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Integral averaging techniques for the oscillation and nonoscillation of solutions of second order ordinary differential equations

Dedicated to Professor Kusano Takaŝi on his 60th birthday

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1. Introduction

In this paper we consider the second order ordinary differential equation

$$x'' + a(t)f(x) = 0 \tag{1.1}$$

under the following conditions: $a(t)$ is continuous on $[t_0, \infty)$, $t_0 > 0$; $f(x)$ is continuous on $\boldsymbol{R}$, $f'(x)$ exists and is continuous on $\boldsymbol{R} - \{0\}$, and

$$xf(x) > 0 \qquad \text{and} \qquad f'(x) > 0 \qquad \text{for every } x \in \boldsymbol{R} - \{0\} .$$

A typical example of (1.1) is the Emden-Fowler equation

$$x'' + a(t)|x|^{\gamma} \operatorname{sgn} x = 0 , \tag{1.2}$$

where γ is a positive constant.

Our interest here is the problem of oscillation and nonoscillation of solutions of equation (1.1). First we give a new necessary condition for the existence of a nonoscillatory solution $x(t)$ of (1.1), and then, as the contrapositive forms of the result, we establish new oscillation criteria for (1.1). The coefficient $a(t)$ in (1.1) is allowed to take both positive and negative values on any interval $[T, \infty)$, $T \geq t_0$. It is known that, for such an $a(t)$, the integral averages of the integral of $a(t)$ play a crucial role. Let $p \in \boldsymbol{R}$ and $p \geq 1$. There is no need to assume that p is an integer. Then we define $A_p(t)$ by

$$A_p(t) = \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-1} a(s) ds , \qquad t \geq t_0 . \tag{1.3}$$

The important oscillation criteria of Wintner [20]–Hartman [4] (see also [5]) for the linear case and Butler [1] for the nonlinear case involve the asymptotic conditions as $t \to \infty$ of the average function $A_2(t)$. In this paper these oscillation criteria are improved by making use of the general average function $A_p(t)$. Among a number of papers dealing with integral averaging techniques in the study of second order oscillations, we refer to the papers [2–6, 8, 19, 20] for the linear case, and [1, 7, 9, 12–18, 21–26] for the nonlinear case.

2. Statement of results

Equation (1.1) is said to be *strongly superlinear* if

$$\int_{1}^{+\infty}\frac{dy}{f(y)}<\infty \quad \text{and} \quad \int_{-1}^{-\infty}\frac{dy}{f(y)}<\infty, \tag{2.1}$$

and it is said to be *strongly sublinear* if

$$\int_{+0}^{1}\frac{dy}{f(y)}<\infty \quad \text{and} \quad \int_{-0}^{-1}\frac{dy}{f(y)}<\infty. \tag{2.2}$$

In this paper we say that equation (1.1) is *strictly superlinear* if it is strongly superlinear and there is a constant $c(f)>1$ such that

$$f'(x)\int_{x}^{+\infty}\frac{dy}{f(y)}\geq c(f)>1 \quad \text{for all } x>0 \tag{2.3}$$

and

$$f'(x)\int_{x}^{-\infty}\frac{dy}{f(y)}\geq c(f)>1 \quad \text{for all } x<0. \tag{2.4}$$

Likewise we say that equation (1.1) is *strictly sublinear* if it is strongly sublinear and there is a positive constant $d(f)$ such that

$$f'(x)\int_{+0}^{x}\frac{dy}{f(y)}\geq d(f)>0 \quad \text{for all } x>0 \tag{2.5}$$

and

$$f'(x)\int_{-0}^{x}\frac{dy}{f(y)}\geq d(f)>0 \quad \text{for all } x<0. \tag{2.6}$$

These conditions (2.3)–(2.4) and (2.5)–(2.6) are effectively used in the papers of Philos and Purnaras [17, 18] and Wong [25]. Equation (1.1) is linear if $f(x)\equiv x$, $x\in \boldsymbol{R}$.

Now let us consider the special case where $f(x)=|x|^{\gamma}\operatorname{sgn} x$, $x\in\boldsymbol{R}$ $(\gamma>0)$. In this case, equation (1.1) becomes the Emden-Fowler equation (1.2). It is easy to see that if $\gamma>1$, then (1.2) is strictly superlinear, and we can take $c(f)=\gamma/(\gamma-1)$ in (2.3) and (2.4). It is also easy to see that if $0<\gamma<1$, then (1.2) is strictly sublinear. We can take $d(f)=\gamma/(1-\gamma)$ in (2.5) and (2.6). If $\gamma=1$, then (1.2) is a linear equation.

We now state the main results.

THEOREM 1. *Let equation* (1.1) *be strictly superlinear, strictly sublinear, or linear. If there is a nonoscillatory solution* $x(t)$ *of equation* (1.1), *then we have either* (I) *or* (II) *in the following*:

(I) $\lim_{t\to\infty} A_p(t)$ *exists and is finite for any* $p \in \boldsymbol{R}$ *with* $p \geq 2$;

(II) $\lim_{t\to\infty} A_q(t) = -\infty$ *for any* $q \in \boldsymbol{R}$ *with*

$$\tag{2.7} \begin{cases} q > \dfrac{2c(f)-1}{c(f)-1} & \textit{in the strictly superlinear case}, \\ q \geq 2 & \textit{in the strictly sublinear case}, \\ q > 2 & \textit{in the linear case}, \end{cases}$$

where $c(f)$ *is the constant appearing in* (2.3) *and* (2.4).

For the Emden-Fowler equation (1.2), we can take $c(f) = \gamma/(\gamma - 1)$. Therefore we obtain the next corollary.

COROLLARY 1. *Suppose that the Emden-Fowler equation* (1.2) *has a nonoscillatory solution* $x(t)$. *Then we have either* (I) *or* (II) *in the following*:

(I) $\lim_{t\to\infty} A_p(t)$ *exists and is finite for any* $p \in \boldsymbol{R}$ *with* $p \geq 2$;

(II) $\lim_{t\to\infty} A_q(t) = -\infty$ *for any* $q \in \boldsymbol{R}$ *with*

$$\begin{cases} q > \gamma + 1 & \textit{in the strictly superlinear case}, \\ q \geq 2 & \textit{in the strictly sublinear case}, \\ q > 2 & \textit{in the linear case}. \end{cases}$$

We see that if $\lim_{t\to\infty} A_2(t) = \lambda$ exists in the extended real line $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$, then $\lim_{t\to\infty} A_p(t) = \lambda$ for any $p \in \boldsymbol{R}$ with $p \geq 2$ (see Lemma 3). Therefore, in the statement (I) of Theorem 1 and Corollary 1, it is enough to consider the case $p = 2$. Similarly, in (II) of Theorem 1 and Corollary 1, it is enough to consider the case $q = 2$ provided that the equation is strictly sublinear.

It is to be noted that (I) and (II) in Theorem 1 and Corollary 1 are mutually exclusive. If neither (I) nor (II) is satisfied, then equation (1.1) does not have any nonoscillatory solutions, in other words, all continuable solutions of (1.1) are oscillatory. Thus we have the following corollary.

COROLLARY 2. *Let equation* (1.1) *be strictly superlinear, strictly sublinear, or linear. Suppose that*

$$\limsup_{t\to\infty} A_q(t) > -\infty$$

for some $q \in \boldsymbol{R}$ *with* (2.7).

(i) *If*

$$\liminf_{t\to\infty} A_p(t) < \limsup_{t\to\infty} A_p(t)$$

for some $p \in \boldsymbol{R}$, $p \geq 2$, *then all continuable solutions of* (1.1) *are oscillatory.*

(ii) *If*

$$\limsup_{t\to\infty} A_p(t) = +\infty$$

for some $p \in \boldsymbol{R}$, $p \geq 2$, *then all continuable solutions of* (1.1) *are oscillatory.*

Note that the function $A_2(t)$ has the following four possibilities:

$$\liminf_{t\to\infty} A_2(t) < \limsup_{t\to\infty} A_2(t)\,; \tag{2.8}$$

$$\lim_{t\to\infty} A_2(t) \text{ exists as a finite number}\,; \tag{2.9}$$

$$\lim_{t\to\infty} A_2(t) = +\infty\,; \tag{2.10}$$

$$\lim_{t\to\infty} A_2(t) = -\infty\,. \tag{2.11}$$

As stated above, we see that if (2.11) is satisfied, then $\lim_{t\to\infty} A_q(t) = -\infty$ for any $q \in \boldsymbol{R}$, $q \geq 2$; and, in particular, $\lim_{t\to\infty} A_q(t) = -\infty$ for any $q \in \boldsymbol{R}$ with (2.7). Thus, under the condition that $\limsup_{t\to\infty} A_q(t) > -\infty$ for some $q \in \boldsymbol{R}$ with (2.7), the case (2.11) does not occur. Corollary 2 implies that, under the same condition on $A_q(t)$, all continuable solutions of (1.1) are oscillatory if (2.8) or (2.10) is satisfied. For the case where (2.9) is satisfied, the oscillation and nonoscillation of solutions of (1.2) are studied in [10, 11].

Let us discuss the special case of Corollary 2. The case $p = q$ immediately yields the following corollary.

Corollary 3. *Let equation* (1.1) *be strictly superlinear, strictly sublinear, or linear.*

(i) *If*

$$\liminf_{t\to\infty} A_q(t) < \limsup_{t\to\infty} A_q(t)$$

for some $q \in \boldsymbol{R}$ *satisfying* (2.7), *then all continuable solutions of* (1.1) *are oscillatory.*

(ii) *If*

$$\limsup_{t\to\infty} A_q(t) = +\infty$$

for some $q \in \boldsymbol{R}$ *satisfying* (2.7), *then all continuable solutions of* (1.1) *are oscillatory.*

It is easy to see that if $\liminf_{t\to\infty} A_r(t) > -\infty$ for some $r \in \boldsymbol{R}$ with $r \geq 1$, then $\liminf_{t\to\infty} A_q(t) > -\infty$ for $q = r + i$, $i = 0, 1, 2, \cdots$; and hence there always exists a $q \in \boldsymbol{R}$ which satisfies $\limsup_{t\to\infty} A_q(t) > -\infty$ and (2.7). Therefore, Corollary 2 gives the next result.

COROLLARY 4. *Let equation* (1.1) *be strictly superlinear, strictly sublinear, or linear. Suppose that*

$$\liminf_{t\to\infty} A_r(t) > -\infty$$

for some $r \in \boldsymbol{R}$, $r \geq 1$.

(i) *If*

$$\liminf_{t\to\infty} A_p(t) < \limsup_{t\to\infty} A_p(t)$$

for some $p \in \boldsymbol{R}$, $p \geq 2$, *then all continuable solutions of* (1.1) *are oscillatory.*

(ii) *If*

$$\limsup_{t\to\infty} A_p(t) = +\infty\,,$$

for some $p \in \boldsymbol{R}$, $p \geq 2$, *then all continuable solutions of* (1.1) *are oscillatory.*

The recent results of Philos and Purnaras [17] and Wong [25] follow from Corollary 4.

For the Emden-Fowler equation (1.2), Corollaries 3 and 4 include the well-known oscillation criteria of Butler [1] (the case $\gamma > 1$), Kamenev [7] and Butler [1] (the case $0 < \gamma < 1$), Wintner [20], Hartman [4, 5] and Kamenev [8] (the case $\gamma = 1$), and Wong [23] (the case $\gamma > 0$). (These oscillation criteria are summarized in [18, 21].) We must notice that, for the case $f(x) = |x|^\gamma \operatorname{sgn} x$ ($x \in \boldsymbol{R}$) with $0 < \gamma < 1$, Corollary 2 is not an essential extension of the results of Kamenev [7] and Butler [1]. As mentioned before, it is enough to consider the case $p = q = 2$ for the strictly sublinear equation, and the case $p = q = 2$ is already obtained in [1, 7]. For the case $f(x) = |x|^\gamma \operatorname{sgn} x$ ($x \in \boldsymbol{R}$) with $\gamma \geq 1$, Corollary 3 ensures the following result: If there is a $q \in \boldsymbol{R}$, $q > \gamma + 1$, satisfying

$$\liminf_{t\to\infty} A_q(t) < \limsup_{t\to\infty} A_q(t)$$

or

$$\limsup_{t\to\infty} A_q(t) = +\infty\,,$$

then all continuable solutions of (1.2) are oscillatory. This result is certainly new. As an example, consider the case

$$(2.12) \qquad a(t)=\frac{d^3}{dt^3}\{t^{\alpha+2}\cos(\log t)\}\,, \qquad t\geq 1\,,$$

where $\alpha>0$ is a positive constant. It is easily seen that, for each $i=1, 2, \cdots$, there are constants $c_i\neq 0$ and θ_i such that

$$A_i(t)=c_it^\alpha\cos(\log t+\theta_i)+O(1)$$

as $t\to\infty$. Thus we conclude by the above result that if $\gamma\geq 1$, then all continuable solutions of (1.2) with (2.12) are oscillatory. Note that Butler's oscillation criterion for $\gamma>1$ [1] cannot be applied to this example since $\liminf_{t\to\infty}A_2(t)=-\infty$.

3. Proof of Theorem

To prove Theorem 1, we need a few lemmas.

LEMMA 1. *Suppose that $b\in C[t_0, \infty)$ and that $p, q\in \boldsymbol{R}$, $p>q+1>0$. If*

$$(3.1) \qquad \lim_{t\to\infty}\frac{b(t)}{t^q}=\lambda$$

exists in the extended real line $\bar{\boldsymbol{R}}=\boldsymbol{R}\cup\{-\infty, +\infty\}$, then

$$(3.2) \qquad \lim_{t\to\infty}\frac{1}{t^{p-1}}\int_{t_0}^t(t-s)^{p-q-2}b(s)ds=B(q+1, p-q-1)\lambda\,,$$

where B is the Beta function.

PROOF. Note that

$$(3.3) \qquad \lim_{t\to\infty}\frac{1}{t^{p-1}}\int_T^t(t-s)^{p-q-2}s^qds=B(q+1, p-q-1)$$

for any $T\geq 0$. We first consider the case where λ is finite. Let $\varepsilon>0$ be an arbitrary positive number. By (3.1), there is a $T=T(\varepsilon)\geq t_0$ such that

$$|b(t)-\lambda t^q|\leq\varepsilon t^q\,, \qquad t\geq T.$$

We define T^* by

$$(3.4) \qquad T^*=T \text{ if } -1<p-q-2<0\,, \text{ and } T^*=t_0 \text{ if } p-q-2\geq 0\,.$$

Then we have

$$\begin{aligned}0&\leq\frac{1}{t^{p-1}}\int_{t_0}^t(t-s)^{p-q-2}|b(s)-\lambda s^q|\,ds\\&\leq\frac{(t-T^*)^{p-q-2}}{t^{p-1}}\int_{t_0}^T|b(s)-\lambda s^q|\,ds+\frac{\varepsilon}{t^{p-1}}\int_T^t(t-s)^{p-q-2}s^qds\end{aligned}$$

for $t > T$. Taking the upper limit as $t \to \infty$ in the above, and noting that (3.3) holds, we see that

$$0 \le \limsup_{t\to\infty} \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-q-2} |b(s) - \lambda s^q| ds \le \varepsilon B(q+1, p-q-1).$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we conclude that

$$\lim_{t\to\infty} \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-q-2} |b(s) - \lambda s^q| ds = 0. \tag{3.5}$$

Then, (3.3) with $T = t_0$ and (3.5) together yield (3.2).

We next consider the case $\lambda = +\infty$. Let $G > 0$ be an arbitrary number. It follows from (3.1) that $b(t) \ge Gt^q$, $t \ge T$, for some $T = T(G) \ge t_0$. We find that

$$\begin{aligned} &\frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-q-2} b(s) ds \\ &\qquad \ge -\frac{(t-T^*)^{p-q-2}}{t^{p-1}} \int_{t_0}^{T} |b(s)| ds + \frac{G}{t^{p-1}} \int_{T}^{t} (t-s)^{p-q-2} s^q ds \end{aligned}$$

for $t > T$, where T^* is given by (3.4). Then, by (3.3), we obtain

$$\liminf_{t\to\infty} \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-q-2} b(s) ds \ge GB(q+1, p-q-1).$$

Since $G > 0$ is arbitrary, this implies that

$$\lim_{t\to\infty} \frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-q-2} b(s) ds = +\infty.$$

Thus (3.2) is also true for the case $\lambda = +\infty$. The case $\lambda = -\infty$ can be similarly verified. The proof of Lemma 1 is complete.

The following equality is useful:

$$\frac{1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-1} b(s) ds = \frac{p-1}{t^{p-1}} \int_{t_0}^{t} (t-s)^{p-2} \left(\int_{t_0}^{s} b(\sigma) d\sigma \right) ds, \tag{3.6}$$

where $b \in C[t_0, \infty)$, $p \in \boldsymbol{R}$, $p > 1$.

LEMMA 2. *Let* $b \in C[t_0, \infty)$, *and let* $p \in \boldsymbol{R}$, $p > 1$. *If*

$$\lim_{t\to\infty} \int_{t_0}^{t} b(s) ds = \int_{t_0}^{\infty} b(s) ds \tag{3.7}$$

exists in the extended real line $\overline{\boldsymbol{R}}$, *then*

$$\lim_{t\to\infty}\frac{1}{t^{p-1}}\int_{t_0}^{t}(t-s)^{p-1}b(s)ds=\int_{t_0}^{\infty}b(s)ds\,. \tag{3.8}$$

PROOF. We have (3.6). Then, an application of Lemma 1 to the case $q=0$ gives

$$\lim_{t\to\infty}\frac{1}{t^{p-1}}\int_{t_0}^{t}(t-s)^{p-1}b(s)ds=(p-1)B(1,p-1)\int_{t_0}^{\infty}b(s)ds\,,$$

and so, by the equality $(p-1)B(1,p-1)=1$, we obtain (3.8). The proof of Lemma 2 is complete.

LEMMA 3. *Let* $a\in C[t_0,\infty)$. *If*

$$\lim_{t\to\infty}A_2(t)=\lambda \tag{3.9}$$

exists in the extended real line $\overline{\boldsymbol{R}}$, *then*

$$\lim_{t\to\infty}A_p(t)=\lambda \tag{3.10}$$

for any $p\in\boldsymbol{R}$, $p\geq 2$. *Here*, $A_p(t)$ *is defined by* (1.3).

PROOF. For the case $p=2$, the assertion (3.10) is trivial. Let $p\in\boldsymbol{R}$, $p>2$. In this case, using (3.6) twice, we see that

$$A_p(t)=\frac{(p-1)(p-2)}{t^{p-1}}\int_{t_0}^{t}(t-s)^{p-3}sA_2(s)ds\,,\qquad t\geq t_0\,.$$

From Lemma 1 applied to the case $q=1$ and $b(t)=tA_2(t)$ it follows that

$$\lim_{t\to\infty}A_p(t)=(p-1)(p-2)B(2,p-2)\lambda\,.$$

Then, because of $(p-1)(p-2)B(2,p-2)=1$, we have (3.10). This completes the proof of Lemma 3.

PROOF OF THEOREM 1. We suppose that equation (1.1) has a nonoscillatory solution $x(t)$ on an interval $[T,\infty)$, $T\geq t_0$. There is no loss of generality in assuming that $x(t)>0$ on $[T,\infty)$. Let $p\in\boldsymbol{R}$, $p\geq 2$. Then, from equation (1.1) it follows that

$$\begin{aligned}\frac{1}{t^{p-1}}\int_{T}^{t}(t-s)^{p-1}a(s)ds&=-\frac{1}{t^{p-1}}\int_{T}^{t}(t-s)^{p-1}\frac{x''(s)}{f(x(s))}ds\\&=-\frac{p-1}{t^{p-1}}\int_{T}^{t}(t-s)^{p-2}\left(\int_{T}^{s}\frac{x''(\sigma)}{f(x(\sigma))}d\sigma\right)ds\end{aligned}$$

for $t \geq T$. Thus, in view of

$$\int_T^s \frac{x''(\sigma)}{f(x(\sigma))} d\sigma = -\frac{x'(T)}{f(x(T))} + \frac{x'(s)}{f(x(s))} + \int_T^s \frac{f'(x(\sigma))[x'(\sigma)]^2}{[f(x(\sigma))]^2} d\sigma$$

for $s \geq T$, we find that

$$(3.11) \qquad \begin{aligned} &\frac{1}{t^{p-1}} \int_T^t (t-s)^{p-1} a(s) ds \\ &= \frac{x'(T)}{f(x(T))} \left(\frac{t-T}{t} \right)^{p-1} - \frac{p-1}{t^{p-1}} \int_T^t (t-s)^{p-2} \frac{x'(s)}{f(x(s))} ds \\ &\quad - \frac{1}{t^{p-1}} \int_T^t (t-s)^{p-1} \frac{f'(x(s))[x'(s)]^2}{[f(x(s))]^2} ds \end{aligned}$$

for $t \geq T$. Define $w \in C^1[T, \infty)$ and $k \in \boldsymbol{Z}$ by

$$w(t) = \int_{x(t)}^{+\infty} \frac{dy}{f(y)}, \qquad k = -1 \qquad \text{in the strictly superlinear case;}$$

$$w(t) = \int_{+0}^{x(t)} \frac{dy}{f(y)}, \qquad k = +1 \qquad \text{in the strictly sublinear case;}$$

and

$$w(t) = \log x(t), \qquad k = +1 \qquad \text{in the linear case.}$$

In each case we have $kw'(t) = x'(t)/f(x(t))$, $t \geq T$. Thus (3.11) may be rewritten as

$$(3.12) \qquad \begin{aligned} &\frac{1}{t^{p-1}} \int_T^t (t-s)^{p-1} a(s) ds \\ &= \frac{x'(T)}{f(x(T))} \left(\frac{t-T}{t} \right)^{p-1} - \frac{(p-1)k}{t^{p-1}} \int_T^t (t-s)^{p-2} w'(s) ds \\ &\quad - \frac{1}{t^{p-1}} \int_T^t (t-s)^{p-1} f'(x(s))[w'(s)]^2 ds \end{aligned}$$

for $t \geq T$. Now let us distinguish the two mutually exclusive cases where the integral

$$I \equiv \int_T^\infty f'(x(s))[w'(s)]^2 ds$$

is finite or is infinite.

The case where I is finite. In this case we will show that (I) occurs.

Taking Lemma 3 into account, we have only to verify the case $p = 2$. To this end, it is sufficient to prove that

$$\lim_{t\to\infty} \frac{1}{t}\int_T^t (t-s)a(s)ds \quad \text{exists and is finite}. \tag{3.13}$$

When $p = 2$, equality (3.12) becomes

$$\frac{1}{t}\int_T^t (t-s)a(s)ds = \frac{x'(T)}{f(x(T))}\frac{t-T}{t} + \frac{kw(T)}{t} - \frac{kw(t)}{t} - \frac{1}{t}\int_T^t (t-s)f'(x(s))[w'(s)]^2 ds \tag{3.14}$$

for $t \ge T$. We claim that

$$\lim_{t\to\infty} \frac{w(t)}{t} = 0. \tag{3.15}$$

First consider the case where equation (1.1) is strictly superlinear or strictly sublinear. Let $\tau \ge T$ be an arbitrary number. Then, for every $t \ge \tau$, we obtain

$$\begin{aligned} 0 < w(t) &= \left\{[w(\tau)]^{1/2} + \int_\tau^t \frac{d}{ds}[w(s)]^{1/2} ds\right\}^2 \\ &= \left\{[w(\tau)]^{1/2} + \frac{1}{2}\int_\tau^t [w(s)]^{-1/2} w'(s)ds\right\}^2 \\ &\le 2w(\tau) + \frac{1}{2}\left(\int_\tau^t [w(s)]^{-1/2} w'(s)ds\right)^2 \\ &\le 2w(\tau) + \frac{1}{2}\left(\int_\tau^t \frac{ds}{f'(x(s))w(s)}\right) \times \left(\int_\tau^t f'(x(s))[w'(s)]^2 ds\right), \end{aligned}$$

where Schwarz's inequality has been used at the last step. From the above inequality it follows that

$$0 < \frac{w(t)}{t} \le \frac{2w(\tau)}{t} + \frac{1}{2t}\left(\int_\tau^t \frac{ds}{f'(x(s))w(s)}\right) \times \left(\int_\tau^\infty f'(x(s))[w'(s)]^2 ds\right)$$

for $t \ge \tau$, which in the upper limit as $t \to \infty$ gives

$$\begin{aligned} 0 &\le \limsup_{t\to\infty} \frac{w(t)}{t} \\ &\le \frac{1}{2 \liminf_{t\to\infty} f'(x(t))w(t)} \int_\tau^\infty f'(x(s))[w'(s)]^2 ds. \end{aligned} \tag{3.16}$$

Note that, by (2.3) and (2.5), we have

$$\liminf_{t\to\infty} f'(x(t))w(t) > 0 .$$

Then, letting $\tau \to \infty$ in (3.16), we easily see that (3.15) is true.

Next consider the linear case. By using Schwarz's inequality, we obtain

$$\begin{aligned}|w(t)| &= \left| w(T) + \int_T^t w'(s)ds \right| \\ &\leq |w(T)| + (t-T)^{1/2}\left(\int_T^t [w'(s)]^2 ds\right)^{1/2} \\ &\leq |w(T)| + (t-T)^{1/2}\left(\int_T^\infty [w'(s)]^2 ds\right)^{1/2}\end{aligned}$$

for $t \geq T$. Then we easily see that (3.15) is also true.

Making use of (3.14) and (3.15), we find that

$$\lim_{t\to\infty} \frac{1}{t}\int_T^t (t-s)a(s)ds = \frac{x'(T)}{f(x(T))} - \int_T^\infty f'(x(s))[w'(s)]^2 ds ,$$

which proves (3.13).

The case where I is infinite. In this case we will show that (II) occurs. It is sufficient to prove that

$$\lim_{t\to\infty} \frac{1}{t^{q-1}}\int_T^t (t-s)^{q-1}a(s)ds = -\infty \tag{3.17}$$

for any $q \in \boldsymbol{R}$ satisfying (2.7).

First we consider the strictly superlinear case. Let $q > [2c(f)-1]/[c(f)-1]$ (>2). We utilize (3.12) with p replaced by q. Set

$$I_1(t) = \frac{1}{t^{q-1}}\int_T^t (t-s)^{q-2}w'(s)ds , \tag{3.18}$$

$$I_2(t) = \frac{1}{t^{q-1}}\int_T^t (t-s)^{q-1}f'(x(s))[w'(s)]^2 ds \tag{3.19}$$

for $t \geq T$. Then, (3.12) with $p = q$ is rewritten as

$$\frac{1}{t^{q-1}}\int_T^t (t-s)^{q-1}a(s)ds = \frac{x'(T)}{f(x(T))}\left(\frac{t-T}{t}\right)^{q-1} - (q-1)kI_1(t) - I_2(t) \tag{3.20}$$

for $t \geq T$. Using Schwarz's inequality, we find

$$[I_1(t)]^2 \leq \frac{1}{t^{q-1}}\int_T^t \frac{(t-s)^{q-3}}{f'(x(s))}ds \cdot I_2(t) , \qquad t \geq T. \tag{3.21}$$

Since (2.3) implies

$$\frac{1}{f'(x(t))} \le \frac{1}{c(f)} \int_{x(t)}^{+\infty} \frac{dy}{f(y)} = \frac{w(t)}{c(f)}, \qquad t \ge T,$$

we have

$$\begin{aligned}[I_1(t)]^2 &\le \frac{1}{c(f)t^{q-1}} \int_T^t (t-s)^{q-3} w(s) ds \cdot I_2(t) \\ &= \frac{1}{(q-2)c(f)t^{q-1}} \{(t-T)^{q-2} w(T) + t^{q-1} I_1(t)\} \cdot I_2(t)\end{aligned}$$

for $t \ge T$. Thus we obtain

$$\left[\frac{I_1(t)}{I_2(t)}\right]^2 \le \frac{1}{(q-2)c(f)} \left\{\frac{(t-T)^{q-2} w(T)}{t^{q-1} I_2(t)} + \frac{I_1(t)}{I_2(t)}\right\}$$

for $t > T$. This inequality may be regarded as a quadratic inequality with respect to $I_1(t)/I_2(t)$. Then we find that

$$\frac{I_1(t)}{I_2(t)} \le \frac{1}{2}\left\{\frac{1}{(q-2)c(f)} + [D(t)]^{1/2}\right\}, \qquad t > T, \tag{3.22}$$

where

$$D(t) = \left(\frac{1}{(q-2)c(f)}\right)^2 + \frac{4w(T)}{(q-2)c(f)} \frac{(t-T)^{q-2}}{t^{q-1} I_2(t)}, \qquad t > T.$$

By Lemma 2, $I_2(t)$ tends to I $(= +\infty)$ as $t \to \infty$, and consequently we have

$$\frac{(t-T)^{q-2}}{t^{q-1} I_2(t)} \to 0 \qquad \text{as } t \to \infty .$$

Taking the upper limit in (3.22), we get

$$\limsup_{t \to \infty} \frac{I_1(t)}{I_2(t)} \le \frac{1}{(q-2)c(f)}. \tag{3.23}$$

Since the condition $q > [2c(f) - 1]/[c(f) - 1]$ is equivalent to $(q-1)/[(q-2)c(f)] - 1 < 0$, it is possible to choose $\varepsilon_1 > 0$ such that

$$\varkappa \equiv (q-1)\left(\frac{1}{(q-2)c(f)} + \varepsilon_1\right) - 1 < 0 .$$

By (3.23), there is a $T_1 \ge T$ satisfying

$$I_1(t) \le \left(\frac{1}{(q-2)c(f)} + \varepsilon_1\right) I_2(t), \qquad t \ge T_1,$$

which implies, together with (3.20), that

$$\frac{1}{t^{q-1}}\int_T^t (t-s)^{q-1}a(s)ds \le \frac{x'(T)}{f(x(T))}\left(\frac{t-T}{t}\right)^{q-1} + \varkappa I_2(t)\,, \qquad t \ge T_1\,. \tag{3.24}$$

Then, noting that $\varkappa < 0$ and $I_2(t) \to +\infty$ $(t \to \infty)$, we get (3.17).

Next we consider the case where equation (1.1) is linear. We verify (3.17) for $q > 2$. As in the discussion for the strictly superlinear case, we have (3.20) and (3.21), where $I_1(t)$ and $I_2(t)$ are defined by (3.18) and (3.19), respectively. It follows from (3.20) and (3.21) that

$$\begin{aligned}\frac{1}{t^{q-1}}\int_T^t (t-s)^{q-1}a(s)ds &\le \frac{x'(T)}{f(x(T))}\left(\frac{t-T}{t}\right)^{q-1} \\ &\quad + (q-1)\left[\frac{(t-T)^{q-2}}{(q-2)t^{q-1}}I_2(t)\right]^{1/2} - I_2(t)\end{aligned}$$

for $t \ge T$. Let $-1 < \varkappa < 0$. Then, since $I_2(t) \to +\infty$ as $t \to \infty$, there is a $T_1 \ge T$ such that (3.24) holds. Thus, as in the strictly superlinear case, we obtain (3.17) where $q > 2$.

Finally we consider the strictly sublinear case. In this case, it is enough to prove

$$\lim_{t\to\infty}\frac{1}{t}\int_T^t (t-s)a(s)ds = -\infty\,. \tag{3.25}$$

We have (3.14). In the right-hand side of (3.14), the third term $-kw(t)/t$ is negative for $t \ge T$, and the last integral term tends to $-I = -\infty$ as $t \to \infty$. Thus (3.25) is evident.

This finishes the proof of Theorem 1.

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