min(u, \alpha)) \leqq D(u)$ by [8; Corollary 1 of Lemma 2]. This implies the second inequality in (2.2). Similarly, we obtain the first inequality.

§3. Kuramochi function $\tilde{g}$

In this section we give a discrete analogue of the Kuramochi function in the theory of Riemann surfaces.

Let $a \in X'$, and take A as $\{a\}$ and φ as 1 in Theorem 2.1. Namely, we consider the function $1_{\{a\}} \in \mathscr{D}$. Obviously, $1_{\{a\}}$ is not a constant function, so that $D(1_{\{a\}}) \neq 0$.

DEFINITION 3.1. We call the following function $\tilde{g}_a = \tilde{g}(\cdot\,, a) \in \mathscr{D}$ *the Kuramochi function with pole at $a \in X'$*:

$$\tilde{g}_a = \tilde{g}(\cdot\,, a) = 1_{\{a\}}/D(1_{\{a\}}). \tag{3.1}$$

THEOREM 3.1. 1) *$0 \leqq \tilde{g}_a \leqq \tilde{g}_a(a)$ on X'.*

2) *$\Delta \tilde{g}_a = -\varepsilon_a$ in X', and hence $\tilde{g}_a$ is harmonic in $X' - \{a\}$ and superharmonic in X'. Furthermore, $\tilde{g}_a > 0$ on the component of X' which contains a, and it vanishes on the other components. Here a component of X' is defined as a maximal connected subset of X'.*

3) *The function $\tilde{g}_a$ is the reproducing kernel of $\mathscr{D}$, i.e.,*

$$(\tilde{g}_a, u) = u(a) \qquad \text{for every } u \in \mathscr{D}. \tag{3.2}$$

4) *If f is a function on X which has finite support $Sf \subset X'$, then*

$$f(a) = -\sum_{x \in X} \Delta f(x) \tilde{g}_a(x). \tag{3.3}$$

5) $\sum_{x \in A_0} \Delta \tilde{g}_a(x) = 1$.

PROOF. 1) follows from Theorem 2.3.

2) Let $h = 1_{\{a\}}$. By (2.1), we have $(\varepsilon_a, h) = D(h)$. By Lemma 1.1, $(\varepsilon_a, h) = -\Delta h(a)$, and hence $D(h) = -\Delta h(a)$. This implies $\Delta \tilde{g}_a(a) = -1$. By Theorem 2.1, $\Delta \tilde{g}_a = 0$ in $X' - \{a\}$. By Theorem 2.3, $\tilde{g}_a \geqq 0$ on X'. In the same way as in the proof of [9; Lemma 2.1], we see that $\tilde{g}_a > 0$ on the component of X' containing a. The rest of the assertion follows from the fact that $1_{\{a\}}$ has the smallest Dirichlet integral among the functions in $\mathscr{D}_{\{a\}}(1)$.

3) By (2.1), $(u - u(a)1_{\{a\}}, 1_{\{a\}}) = 0$, from which (3.2) follows.

We have 4) using 2) and Lemma 1.1.

5) Since $1 - \varepsilon_{A_0} \in \mathscr{D}$, $1 = (\tilde{g}_a, 1 - \varepsilon_{A_0}) = -(\tilde{g}_a, \varepsilon_{A_0}) = \sum_{x \in A_0} \Delta \tilde{g}_a(x)$ by 3) and Lemma 1.1.

COROLLARY 3.1. *$\tilde{g}_a(b) = \tilde{g}_b(a)$ for $a, b \in X'$.*

PROOF. Setting $u = \tilde{g}_b$ in (3.2) and next changing the role of a and b, we have the assertion.

Since $\tilde{g}_a \in \mathscr{D}$, we can define $(\tilde{g}_a)_A$ for any subset A of X'. We have

THEOREM 3.2. *Let A be a subset of X'. If $a \in A$, then $(\tilde{g}_a)_A = \tilde{g}_a$ on X. If $a \notin A$ then $(\tilde{g}_a)_A \leqq \tilde{g}_a$ on X under the additional assumption that A is finite.*

PROOF. For simplicity we put $\tilde{g} = \tilde{g}_a$ and $\tilde{g}_A = (\tilde{g}_a)_A$. If $a \in A$, then $\tilde{g}_A = \tilde{g}$ by Theorem 2.2. Next, let $a \notin A$ and put $u = \tilde{g} - \tilde{g}_A$. By Theorem 2.2, $u_{A \cup \{a\}} = u$. Since $u = 0$ on A, Theorem 2.3 implies that $u \geqq \min(0, u(a))$. Since $(\Delta u)(a) = -1$ (Theorem 3.1, 2) and Theorem 2.1), u cannot attain its minimum at $x = a$ (cf. the proof of [9; Lemma 2.1]). Hence $u \geqq 0$, i.e., $\tilde{g} \geqq \tilde{g}_A$.

Now we give a characterization of the Kuramochi function $\tilde{g}_a$ in terms of flows from A_0 to $\{a\}$.

For $w \in L(Y)$, let us put

$$I(w; x) = \sum_{y \in Y} K(x, y)w(y), \quad H(w) = \sum_{y \in Y} r(y)w(y)^2.$$

We say that $w \in L(Y)$ is a flow from A_0 to $\{a\}$ if $I(w) := -\sum_{x \in A_0} I(w; x) = I(w; a)$ and $I(w; x) = 0$ for all $x \in X' - \{a\}$. Denote by $F(A_0, \{a\})$ the set of all flows from A_0 to $\{a\}$ and by $F_0(A_0, \{a\})$ the closure of the set $\{w \in F(A_0, \{a\}); Sw \text{ is finite}\}$ in the Hilbert space $L_2(Y; r) := \{w \in L(Y); H(w) < \infty\}$ with the norm $[H(\cdot)]^{1/2}$.

Given $x, x' \in X$, a path P from x to x' is the triple $(C_X(P), C_Y(P), p)$ of a finite ordered set $C_X(P) = \{x_0, x_1, x_2, \dots, x_n\}$ of nodes, a finite ordered set $C_Y(P) = \{y_1, y_2, y_3, \dots, y_n\}$ of arcs and a function $p \in L(Y)$, called the path index of P, such that

$x_0 = x$, $x_n = x'$, $x_i \neq x_j$ $(i \neq j)$,
$\{x \in X; K(x, y_i) \neq 0\} = \{x_{i-1}, x_i\}$ for $i = 1, 2, \dots, n$,
$p(y) = 0$ if $y \notin C_Y(P)$ and $p(y_i) = -K(x_{i-1}, y_i)$ for $i = 1, 2, \dots, n$.
Consider the following extremum problem:

$$\text{(3.4)} \qquad \text{Find} \quad e_a = \inf\{H(w);\ w \in F_0(A_0, \{a\}) \text{ and } I(w) = 1\}.$$

It is known ([8, p. 244]) that problem (3.4) has a unique optimal solution $\hat{w}$. Then, for $x' \in A_0$ we can define a function $v_{x'} \in L(X)$ by

$$v_{x'}(x') = 0 \text{ and } v_{x'}(x) = \sum_{y \in Y} r(y) p(y) \hat{w}(y) \qquad \text{for } x \neq x',$$

where p is the path index of a path P from x' to x. This function does not depend on the choice of P (see [8, p. 247]).

THEOREM 3.3. *$\tilde{g}_a(x) = \min\{|v_{x'}(x)|;\ x' \in A_0\}$ and $\tilde{g}_a(a) = e_a$.*

PROOF. Let $\hat{u}(x) = \min\{|v_{x'}(x)|;\ x' \in A_0\}$. Then $e_a^{-1}\hat{u} \in \mathscr{D}_{\{a\}}(1)$ and $|d\hat{u}| \leqq \hat{w}$ (cf. [6; Lemma 12]). As in the proof of [6; Theorem 11], we see that $e_a D(1_a) = 1$, i.e., $\tilde{g}_a(a) = e_a$, and that

$$D(e_a^{-1}\hat{u}) = e_a^{-2} D(\hat{u}) \leqq e_a^{-2} H(\hat{w}) = e_a^{-1} = D(1_a).$$

Therefore, $e_a^{-1}\hat{u} = 1_{\{a\}}$ by Theorem 2.1, and hence $\tilde{g}_a = \hat{u}$.

EXAMPLE 3.1 (Fig. 1). Let $X = \{x_0, x_1, x_2, \dots\}$; $Y = \{y_1, y_2, y_3, \dots\}$; $K(x_{n-1}, y_n) = -1$, $K(x_n, y_n) = 1$ for $n = 1, 2, 3, \dots$ and $K(x, y) = 0$ for any other pair of (x, y). Put $A_0 = \{x_0\}$. Then $F(A_0, \{x_n\}) = \{\hat{w}_n\}$, where $\hat{w}_n(y_i) = 1$ for $i = 1, 2, 3, \dots, n$ and $\hat{w}_n(y) = 0$ for any other $y \in Y$. Therefore, by Theorem 3.3 we have

$$\text{(3.5)} \qquad \tilde{g}_{x_n}(x_k) = \begin{cases} 0 & \text{for } k = 0 \\ \sum_{j=1}^{k} r(y_j) & \text{for } k = 1, 2, 3, \dots, n \\ \sum_{j=1}^{n} r(y_j) & \text{for } k = n+1, n+2, \dots. \end{cases}$$

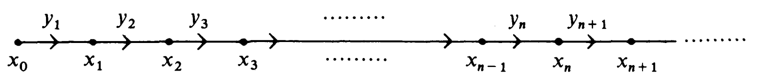

Fig. 1

EXAMPLE 3.2 (Fig. 2). Let $X = \{x_0\} \cup \{x_i^{(j)};\ i = 1, 2, 3, \dots \text{ and } j = 0, 1, 2, \dots\}$; $Y = \{y_i^{(j)};\ i = 1, 2, 3, \dots \text{ and } j = 0, 1, 2, \dots\}$;

$$K(x, y) = \begin{cases} -1 & \text{for } (x, y) = (x_0, y_1^{(0)}),\ (x_i^{(0)}, y_{i+1}^{(0)}) \text{ or } (x_i^{(j)}, y_i^{(j+1)}) \\ & \qquad (i = 1, 2, 3, \dots \text{ and } j = 1, 2, 3, \dots) \\ 1 & \text{for } (x, y) = (x_i^{(j)}, y_i^{(j)}) \quad (i = 1, 2, 3, \dots \text{ and } j = 0, 1, 2, \dots) \\ 0 & \text{for any other pair of } (x, y). \end{cases}$$

Put $A_0 = \{x_0\}$. Then, for $m \geqq 0$ and $n \geqq 1$, $F_0(A_0, \{x_n^{(m)}\}) = \{\hat{w}_{m,n}\}$, where $\hat{w}_{m,n}(y_i^{(0)}) = 1$ for $i = 1, 2, 3, \dots, n-1$, $\hat{w}_{m,n}(y_n^{(j)}) = 1$ for $j = 0, 1, 2, \dots, m$ and $\hat{w}_{m,n}(y) = 0$ for any other $y \in Y$. By Theorem 3.3 we have

$$\text{(3.6)} \qquad \tilde{g}_{x_n^{(m)}}(x) = \begin{cases} s_k & \text{for } x \in X_k & (k = 0, 1, 2, \dots, n-1) \\ s_{n-1} + s_n^{(k)} & \text{for } x = x_n^{(k)} & (k = 0, 1, 2, \dots, m) \\ s_{n-1} + s_n^{(m)} & \text{for } x = x_n^{(k)} & (k = m+1, m+2, \dots) \\ s_n & \text{for } x \in X_k & (k = n+1, n+2, \dots), \end{cases}$$

where $X_0 = \{x_0\}$, $X_k = \{x_k^{(j)}; j = 0, 1, 2, \dots\}$, $s_0 = 0$, $s_n = \sum_{j=1}^{n} r(y_j^{(0)})$ and $s_n^{(k)} = \sum_{j=0}^{k} r(y_n^{(j)})$, $k = 0, 1, 2, \dots$, $n = 1, 2, 3, \dots$.

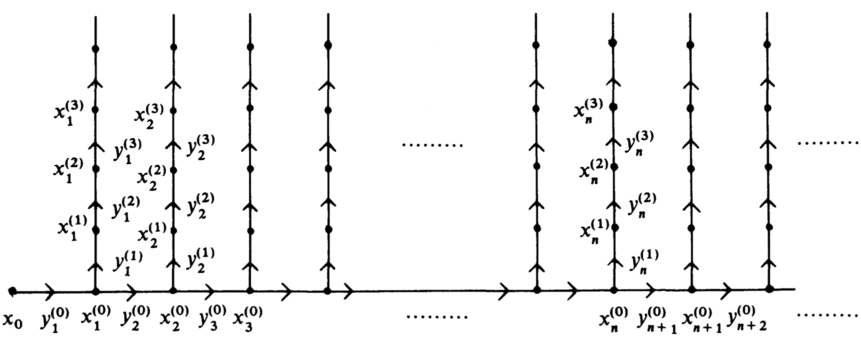

Fig. 2

EXAMPLE 3.3 (Fig. 3). Let $X_{(0)} = \{x_0\}$, $X_{(n)} = \{x(i_1, \dots, i_n); i_1, \dots, i_n = 0 \text{ or } 1\}$, $Y_{(n)} = \{y(i_1, \dots, i_n); i_1, \dots, i_n = 0 \text{ or } 1\}$; $X = \bigcup_{k=0}^{\infty} X_{(k)}$; $Y = \bigcup_{k=1}^{\infty} Y_{(k)}$;

$$K(x, y) = \begin{cases} -1 & \text{for } (x, y) = (x_0, y(0)),\ (x_0, y(1)),\ (x(i_1, \dots, i_n), y(i_1, \dots, i_n, 0)) \\ & \quad \text{or } (x(i_1, \dots, i_n), y(i_1, \dots, i_n, 1)) \quad (n = 1, 2, 3, \dots) \\ 1 & \text{for } (x, y) = (x(i_1, \dots, i_n), y(i_1, \dots, i_n)) \quad (n = 1, 2, 3, \dots) \\ 0 & \text{for any other pair of } (x, y). \end{cases}$$

Put $A_0 = \{x_0\}$. Then $F_0(A_0, \{x(i_1, \dots, i_n)\}) = \{\hat{w}_{i_1, \dots, i_n}\}$, where $\hat{w}_{i_1, \dots, i_n}(y)$ is equal to 1 for $y = y(i_1, \dots, i_k)$ $(k = 1, 2, \dots, n)$ and to 0 for any other $y \in Y$. Hence, we have

(3.7)

$$\tilde{g}_{x(i_1,\dots,i_n)}(x) = \begin{cases} \sum_{k=1}^{m} r(y(i_1,\dots,i_k)) & \text{for } x \in C(i_1,\dots,i_m)\ (m = 1, 2,\dots, n) \\ \sum_{k=1}^{n} r(y(i_1,\dots,i_k)) & \text{for } x \in X(i_1,\dots,i_n) \\ 0 & \text{for } x = x_0 \text{ or } x \in X(1-i_1), \end{cases}$$

where $C(i_1,\dots,i_m) = \{x(i_1,\dots,i_m),\ x(i_1,\dots,i_m, 1-i_{m+1})\} \cup (\bigcup_{\ell=2}^{\infty} \{x(i_1,\dots,i_m, 1-i_{m+1}, j_{m+2},\dots,j_{m+\ell}); j_{m+2},\dots,j_{m+\ell} = 0 \text{ or } 1\})$ $(m = 1, 2,\dots, n-1)$; $X(i_1,\dots,i_n) = \{x(i_1,\dots,i_n)\} \cup (\bigcup_{\ell=1}^{\infty} \{x(i_1,\dots,i_n, j_{n+1},\dots,j_{n+\ell}); j_{n+1},\dots,j_{n+\ell} = 0 \text{ or } 1\})$.

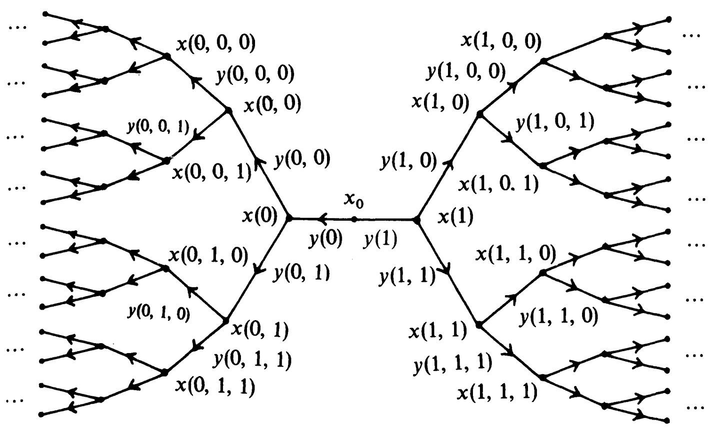

Fig. 3

EXAMPLE 3.4 (Fig. 4). Let $X = \{x_0, x_1, x_1', x_2, x_2',\dots\}$; $Y = \{y_1, y_1', y_1'', y_2, y_2', y_2'',\dots\}$;

$$K(x, y) = \begin{cases} -1 & \text{for } (x, y) = (x_0, y_1), (x_0, y_1'), \\ & \qquad (x_i, y_{i+1}), (x_i', y_{i+1}') \text{ or } (x_i, y_i'') \quad (i = 1, 2, 3,\dots) \\ 1 & \text{for } (x, y) = (x_i, y_i), (x_i', y_i') \text{ or } (x_i', y_i'') \quad (i = 1, 2, 3,\dots) \\ 0 & \text{for any other pair of } (x, y). \end{cases}$$

Put $A_0 = \{x_0\}$. We can not apply Theorem 3.3 to this case. However, solving simultaneous difference equations we have the following results.

1) The case $r(y) = 1$ for any $y \in Y$. In this case we have

$$\begin{cases} \dfrac{k}{2} + \alpha(k, n) & \text{for } x = x_k \quad (k = 0, 1,\dots, n) \\ \dfrac{n}{2} + \alpha(n, k) & \text{for } x = x_k \quad (k = n+1, n+2,\dots) \end{cases}$$

$$(3.8)\qquad \tilde{g}_{x_n}(x)=\begin{cases}\dfrac{k}{2}-\alpha(k,n) & \text{for } x=x'_k \quad (k=1,2,\dots,n)\\[2ex] \dfrac{n}{2}-\alpha(n,k) & \text{for } x=x'_k \quad (k=n+1,n+2,\dots),\end{cases}$$

where $\alpha(k,n)=\{(2+\sqrt{3})^k-(2-\sqrt{3})^k\}/\{4\sqrt{3}(2+\sqrt{3})^n\}$; and $\tilde{g}_{x'_n}(x)=\tilde{g}_{x_n}(\hat{x})$ ($\hat{x}=x'_n$ if $x=x_n$ and $\hat{x}=x_n$ if $x=x'_n$).

2) The case $r(y_i)=r(y'_i)=1$ $(i=1,2,3,\dots)$ and $r(y''_i)=i(i+2)^2$ $(i=1,2,3,\dots)$. In this case we have

$$(3.9)\qquad \tilde{g}_{x_n}(x)=\begin{cases}\dfrac{k}{2}+\beta(n,k) & \text{for } x=x_k \quad (k=0,1,\dots,n)\\[2ex] \dfrac{n}{2}+\beta(k,n) & \text{for } x=x_k \quad (k=n+1,n+2,\dots)\\[2ex] \dfrac{k}{2}-\beta(n,k) & \text{for } x=x'_k \quad (k=1,2,\dots,n)\\[2ex] \dfrac{n}{2}-\beta(k,n) & \text{for } x=x'_k \quad (k=n+1,n+2,\dots),\end{cases}$$

where $\beta(n,0)=0$, $\beta(n,k)=\dfrac{1}{2}\dfrac{n+2}{n+1}\dfrac{k+2}{k+1}\left(\sum_{i=1}^{k}\dfrac{i}{i+2}\right)$; and $\tilde{g}_{x'_n}(x)=\tilde{g}_{x_n}(\hat{x})$.

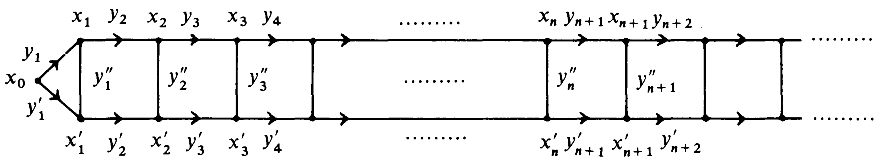

Fig. 4

§4. Definition of the Kuramochi boundary of N

In this section we use notation $\tilde{g}(x,a)$ for $\tilde{g}_a(x)$. We say that a sequence $\{x_j\}$ in X' tends to the boundary of N if, for any finite subset A of X' there exists j_0 such that $x_j\notin A$ for all $j\geqq j_0$. Given such a sequence $\{x_j\}$, we see by Corollary 3.1 and Theorem 3.1, 1) that $\{\tilde{g}(x,x_j)\}_j$ is bounded for each $x\in X'$. Consequently, $\{\tilde{g}(\cdot,x_j)\}$ has a convergent subsequence. If $\{\tilde{g}(\cdot,x_j)\}$ converges, then $\{x_j\}$ will be called a fundamental sequence. When the limit functions of two fundamental sequences $\{\tilde{g}(\cdot, x_j)\}$ and $\{\tilde{g}(\cdot, x_j')\}$ are equal to each other, we say that $\{x_j\}$ and $\{x_j'\}$ are equivalent and call an equivalence class a *Kuramochi boundary point* of N. We call the set of all Kuramochi boundary points *the Kuramochi boundary* of N and denote it by ∂N. If $x \in X$, $z \in \partial N$ and $\{x_j\}$ in X' determines z, then we set $\tilde{g}(x, z) = \tilde{g}_z(x) = \lim_{j\to\infty} \tilde{g}(x, x_j)$, which does not depend on the choice of fundamental sequence $\{x_j\}$. Obviously, $\tilde{g}(x, z) = 0$ for $x \in A_0$ and $\tilde{g}(\cdot, z)$ is harmonic on X'. By Theorem 3.1, 5) we see that $\sum_{x\in A_0} \Delta\tilde{g}(x, z) = 1$ for any $z \in \partial N$.

We denote $X' \cup \partial N$ by $\tilde{X}$ and introduce a metric by

$$d(x_1, x_2) = \sum_{x\in X'} \alpha(x) \frac{|\tilde{g}(x, x_1) - \tilde{g}(x, x_2)|}{1 + |\tilde{g}(x, x_1) - \tilde{g}(x, x_2)|}$$

for $x_1, x_2 \in \tilde{X}$, where $\alpha(x)$ is a positive function on X' such that $\sum_{x\in X'} \alpha(x) < \infty$. It is easy to see that with this metric, $\tilde{X}$ is a compact metric space and that the topology induced by d on X' is the discrete topology. By our definition a sequence $\{x_j\} \subset \tilde{X}$ converges to an $x \in \tilde{X}$ in d if and only if $\tilde{g}(\cdot, x_j) \to \tilde{g}(\cdot, x)$ as $j \to \infty$; in particular, $\tilde{g}(x, z)$ is a continuous function of z for each $x \in X'$.

Remark. As in the continuous case, we can show that the definition of boundary points of N does not depend on the choice of A_0 (cf. [6; Theorem 12]).

The Kuramochi boundaries of the networks given in § 3 are described as follows:

Example 4.1. Let N and A_0 be the same as in Example 3.1 with $r(y_i) = r_i$. We can see that $\lim_{j\to\infty} \tilde{g}(x_k, x_j)$ is equal to 0 for $k = 0$ and to $\sum_{i=1}^{k} r_i$ for $k = 1, 2, 3, \ldots$. This implies that $\{x_j\}$ is a fundamental sequence and that ∂N consists of only one point.

Example 4.2. Let N and A_0 be the same as in Example 3.2. We can easily verify that $\{x_j^{(0)}\}_j$ and $\{x_n^{(j)}\}_j$ for $n = 1, 2, 3, \ldots$ are fundamental sequences. Let $z^{(0)}$ and z_n be the boundary points determined by $\{x_j^{(0)}\}_j$ and $\{x_n^{(j)}\}_j$ respectively. Then we have

$$\tilde{g}(x, z^{(0)}) = s_k \quad \text{for} \quad x \in X_k \qquad (k = 0, 1, 2, \ldots),$$

and

$$\tilde{g}(x, z_n) = \begin{cases} s_k & \text{for } x \in X_k & (k = 0, 1, 2, \ldots, n-1) \\ s_{n-1} + s_n^{(k)} & \text{for } x = x_n^{(k)} & (k = 0, 1, 2, \ldots) \\ s_n & \text{for } x \in X_k & (k = n+1, n+2, \ldots). \end{cases}$$

It is easy to show that any fundamental sequence is equivalent to one of the above ones. Hence we know that $\partial N=\{z^{(0)}, z_1, z_2,\dots\}$. Each z_n is isolated in ∂N and $\lim_{n\to\infty} z_n=z^{(0)}$.

EXAMPLE 4.3. Let N and A_0 be the same as in Example 3.3. Let $\{i_n\}$ be a sequence whose elements consist of 0 or 1. Then, it is evident that $\{x(i_1, i_2,\dots, i_n)\}_n$ forms a fundamental sequence. In fact, let $z(\{i_n\})\in\partial N$ be determined by $\{x(i_1, i_2,\dots, i_n)\}_n$. Then we have by (3.7)

$$\tilde{g}(x, z(\{i_n\}))=\begin{cases}\sum_{k=1}^{m} r(y(i_1,\dots, i_k)) & \text{for } x\in C(i_1,\dots, i_m)\ (m=1, 2, 3,\dots)\\ 0 & \text{for } x=x_0 \text{ or } x\in X(1-i_1).\end{cases}$$

Identifying $z(\{i_n\})$ with $\{i_n\}$, we see that $\partial N=\{\{i_n\}_{n=1}^{\infty};\ i_n=0 \text{ or } 1\}$, and in fact it is homeomorphic to the Cantor set in $[0, 1]$.

EXAMPLE 4.4. Let N and A_0 be the same as in Example 3.4. Then, in the case 1) we know that ∂N consists of only one point z and that $\tilde{g}(x, z)=k/2$ for $x=x_k$, x_k' $(k=1, 2, 3,\dots)$. In the case 2), we see that $\{x_j\}$ and $\{x_j'\}$ are different fundamental sequences. Let z and z' be the Kuramochi boundary points determined by $\{x_j\}$ and $\{x_j'\}$ respectively. Then we have

$$\tilde{g}_z(x)=\begin{cases}\dfrac{1}{2}\left\{k+\dfrac{k+2}{k+1}\sum_{i=1}^{k}\dfrac{i}{i+2}\right\} & \text{for } x=x_k \quad (k=1, 2, 3,\dots)\\ \dfrac{1}{2}\left\{k-\dfrac{k+2}{k+1}\sum_{i=1}^{k}\dfrac{i}{i+2}\right\} & \text{for } x=x_k' \quad (k=1, 2, 3,\dots)\end{cases}$$

and $\tilde{g}_{z'}(x)=\tilde{g}_z(\hat{x})$ $(\hat{x}=x_k'$ if $x=x_k$ and $\hat{x}=x_k$ if $x=x_k')$. We see that $\partial N=\{z, z'\}$ in this case.

§5. Extremal distance and Dirichlet principle related to the Kuramochi boundary

For a subset A of X, a path from A to the ideal boundary of N is the triple $(C_X(P), C_Y(P), p)$ of an infinite ordered set $C_X(P)=\{x_0, x_1, x_2,\dots\}$ of nodes, an infinite ordered set $C_Y(P)=\{y_1, y_2, y_3,\dots\}$ of arcs and the path index p of P such that $C_X(P)\cap A=\{x_0\}$, $x_i\neq x_j$ for $i\neq j$, $\{x\in X; K(x, y_i)\neq 0\}=\{x_{i-1}, x_i\}$ for $i=1, 2,\dots, p(y)=0$ if $y\notin C_Y(P)$ and $p(y_i)=-K(x_{i-1}, y_i)$ for $i=1, 2,\dots$. We denote by $P_{A,\infty}$ the set of all paths from A to the ideal boundary of N. We put $P_\infty=\bigcup_{x\in X} P_{\{x\},\infty}$.

Let Γ be a family of paths in N. The extremal length $\lambda(\Gamma)$ of Γ is defined by

$$\lambda(\Gamma)^{-1}=\inf\{H(W);\ W\in E(\Gamma)\},$$

where

$$E(\Gamma) = \{W \in L^+(Y);\ H(W) < \infty \text{ and } \sum_{y \in C_Y(P)} r(y)W(y) \geqq 1 \text{ for all } P \in \Gamma\}.$$

In case $\Gamma = \emptyset$, we set $\lambda(\Gamma) = \infty$. We say that a property holds for almost every path of Γ if there is $\Gamma' \subset \Gamma$ such that $\lambda(\Gamma - \Gamma') = \infty$ and the property holds for all $P \in \Gamma'$. Note that if $\{\Gamma_n\}$ is a countable set of families of paths and $\lambda(\Gamma_n) = \infty$ for each n, then $\lambda(\bigcup_n \Gamma_n) = \infty$.

For $P \in P_\infty$ with $C_X(P) = \{x_0, x_1, x_2, \dots\}$ and $u \in L(X)$, we write $u(P) = \lim_{k\to\infty} u(x_k)$ if exists. We know ([3; Theorem 3.1]) that $u(P)$ exists for almost every $P \in P_\infty$ if u is a Dirichlet finite function.

We say that a path $P \in P_{A_0,\infty}$ with $C_X(P) = \{x_0, x_1, x_2, \dots\}$ tends to a point of ∂N if $\{x_k\}$ is a fundamental sequence, i.e., if $\{\tilde{g}_{x_k}\}$ is convergent.

THEOREM 5.1 (cf. [2; Proposition 1]). *Almost every path of $P_{A_0,\infty}$ tends to one point of ∂N.*

PROOF. Let Γ be the set of all paths of $P_{A_0,\infty}$ which do not tend to any point of ∂N. Then, $\Gamma = \bigcup_{x \in X'} \Gamma_x$, where $\Gamma_x = \{P \in P_{A_0,\infty};\ \tilde{g}(x, P)$ does not exist$\}$. On the other hand, we know that for each $x \in X'$, $\tilde{g}(\cdot\,, x)$ is Dirichlet finite, and hence by Corollary 3.1, so is $\tilde{g}(x, \cdot)$. Thus we see that $\lambda(\Gamma_x) = \infty$, and hence $\lambda(\Gamma) = \infty$ since X' is a countably infinite set.

Now we prove a kind of Dirichlet principle similar to [7; Theorem 1]:

THEOREM 5.2. *Let Γ be a subfamily of P_∞ with $\lambda(\Gamma) < \infty$, and let $\phi(P)$ be a function defined for almost every path P of Γ. Let*

$$\mathscr{D}_\phi^\Gamma = \{u \in \mathscr{D};\ u(P) = \phi(P) \text{ for almost every } P \in \Gamma\}.$$

If $\mathscr{D}_\phi^\Gamma \neq \emptyset$, then there exists a unique function H_ϕ^Γ which minimizes $D(u)$ among the functions u in $\mathscr{D}_\phi^\Gamma$. H_ϕ^Γ is harmonic on X' and is characterized by

$$H_\phi^\Gamma \in \mathscr{D}_\phi^\Gamma \text{ and } (H_\phi^\Gamma, u) = 0 \qquad \text{for all } u \in \mathscr{D}_0^\Gamma. \tag{5.1}$$

PROOF. It is enough to show that $\mathscr{D}_\phi^\Gamma$ is a colsed convex subset of $\mathscr{D}$. Obviously, $\mathscr{D}_\phi^\Gamma$ is convex. Now let $\{u_n\}$ be a sequence in $\mathscr{D}_\phi^\Gamma$ such that $D(u_n - u_0) \to 0$ for some $u_0 \in \mathscr{D}$. We shall show that $u_0 \in \mathscr{D}_\phi^\Gamma$. Let $W_n(y) = |du_n(y) - du_0(y)|$. Then $H(W_n) = D(u_n - u_0) \to 0\,(n \to \infty)$. Set

$$\Gamma' = \{P \in \Gamma;\ u_0(P) \text{ exists and } u_n(P) = \phi(P) \text{ for all } n\}.$$

Then $\lambda(\Gamma - \Gamma') = \infty$. We can find a subfamily Γ'' of Γ' and a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that $\lim_{k\to\infty} \sum_{y \in C_Y(P)} r(y) W_{n_k}(y) = 0$ for all $P \in \Gamma''$ and $\lambda(\Gamma' - \Gamma'') = \infty$ (cf. [4; Lemma 1.3]). Then $\lambda(\Gamma - \Gamma'') = \infty$ and for each $P \in \Gamma''$ we have

$$\sum_{y\in C_Y(P)} r(y) W_{n_k}(y) \geqq \lim_{i\to\infty} |u_{n_k}(x_{i+1}) - u_0(x_{i+1})| - |u_{n_k}(x_0) - u_0(x_0)|$$
$$= |u_{n_k}(P) - u_0(P)| - |u_{n_k}(x_0) - u_0(x_0)|$$
$$= |\phi(P) - u_0(P)| - |u_{n_k}(x_0) - u_0(x_0)|,$$

where $C_X(P) = \{x_0, x_1, \dots\}$. Letting $k\to\infty$, we conclude that $u_0(P) = \phi(P)$ for $P\in\Gamma''$. Hence $u_0 \in \mathscr{D}_\phi^\Gamma$, which implies that $\mathscr{D}_\phi^\Gamma$ is closed.

Let Z be a subset of ∂N and let

$$P_{A_0,Z} = \{P\in P_{A_0,\infty};\ P \text{ tends to a point in } Z\}.$$

Then, Theorem 5.2 shows that $H_1^{P_Z}$ is the optimal solution of the following extremal problem:

(5.2)
Find $e(A_0, Z) = \inf\{D(u);\ u\in\mathscr{D}$ and $u(P) = 1$ for almost every $P\in P_{A_0,Z}\}$.

Thus, $e(A_0, Z) = D(H_1^{P_Z})$. Here, P_Z implies $P_{A_0,Z}$. We shall show

THEOREM 5.3. $e(A_0, Z) = \lambda(P_{A_0,Z})^{-1}$.

The value $\lambda(P_{A_0,Z})$ may be called the extremal distance between A_0 and Z.

In order to prove Theorem 5.3, we consider the following extremal problems: Let $c\in L^+(Y)$.

(5.3) (Min-work problem) Find $N(P_{A_0,Z}; c) = \inf\{\sum_{y\in C_Y(P)} c(y);\ P\in P_{A_0,Z}\}$;

(5.4) (Max-potential problem) Find

$$N^*(A_0, Z; c) = \sup\{\inf\{u(x);\ x\in A_0\} - \sup\{u(P);\ P\in\Gamma_{A_0,Z;c}\};\ u\in S_c^*\},$$

where

$$S_c^* = \{u\in L(X);\ |\sum_{x\in X} K(x, y)u(x)| \leqq c(y) \text{ for all } y\in Y\}$$

and

$$\Gamma_{A_0,Z;c} = \{P\in P_{A_0,Z};\ \sum_{y\in C_Y(P)} c(y) < \infty\}.$$

In the same way as [4; Theorem 2.1] we obtain

LEMMA 5.1. *If* $\Gamma_{A_0,Z;c} \neq \emptyset$, *then* $N(P_{A_0,Z}; c) = N^*(A_0, Z; c)$ *and the problem* (5.4) *has an optimal solution* $\hat{u}(x)$ *such that* $\hat{u}(P) = 0$ *for all* $P\in\Gamma_{A_0,Z;c}$.

Using this lemma, we can prove Theorem 5.3 in the same way as in the proof of [4; Theorem 2.2].

§6. Extremal width and extremal flow problem with respect to Kuramochi boundary

First we recall the notions of cuts and extremal width.

Let A and B be mutually disjoint subsets of X. A cut between A and B is a nonempty subset Q of Y such that there exist mutually disjoint subsets $Q(A)$ and $Q(B)$ of X satisfying the following conditions:

$$Q(A) \supset A, \; Q(B) \supset B, \; Q(A) \cup Q(B) = X$$

and

$$Q = \{y \in Y; \, e(y) \cap Q(A) \neq \emptyset \text{ and } e(y) \cap Q(B) \neq \emptyset\},$$

where $e(y) = \{x \in X \, ; \, K(x, y) \neq 0\}$. Let $Q_{A,B}$ be the set of all cuts between A and B. A cut in N is a subset Q of Y such that $Q \in Q_{A,B}$ for some mutually disjoint subsets A and B of X.

Let Λ be a family of cuts in N. The extremal width $\mu(\Lambda)$ of Λ is defined by

$$\mu(\Lambda)^{-1} = \inf \{H(W); \, W \in E^*(\Lambda)\},$$

where

$$E^*(\Lambda) = \{W \in L^+(Y); \, H(W) < \infty \text{ and } \textstyle\sum_{y \in Q} W(y) \geqq 1 \text{ for all } Q \in \Lambda\}.$$

If $\Lambda = \emptyset$, we set $\mu(\Lambda) = \infty$.

Now, let Z be a closed subset of ∂N and let $Z_{(m)} = \{x \in X'; \, d(x, Z) \leqq 1/m\}$, where d is the metric introduced in §4. Set $Q_{A_0,Z} = \bigcup_{m=1}^{\infty} Q_{A_0,Z_{(m)}}$. We shall show that the extremal width $\mu(Q_{A_0,Z})$ is given by the value of an extremal flow problem.

Denote by $F(A_0, Z_{(m)})$ the set of all flows from A_0 to $Z_{(m)}$, i.e.,

$$F(A_0, Z_{(m)}) = \{w \in L(Y); \, I(w; x) = 0 \text{ for all } x \in X' - Z_{(m)},$$
$$\textstyle\sum_{x \in Z_{(m)}} |I(w; x)| < \infty \text{ and } I(w; A_0) + \sum_{x \in Z_{(m)}} I(w; x) = 0\}$$

and let $F_0(A_0, Z_{(m)})$ be the closure of $\{w \in F(A_0, Z_{(m)}); \, Sw \text{ is finite}\}$ in $L_2(Y; r)$ (cf. §3). Put $F_0(A_0, Z) = \bigcap_{m=1}^{\infty} F_0(A_0, Z_{(m)})$ and call $w \in F_0(A_0, Z)$ a flow from A_0 to Z. We condider the following extremal problems:

(6.1) Find $e^*(A_0, Z_{(m)}) = \inf \{H(w); \, w \in F_0(A_0, Z_{(m)}) \text{ and } I(w; A_0) = -1\}$;

(6.2) Find $e^*(A_0, Z) = \inf \{H(w); \, w \in F_0(A_0, Z) \text{ and } I(w; A_0) = -1\}$.

By a slight modification of the proof of [5; Theorem 4.1], we can prove

LEMMA 6.1. *$e^*(A_0, Z_{(m)}) = \mu(Q_{A_0,Z_{(m)}})^{-1}$ for every m.*

Also, in the same way as [4; Theorem 3.1], we have

LEMMA 6.2. $\lim_{m\to\infty} e^*(A_0, Z_{(m)}) = e^*(A_0, Z)$.

Using these lemmas, we obtain the following theorem in the same way as in the proof of [4; Theorem 4.1]:

THEOREM 6.1. $e^*(A_0, Z) = \mu(Q_{A_0,Z})^{-1}$.

REMARK. If $u \in \mathscr{D}_{Z(m)}(1)$, then $u(P) = 1$ for every $P \in P_{A_0,Z}$. Hence $D(1_{Z(m)}) \geqq e(A_0, Z)$. On the other hand, as in the proof of [8; Theorem 11], we can show that $D(1_{Z(m)}) = e^*(A_0, Z_{(m)})^{-1}$. Therefore, by Lemma 6.2 and Theorems 5.3 and 6.1, we have

$$\lambda(P_{A_0,Z})^{-1} = e(A_0, Z) \leqq e^*(A_0, Z)^{-1} = \mu(Q_{A_0,Z}).$$

We do not know whether the equality $e(A_0, Z) = e^*(A_0, Z)^{-1}$ holds or not.

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