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# On oscillation of half-linear functional differential equations with deviating arguments

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## 0. Introduction

This paper is devoted to the study of the oscillatory behavior of half-linear functional differential equations of the type

$$(|x'(t)|^{\alpha-1}x'(t))' = \sum_{i=1}^{n} p_i(t)|x(g_i(t))|^{\alpha-1}x(g_i(t)), \tag{A}$$

which can be written as

$$(|x'(t)|^{\alpha}\,\mathrm{sgn}\,x'(t))' = \sum_{i=1}^{n} p_i(t)|x(g_i(t))|^{\alpha}\,\mathrm{sgn}\,x(g_i(t)),$$

where $\alpha > 0$ is a constant, $p_i: [0, \infty) \to [0, \infty)$ is a continuous function such that $\sup\{p_i(t): t \geq T\} > 0$ for any $T \geq a$, $i = 1, 2, \ldots, n$, and $g_i: [0, \infty) \to R$ is a continuously differentiable function satisfying $g_i'(t) \geq 0$ for $t \geq a$ and $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \ldots, n$.

By a solution of (A) we mean a function $x \in C^1[T_x, \infty)$, $T_x \geq a$, which has the property $|x'|^{\alpha-1}x' \in C^1[T_x, \infty)$ and satisfies the equation for all sufficiently large $t \geq T_x$. Our attention will be restricted to those solutions $x(t)$ of (A) which satisfy $\sup\{|x(t)|: t \geq T\} > 0$ for all $T \geq T_x$. It is assumed that (A) does possess such a solution. A solution is said to be oscillatory if it has a sequence of zeros clustering at $t = \infty$; otherwise a solution is said to be nonoscillatory.

The half-linear ordinary differential equation

$$(|x'(t)|^{\alpha-1}x'(t))' = p(t)|x(t)|^{\alpha-1}x(t), \qquad p(t) \geq 0, \tag{B}$$

to which (A) reduces when $g_i(t) \equiv t$, $i = 1, 2, \ldots, n$, is nonoscillatory in the sense that all of its solutions are nonoscillatory; see Elbert [1]. However, the presence of at least one deviating argument $g_i(t) \not\equiv t$ in (A) may generate oscillation of some or all of its solutions as the following example shows.

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EXAMPLE. Let $S_\alpha(t)$ denote the solution of the equation

$$(|x'(t)|^{\alpha-1}x'(t))' + \alpha|x(t)|^{\alpha-1}x(t) = 0 . \qquad \alpha > 0 ,$$

satisfying the initial conditions $x(0) = 0$, $x'(0) = 1$. As was shown by Elbert [1], $S_\alpha(t)$ uniquely exists on $(-\infty, \infty)$ and is periodic with period $\pi_\alpha$, where

$$\pi_\alpha = \frac{\dfrac{2\pi}{\alpha+1}}{\sin\left(\dfrac{\pi}{\alpha+1}\right)} .$$

Furthermore $S_\alpha(t)$ satisfies

$$S_\alpha(t-\pi_\alpha) = S_\alpha(t+\pi_\alpha) = -S_\alpha(t) , \qquad t \in (-\infty, \infty) .$$

It follows that $S_\alpha(t)$ is an oscillatory solution of the functional differential equations

$$(|x'(t)|^{\alpha-1}x'(t))' = \alpha|x(t-\pi_\alpha)|^{\alpha-1}x(t-\pi_\alpha) ,$$

$$(|x'(t)|^{\alpha-1}x'(t))' = \alpha|x(t+\pi_\alpha)|^{\alpha-1}x(t+\pi_\alpha) .$$

In the first part of the paper we investigate the phenomena of oscillation of solutions of (A) generated by the deviating argument $g_i(t)$. As a result, it is shown that all bounded [respectively unbounded] solutions of (A) oscillate if one of the $g_i(t)$ is retarded [respectively advanced] and deviation $|g_i(t) - t|$ is large enough in some sense, and that all solutions of (A) oscillate if both retarded and advanced arguments with sufficiently large deviations are present.

In the second part of the paper the nonoscillatory behavior of (A) is studied in some detail. We establish criteria for the existence of both bounded and unbounded nonoscillatory solutions with specified asymptotic properties of the equation (A). The results developed therein show that basic aspects of the existence of nonoscillatory solutions of the functional differential equation (A) with deviating arguments are shared with the corresponding ordinary differential equation (B) without deviating arguments.

## 1. Oscillation of solutions

We begin by considering functional differential inequalities of the form

$$\{(|x'(t)|^{\alpha-1}x'(t))' - p(t)|x(g(t))|^{\alpha-1}x(g(t))\} \operatorname{sgn} x(g(t)) \geq 0 , \tag{1.1}$$

where $\alpha > 0$ is a constant, $p : [a, \infty) \to [0, \infty)$ is a continuous function such that $\sup\{p(t) : t \geq T\} > 0$ for any $T \geq a$ and $g : [a, \infty) \to R$ is a continuously differentiable function satisfying $g'(t) \geq 0$ for $t \geq a$ and $\lim_{t\to\infty} g(t) = \infty$.

Let $x(t)$ be a nonoscillatory solution of (1.1). It is easy to see that $x'(t)$ is eventually of constant sign, so that

$$\text{(1.2)} \qquad \text{either} \quad x(t)x'(t) < 0 \quad \text{or} \quad x(t)x'(t) > 0 \quad \text{for } t \geq T,$$

provided $T$ is sufficiently large. Clearly, $x(t)$ is bounded or unbounded according to whether the first or the second inequality in (1.2) holds. Note that if $x(t) > 0$ for $t \geq T$, then (1.1) implies that $x'(t)$ is increasing for $t \geq T^*$, where $T^* > T$ is chosen so large that $\lim_{t \geq T^*} g(t) \geq T$, and hence $x(t)$ is a convex function on $[T^*, \infty)$.

In the case where $g(t)$ is a retarded argument it may happen that (1.1) admits no bounded nonoscillatory solutions as the following theorem shows.

THEOREM 1.1. *Suppose that* $g(t) < t$ *for* $t \geq a$ *and that either*

$$\text{(1.3)} \qquad \limsup_{t \to \infty} \int_{g(t)}^{t} p(s)(g(t) - g(s))^\alpha ds > 1$$

*or*

$$\text{(1.4)} \qquad \limsup_{t \to \infty} \int_{g(t)}^{t} \left( \int_{s}^{t} p(r)dr \right)^{1/\alpha} ds > 1 .$$

*Then* (1.1) *has no bounded nonoscillatory solutions.*

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (1.1). Without loss of generality we may suppose that $x(t)$ is eventually positive. Thus, there is $T > a$ such that $x(t) > 0$ and $x'(t) < 0$ for $t \geq T$.

Suppose first that (1.3) holds. Let $T^* > T$ be such that $\inf_{t \geq T^*} g(t) \geq T$. Since $x(t)$ is convex on $[T^*, \infty)$, we have

$$x(\sigma) \geq x(\tau) + x'(\tau)(\sigma - \tau) \geq -x'(\tau)(\tau - \sigma), \qquad \tau \geq \sigma \geq T^* .$$

Substituting $g(s)$ and $g(t)$ for $\sigma$ and $\tau$, respectively, in the above, we obtain

$$x(g(s)) \geq -x'(g(t))(g(t) - g(s)), \qquad t \geq s \geq T^* ,$$

which implies

$$p(s)(x(g(s)))^\alpha \geq (-x'(g(t)))^\alpha (g(t) - g(s))^\alpha , \qquad t \geq s \geq T^* .$$

Replace the left hand side of the above by

$$(|x'(s)|^{\alpha-1} x'(s))' = -((-x'(s))^\alpha)'$$

and integrate from $g(t)$ to $t$. We then have

$$(-x'(g(t)))^\alpha - (-x'(t))^\alpha \geq (-x'(g(t)))^\alpha \int_{g(s)}^{t} p(s)(g(t) - g(s))^\alpha ds , \qquad t \geq T^* ,$$

whence it follows that

$$(-x'(g(t)))^{\alpha}\left[\int_{g(t)}^{t} p(s)(g(t)-g(s))^{\alpha}ds - 1\right] \leq 0\,, \qquad t \geq T^*\,.$$

But this is inconsistent with (1.3).

Suppose next that (1.4) holds. Integration of (1.1) over $[\sigma, t]$ gives

$$\begin{aligned}(-x'(\sigma))^{\alpha} &= (-x'(t))^{\alpha} + \int_{\sigma}^{t} p(r)(x(g(r)))^{\alpha}dr \\ &\geq \int_{\sigma}^{t} p(r)(x(g(r)))^{\alpha}dr\,, \qquad t \geq \sigma \geq T^*\,,\end{aligned}$$

which implies

$$-x'(\sigma) \geq \left(\int_{\sigma}^{t} p(r)(x(g(r)))^{\alpha}dr\right)^{1/\alpha}, \qquad t \geq \sigma \geq T^*\,. \tag{1.5}$$

Substituting (1.5) into

$$x(s) = x(t) + \int_{s}^{t} (-x'(\sigma))d\sigma\,, \qquad t \geq s \geq T^*\,, \tag{1.6}$$

we have

$$x(s) \geq \int_{s}^{t}\left(\int_{\sigma}^{t} p(r)(x(g(r)))^{\alpha}dr\right)^{1/\alpha} d\sigma\,, \qquad t \geq s \geq T^*\,.$$

Putting $s = g(t)$ in (1.6) and using the fact that $x(g(t))$ is decreasing, we conclude that

$$x(g(t))\left[\int_{g(t)}^{t}\left(\int_{\sigma}^{t} p(r)dr\right)^{1/\alpha} d\sigma - 1\right] \leq 0\,, \qquad t \geq T^*\,,$$

which contradicts (1.4). Thus the proof of Theorem 1.1 is complete.

A duality to Theorem 1.1 holds in the case where $g(t)$ is an advanced argument.

THEOREM 1.2. *Suppose that $g(t) > t$ for $t \geq a$ and that either*

$$\limsup_{t\to\infty} \int_{t}^{g(t)} p(s)(g(s)-g(t))^{\alpha}ds > 1 \tag{1.7}$$

*or*

$$\limsup_{t\to\infty} \int_{t}^{g(t)}\left(\int_{t}^{s} p(r)dr\right)^{1/\alpha} ds > 1\,. \tag{1.8}$$

*Then* (1.1) *has no unbounded nonoscillatory solutions.*

PROOF. Let $x(t)$ be an unbounded nonoscillatory solution of (1.1) which may be assumed to be eventually positive. There is a $T > a$ such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq T$.

Suppose that (1.7) holds. The convexity of $x(t)$ means

$$x(\sigma) \geq x(\tau) + x'(\tau)(\sigma - \tau) \geq x'(\tau)(\sigma - \tau), \qquad \sigma \geq \tau \geq T^*,$$

where $T^*$ is as in the proof of Theorem 1.1. Letting $\sigma = g(s)$, $\tau = g(t)$ in the above, we see that

$$((x'(s))^\alpha)' = p(s)(x(g(s)))^\alpha \geq (x'(g(t)))^\alpha p(s)(g(s) - g(t))^\alpha, \qquad s \geq t \geq T^*.$$

Integrating the above on $[t, g(t)]$ yields

$$(x'(g(t)))^\alpha \left[ \int_t^{g(t)} p(s)(g(s) - g(t))^\alpha ds - 1 \right] \leq 0, \qquad t \geq T^*,$$

which is a contradiction because of (1.7).

Suppose that (1.8) holds. Combining the inequality

$$x'(\sigma) \geq \left( \int_t^\sigma p(r)(x(g(r)))^\alpha dr \right)^{1/\alpha}, \qquad \sigma \geq t \geq T^*,$$

with the relation

$$x(s) = x(t) + \int_t^s x'(\sigma) d\sigma, \qquad s \geq t \geq T^*,$$

we obtain

$$x(s) \geq \int_t^s \left( \int_t^\sigma p(r)(x(g(r)))^\alpha dr \right)^{1/\alpha} d\sigma, \qquad s \geq t \geq T^*.$$

Putting $s = g(t)$ and noting that $x(g(t))$ is increasing, we obtain the following contradiction to (1.8):

$$x(g(t)) \left[ \int_t^{g(t)} \left( \int_t^\sigma p(r) dr \right)^{1/\alpha} d\sigma - 1 \right] \leq 0, \qquad t \geq T^*.$$

This completes the proof.

One of our main results now follows from the above theorems.

THEOREM 1.3. (i) *All bounded solutions of* (A) *are oscillatory if there is an* $i \in \{1, 2, \dots, n\}$ *such that* $g_i(t) < t$ *for* $t \geq a$ *and one of the following inequalities holds*:

$$\limsup_{t\to\infty} \int_{g_i(t)}^t p_i(s)(g_i(t) - g_i(s))^\alpha ds > 1\,; \tag{1.9}$$

(1.10)
$$\limsup_{t\to\infty} \int_{g_i(t)}^{t} \left( \int_s^t p_i(r)dr \right)^{1/\alpha} ds > 1 .$$

(ii) *All unbounded solutions of* (A) *are oscillatory if there is a* $j \in \{1, 2, \ldots, n\}$ *such that* $g_j(t) > t$ *for* $t \geq a$ *and one of the following inequalities holds*:

(1.11)
$$\limsup_{t\to\infty} \int_t^{g_j(t)} p_j(s)(g_j(s) - g_j(t))^\alpha ds > 1 ;$$

(1.12)
$$\limsup_{t\to\infty} \int_t^{g_j(t)} \left( \int_t^s p_j(r)dr \right)^{1/\alpha} ds > 1 .$$

(iii) *All solutions of* (A) *are oscillatory if there are* $i$ *and* $j \in \{1, 2, \ldots, n\}$ *such that* $g_i(t)$ *and* $g_j(t)$ *satisfy the conditions of* (i) *and* (ii) *respectively.*

PROOF. (i) Suppose to the contrary that (A) has a bounded nonoscillatory solution $x(t)$. Then, from (A) we see that $x(t)$ satisfies the differential inequality

(1.13)
$$\{(|x'(t)|^{\alpha-1}x'(t))' - p_i(t)|x(g_i(t))|^{\alpha-1}x(g_i(t))\} \operatorname{sgn} x(g_i(t)) \geq 0 ,$$

for all sufficiently large $t$. This, however, is impossible, because the possibility of the existence of bounded nonoscillatory solutions for (1.13) is excluded by Theorem 1.1.

(ii) An unbounded nonoscillatory solution $x(t)$ of (A), if exists, satisfies the differential inequality (1.13) with $i$ replaced by $j$ for sufficiently large $t$. But this is impossible because of Theorem 1.2, and so every unbounded solution of (A) must be oscillatory.

(iii) The final statement of Theorem 1.3 is an immediate consequence of (i) and (ii).

EXAMPLE 1.1. Consider the equations

(1.14)
$$(|x'(t)|^{\alpha-1}x'(t))' = k|x(t-\sigma)|^{\alpha-1}x(t-\sigma) ,$$

(1.15)
$$(|x'(t)|^{\alpha-1}x'(t))' = l|x(t+\tau)|^{\alpha-1}x(t+\tau) ,$$

(1.16)
$$(|x'(t)|^{\alpha-1}x'(t))' = k|x(t-\sigma)|^{\alpha-1}x(t-\sigma) + l|x(t+\tau)|^{\alpha-1}x(t+\tau) ,$$

where $\alpha$, $k$, $l$, $\sigma$ and $\tau$ are positive constants. The conditions of (i) of Theorem 1.3 are satisfied if

(1.17)
$$k\sigma^{\alpha+1} > \alpha + 1 \qquad \text{or} \qquad k^{1/\alpha}\sigma^{(\alpha+1)/\alpha} > \frac{\alpha+1}{\alpha} ,$$

so that all bounded solutions of (1.14) are oscillatory. Similarly, from (ii) of Theorem 1.3 it follows that all unbounded solutions of (1.5) are oscillatory if

$$l\tau^{\alpha+1} > \alpha + 1 \quad \text{or} \quad l^{1/\alpha}\tau^{(\alpha+1)/\alpha} > \frac{\alpha+1}{\alpha}. \tag{1.18}$$

The last statement of Theorem 1.3 implies that all solutions of (1.17) are oscillatory provided both (1.17) and (1.18) are satisfied.

EXAMPLE 1.2. Consider the equations

$$(|x'(t)|^{-1/2}x'(t))' = t^{-3/2}\left|x\left(\frac{t}{\theta}\right)\right|^{-1/2}x\left(\frac{t}{\theta}\right), \tag{1.19}$$

$$(|x'(t)|^{-1/2}x'(t))' = t^{-3/2}|x(\theta t)|^{-1/2}x(\theta t), \tag{1.20}$$

where $\theta > 1$ is a constant, which are special cases of (A) with $i = 1$, $\alpha = \frac{1}{2}$, $p_i(t) = t^{-3/2}$ and $g_1(t) = \frac{t}{\theta}$ or $g_1(t) = \theta t$. In the case of $g_1(t) = \frac{t}{\theta}$ we have

$$\int_{g_1(t)}^{t}\left(\int_s^t p_1(r)dr\right)^{1/\alpha} ds = \int_{t/\theta}^{t}\left(\int_s^t r^{-3/2}dr\right)^2 ds = 4(\ln\theta + 4\theta^{-1/2} - \theta^{-1} - 3),$$

and in the case of $g_1(t) = \theta t$ we have

$$\int_t^{g_1(t)}\left(\int_t^s p_1(r)dr\right)^{1/\alpha} ds = \int_t^{\theta t}\left(\int_t^s r^{-3/2}dr\right)^2 ds = 4(\ln\theta + \theta - 4\theta^{1/2} + 3).$$

Since the above integrals tend to $\infty$ as $\theta \to \infty$ the conditions (1.10) and (1.12) of Theorem 1.3 are satisfied provided $\theta$ is taken sufficiently large. It follows that all bounded solutions of (1.19) and all unbounded solutions of (1.20) are oscillatory if $\theta$ is sufficiently large. It can be shown that, for sufficiently large values of $\theta > 1$ and $\theta' > 1$, all solutions of the equation

$$(|x'(t)|^{-1/2}x'(t))' = t^{-3/2}\left|x\left(\frac{t}{\theta}\right)\right|^{-1/2}x\left(\frac{t}{\theta}\right) + t^{-3/2}|x(\theta' t)|^{-1/2}x(\theta' t) \tag{1.21}$$

are oscillatory.

As examples of (A) to which the criteria (1.9) and (1.11) easily apply we give the equations

$$(|x'(t)|x'(t))' = t^{-3}\left|x\left(\frac{t}{\theta}\right)\right|x\left(\frac{t}{\theta}\right),$$

$$(|x'(t)|x'(t))' = t^{-3}|x(\theta t)|x(\theta t),$$

$$(|x'(t)|x'(t))' = t^{-3}\left|x\left(\frac{t}{\theta}\right)\right|x\left(\frac{t}{\theta}\right) + t^{-3}|x(\theta' t)|x(\theta' t).$$

REMARK 1.1. The above theorems extend the results of Ladas, Ladde and Papadakis [4] for the linear delay equation

$$x''(t) = \sum_{i=1}^{n} p_i(t)x(g_i(t))$$

as well as the second order versions of the basic results of Koplatadze and Čanturija [2], Kusano [3], Ladas, Lakshmikantham and Papadakis [6] concerning higher order linear functional differential equations with deviating arguments.

## 2. Nonoscillation of solutions

We are now interested in the existence and asymptotic behavior of nonoscillatory solutions of (A). If $x(t)$ is a nonoscillatory solution of (A), then there is a $t_0 > a$ such that either

$$x(t)x'(t) > 0 \qquad \text{for } t \geq t_0\,, \tag{2.1}$$

or

$$x(t)x'(t) < 0 \qquad \text{for } t \geq t_0\,. \tag{2.2}$$

If (2.1) holds, then $x(t)$ is unbounded and the limit $x'(\infty) = \lim_{t\to\infty} x'(t)$, either finite or infinite, exists; if (2.2) holds, then $x(t)$ is bounded and the finite limit $x(\infty) = \lim_{t\to\infty} x(t)$ exists.

In what follows we need only to consider eventually positive solutions of (A), since if $x(t)$ satisfies (A), then so does $-x(t)$. Let $x(t)$ be an eventually positive solution of (A) satisfying (2.1) and having a finite limit $x'(\infty) = \lim_{t\to\infty} x'(t) > 0$. Two integrations of (A) then yields

$$x(t) = x(t_1) + \int_{t_1}^{t} \left( (x'(\infty))^\alpha - \int_s^\infty \sum_{i=1}^{n} p_i(r)(x(g_i(r)))^\alpha dr \right)^{1/\alpha} ds\,, \qquad t \geq t_1\,, \tag{2.3}$$

where $t_1 > t_0$ is chosen so that $\inf_{t\geq t_1} g_i(t) \geq t_0$, $i = 1, 2, \ldots, n$. Let $x(t)$ be an eventually positive solution of (A) satisfying (2.2). Then, we have

$$x(t) = x(\infty) + \int_t^\infty \left( \int_s^\infty \sum_{i=1}^{n} p_i(r)(x(g_i(r)))\,dr \right)^{1/\alpha} ds\,, \qquad t \geq t_1\,, \tag{2.4}$$

after integrating (A) twice from $t$ to $\infty$.

Based on these integral representations (2.3), (2.4) we can prove the following existence theorems.

THEOREM 2.1. *The equation* (A) *has a nonoscillatory solution* $x(t)$ *such that* $\lim_{t\to\infty} \dfrac{x(t)}{t} =$ *constant* $\neq 0$ *if and only if*

$$\int^\infty p_i(t)(g_i(t))^\alpha dt < \infty\,, \qquad i = 1, 2, \ldots, n\,. \tag{2.5}$$

THEOREM 2.2. *The equation* (A) *has a nonoscillatory solution* $x(t)$ *such that* $\lim_{t\to\infty} x(t) = constant \neq 0$ *if and only if*

$$\int^{\infty} \left( \int_t^{\infty} p_i(s)ds \right)^{1/\alpha} dt < \infty\,, \qquad i = 1, 2, \ldots, n\,. \tag{2.6}$$

PROOF OF THEOREM 2.1. (The "only if" part) Let $x(t)$ be a nonoscillatory solution of (A) satisfying $\lim_{t\to\infty} \frac{x(t)}{t} = c > 0$. Then from (2.3) we see that

$$\int^{\infty} \sum_{i=1}^{n} p_i(t)(x(g_i(t)))^{\alpha} dt < \infty\,.$$

This, combined with the relation $\lim_{t\to\infty} \frac{x(g_i(t))}{g_i(t)} = c$, $i = 1, 2, \ldots, n$, immediately implies (2.5).

(The "if" part) Suppose that (2.5) holds. Let $k > 0$ be fixed arbitrarily and take $T > a$ so large that

$$T_* = \min_i \left\{ \inf_{t \geq T} g_i(t) \right\} \geq a \tag{2.7}$$

and

$$\sum_{i=1}^{n} \int_T^{\infty} p_i(t)(g_i(t))^{\alpha} dt \leq \frac{2^{\alpha} - 1}{2^{\alpha}}\,.$$

Consider the set $X \subset C[T_*, \infty)$ and the mapping $F : X \to C[T_*, \infty)$ defined by

$$X = \left\{ x \in C[T_*, \infty) : \frac{k}{2}(t - T) \leq x(t) \leq k(t - T),\, t \geq T;\, x(t) = 0,\, T_* \leq t \leq T \right\}$$

and

$$(Fx)(t) = \int_T^t \left( k^{\alpha} - \int_s^{\infty} \sum_{i=1}^{n} p_i(r)(x(g_i(r)))^{\alpha} dr \right)^{1/\alpha} ds\,, \qquad t \geq T\,,$$

$$(Fx)(t) = 0\,, \qquad T_* \leq t \leq T\,.$$

It is clear that $X$ is a closed convex subset of the Fréchet space $C[T, \infty)$ of continuous functions on $[T_*, \infty)$ with the usual metric topology and that $F$ is well defined and continuous on $X$. It can be shown without difficulty that $F$ maps $X$ into itself and $F(X)$ is relatively compact in $C[T_*, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, $F$ has a fixed element $x$ in $X$, which satisfies

$$x(t) = \int_T^t \left( k^\alpha - \int_s^\infty \sum_{i=1}^n p_i(r)(x(g_i(r)))^\alpha dr \right)^{1/\alpha} ds\,, \qquad t \geq T.$$

Differentiation shows that $x(t)$ satisfies (A) for $t \geq T$ and $\lim_{t\to\infty} \dfrac{x(t)}{t} = \lim_{t\to\infty} x'(t) = k$.

PROOF OF THEOREM 2.2. The truth of the "only if" part follows readily from (2.4).

To prove the "if" part, suppose that (2.6) is satisfied. Choose $T > a$ so that (2.7) holds and

$$\int_T^\infty \left( \int_t^\infty \sum_{i=1}^n p_i(s) ds \right)^{1/\alpha} \leq \frac{1}{2}$$

and define $Y \subset C[T_*, \infty)$ and $G: Y \to C[T_*, \infty)$ by

$$Y = \{ y \in C[T_*, \infty) : k \leq y(t) \leq 2k\,, \quad t \geq T_* \}\,,$$

$k > 0$ being a fixed constant, and

$$(Gy)(t) = k + \int_t^\infty \left( \int_s^\infty \sum_{i=1}^n p_i(r)(y(g_i(r)))^\alpha dr \right)^{1/\alpha} ds\,, \qquad t \geq T,$$

$$(Gy)(t) = (Gy)(T)\,, \qquad T_* \leq t \leq T.$$

As in the proof of Theorem 2.1 one can verify that $G$ maps $Y$ into a relatively compact subset of $Y$, so that there exists a $y \in Y$ such that

$$y(t) = k + \int_t^\infty \left( \int_s^\infty \sum_{i=1}^n p_i(r)(y(g_i(r)))^\alpha dr \right)^{1/\alpha} ds\,, \qquad t \geq T.$$

Differentiating this equation twice, one sees that $y(t)$ satisfies (A) for $t \geq T$. Since $y(t) \to k$ as $t \to \infty$, $y(t)$ is a solution of (A) with the desired asymptotic property. This completes the proof.

It remains to discuss the existence of an unbounded nonoscillatory solution $x(t)$ of (A) with the property $\lim_{t\to\infty} \dfrac{|x(t)|}{t} = \infty$ and of a bounded solution $x(t)$ of (A) with the property $\lim_{t\to\infty} x(t) = 0$. This is a difficult problem and there seems to be no general criteria for the existence of such solutions. Below we confine our attention to the case where at least one of the $g_i(t)$ is retarded and show that some sufficient conditions can be derived under which (A) has a nonoscillatory solution tending to zero as $t \to \infty$. (Such solution is often referred to as a decaying nonoscillatory solution.) Our derivation is based on the following theorem which is essentially due to Philos [7].

THEOREM 2.3. *Suppose that there is an* $i_0 \in \{1, 2, \dots, n\}$ *such that*

$$g_{i_0}(t) < t \quad \text{and} \quad p_{i_0}(t) > 0 \quad \text{for } t \geq a. \tag{2.8}$$

*Suppose, in addition, that there exists a positive decreasing function* $\phi(t)$ *on* $[t_0, \infty)$ *satisfying*

$$\phi(t) \geq \int_t^\infty \left( \int_s^\infty \sum_{i=1}^n p_i(r)(\phi(g_i(r)))^\alpha dr \right)^{1/\alpha} ds, \qquad t \geq t_0, \tag{2.9}$$

*where* $t_0$ *is chosen so that* $\inf_{t \geq t_0} g_i(t) \geq a$, $i = 1, 2, \dots, n$. *Then* (A) *has a nonoscillatory solution tending to zero as* $t \to \infty$.

PROOF. Let $Z$ denote the set

$$Z = \{z \in C[t_0, \infty) : 0 \leq z \leq \phi(t), \quad t \geq t_0\}.$$

With each $z \in Z$ we associate the function $\tilde{z} \in C[a, \infty)$ defined by

$$\tilde{z}(t) = \begin{cases} z(t) & \text{for } t \geq t_0; \\ z(t_0) + [\phi(t) - \phi(t_0)] & \text{for } a \leq t \leq t_0. \end{cases} \tag{2.10}$$

Define the mapping $H : Z \to C[t_0, \infty)$ by

$$(Hz)(t) = \int_t^\infty \left( \int_s^\infty \sum_{i=1}^n p_i(r)(\tilde{z}(g_i(r)))^\alpha dr \right)^{1/\alpha} ds, \qquad t \geq t_0.$$

Then $H$ is shown to be a continuous mapping which sends $Z$ into a relatively compact subset of $Z$. It follows therefore that there exists a $z \in Z$ such that $z = Hz$, i.e.,

$$z(t) = \int_t^\infty \left( \int_s^\infty \sum_{i=1}^n p_i(r)(\tilde{z}(g_i(r)))^\alpha dr \right)^{1/\alpha} ds, \qquad t \geq t_0.$$

Differentiating the above twice shows that

$$(-(-z'(t))^\alpha)' = \sum_{i=1}^n p_i(t)(\tilde{z}(g_i(t)))^\alpha, \qquad t \geq t_0,$$

which, in view of (2.10), implies that $z(t)$ is a solution of (A) for all sufficiently large $t$. That $z(t) > 0$ for $t \geq t_0$ can be verified exactly as in Philos [7: p. 170], and so the details are omitted. This completes the proof.

In order to apply Theorem 2.3 to construct decaying nonoscillatory solutions of (A) we distinguish the three cases:

$$\int^\infty \sum_{i=1}^n p_i(t)dt < \infty \quad \text{and} \quad \int^\infty \left( \int_t^\infty \sum_{i=1}^n p_i(s)ds \right)^{1/\alpha} dt < \infty; \tag{2.11}$$

(2.12) $$\int^{\infty} \sum_{i=1}^{n} p_i(t)dt < \infty \qquad \text{but} \qquad \int^{\infty} \left( \int_t^{\infty} \sum_{i=1}^{n} p_i(s)ds \right)^{1/\alpha} dt = \infty\,;$$

(2.13) $$\int^{\infty} \sum_{i=1}^{n} p_i(t)dt = \infty\,.$$

The condition (2.11), which is nothing else but (2.6), always guarantees the existence of a decaying nonoscillatory solution of (A).

THEOREM 2.4. *Suppose that* (2.8) *holds for some* $i_0 \in \{1, 2, \ldots, n\}$. *If* (2.6) *is satisfied, then* (A) *possesses a nonoscillatory solution tending to zero as* $t \to \infty$.

PROOF. Let $t_0$ be large enough so that $\min_i \{\inf_{t \geq t_0} g_i(t)\} \geq \max\{a, 1\}$ and

(2.14) $$\int_{t_0}^{\infty} \left( \int_s^{\infty} \sum_{i=1}^{n} p_i(r)dr \right)^{1/\alpha} ds \leq \frac{1}{2}\,.$$

Choose $\phi(t) = 1 + \dfrac{1}{t}$. Using (2.14), we see that $\phi(t)$ satisfies (2.9):

$$\int_t^{\infty} \left( \int_s^{\infty} \sum_{i=1}^{n} p_i(r)(\phi(g_i(r)))^{\alpha} dr \right)^{1/\alpha} ds = \int_t^{\infty} \left( \int_s^{\infty} \sum_{i=1}^{n} p_i(r) \left( 1 + \frac{1}{g_i(r)} \right)^{\alpha} dr \right)^{1/\alpha} ds$$
$$\leq 2 \int_t^{\infty} \left( \int_s^{\infty} \sum_{i=1}^{n} p_i(r)dr \right)^{1/\alpha} ds \leq 1 < \phi(t)\,, \qquad t \geq t_0\,.$$

The conclusion follows from Theorem 2.3.

We now state existence theorems of decaying nonoscillatory solutions which are applicable to the cases (2.12) and (2.13).

THEOREM 2.5. *Suppose that* (2.8) *holds for some* $i_0 \in \{1, 2, \ldots, n\}$ *and that*

(2.15) $$\limsup_{t \to \infty} \int_{g_*(t)}^{t} \left( \int_s^{\infty} \sum_{i=1}^{n} p_i(r)dr \right)^{1/\alpha} ds < \frac{1}{e}\,,$$

*where* $g_*(t) = \min_i g_i(t)$. *Then* (A) *possesses a nonoscillatory solution tending to zero as* $t \to \infty$.

PROOF. We put

$$P(t) = \left( \int_t^{\infty} \sum_{i=1}^{n} p_i(s)ds \right)^{1/\alpha}$$

and choose $t_0 > 0$ so that $\inf_{t \geq t_0} g_*(t) \geq a$ and

(2.16) $$P_{t_0} := \sup_{t \geq t_0} \int_{g_*(t)}^{t} P(s)ds \leq \frac{1}{e}\,.$$

Define

$$\phi(t)=\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right).$$

Since, for $i=1,\ 2,\ \dots,\ n$,

$$\phi(g_i(t))=\exp\left(\frac{1}{P_{t_0}}\int_{g_i(t)}^t P(s)ds\right)\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right)$$
$$\le e\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right)=e\phi(t)\,,\qquad t\ge t_0\,,$$

we have, in view of (2.16),

$$\int_t^\infty\left(\int_s^\infty\sum_{i=1}^n p_i(r)(\phi(g_i(r)))^\alpha dr\right)^{1/\alpha}ds\le e\int_t^\infty P(s)\phi(s)ds$$
$$=e\int_t^\infty P(s)\exp\left(-\frac{1}{P_{t_0}}\int_a^s P(r)dr\right)ds\le eP_{t_0}\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right)$$
$$=eP_{t_0}\phi(t)\le\phi(t)\,,\qquad t\ge t_0\,.$$

From Theorem 2.3 it follows that (A) has a decaying nonoscillatory solution.

THEOREM 2.6. *Suppose that* (2.8) *holds for some* $i_0\in\{1,2,\dots,n\}$. *Further, suppose that there exists a* $t_0>a$ *such that* $\inf_{t\ge t_0}g_*(t)\ge a$,

$$P_{t_0}:=\inf_{t\ge t_0}P(t)>0\qquad\textit{and}\qquad\sup_{t\ge t_0}\int_{g_*(t)}^t\sum_{i=1}^n p(s)ds\le\frac{\alpha+1}{e}\left(\frac{P_{t_0}}{\alpha}\right)^{\alpha/(\alpha+1)}.\tag{2.17}$$

*Then* (A) *possesses a nonoscillatory solution tending to zero as* $t\to\infty$.

PROOF. Put

$$Q_{t_0}=\sup_{t\ge t_0}\int_{g_*(t)}^t\sum_{i=1}^n p_i(s)ds\qquad\text{and}\qquad\phi(t)=\exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}}\int_a^t\sum_{i=1}^n p_i(s)ds\right).$$

We see that

$$\phi(g_i(t))\le\exp\left(\frac{\alpha+1}{\alpha}\right)\phi(t)\,,\qquad t\ge t_0\,,\qquad i=1,\ 2,\ \dots,\ n\,,$$

and hence that

$$\int_t^\infty\sum_{i=1}^n p_i(s)(\phi(g_i(s)))^\alpha ds\le e^{\alpha+1}\int_t^\infty\left(\sum_{i=1}^n p_i(s)\right)(\phi(s))^\alpha ds$$
$$=e^{\alpha+1}\int_t^\infty\sum_{i=1}^n p_i(s)\exp\left(-\frac{\alpha+1}{Q_{t_0}}\int_a^s\sum_{i=1}^n p_i(r)dr\right)ds$$
$$\le\frac{Q_{t_0}}{\alpha+1}e^{\alpha+1}\exp\left(-\frac{\alpha+1}{Q_{t_0}}\int_a^t\sum_{i=1}^n p_i(s)ds\right),\qquad t\ge t_0\,.$$

Consequently, we obtain

$$\int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r)(\phi(g_i(r)))^\alpha dr\right)^{1/\alpha} ds$$

$$\leq \left(\frac{Q_{t_0}}{\alpha+1}\right)^{1/\alpha} e^{(\alpha+1)/\alpha} \int_t^\infty \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^s \sum_{i=1}^n p_i(r)dr\right) ds$$

$$\leq \frac{1}{P_{t_0}} \left(\frac{Q_{t_0}}{\alpha+1}\right)^{1/\alpha} e^{(\alpha+1)/\alpha} \int_t^\infty \sum_{i=1}^n p_i(s) \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^s \sum_{i=1}^n p_i(r)dr\right) ds$$

$$\leq \frac{\alpha Q_{t_0}}{(\alpha+1)P_{t_0}} \left(\frac{Q_{t_0}}{\alpha+1}\right)^{1/\alpha} e^{(\alpha+1)/\alpha} \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^t \sum_{i=1}^n p_i(s)ds\right) \leq \phi(t), \qquad t \geq t_0,$$

where (2.17) has been used. This establishes the existence of a strictly decre sing function $\phi(t) > 0$ satisfying (2.9), and so the proof is complete via Theorem 2.3.

REMARK 2.1. Theorem 2.4 and Theorem 2.6 extend slightly the second order version of Proposition 3 and Proposition 1, respectively, of Philos [7; pp. 172–175].

EXAMPLE 2.1. Consider the equation

$$(|x'(t)|x'(t))' = t^{-\lambda}\left|x\left(\frac{t}{\theta}\right)\right|x\left(\frac{t}{\theta}\right), \tag{2.18}$$

where $\lambda > 1$, $\theta > 1$ are constants. This is a special case of (A) in which $\alpha = 2$, $n = 1$, $p_1(t) = t^{-\lambda}$ and $g_1(t) = \dfrac{t}{\theta}$.

(i) Let $\lambda > 3$. Then, both (2.5) and (2.6) hold for (2.18), and so by Theorem 2.1 and Theorem 2.2, (2.18) has nonoscillatory solutions $x_1(t)$ and $x_2(t)$ such that $\lim_{t\to\infty} \dfrac{x_1(t)}{t} = \text{const.} \neq 0$ and $\lim_{t\to\infty} x_2(t) = \text{const.} \neq 0$ regardless of the values of $\theta > 1$.

(ii) Let $\lambda = 3$. An easy computation shows that (2.15) is satisfied for (2.18) if $1 < \theta < \exp\left(\dfrac{\sqrt{2}}{e}\right)$, since

$$\int_{g_1(t)}^t \left(\int_s^\infty p_1(r)dr\right)^{1/\alpha} ds = \int_{t/\theta}^t \left(\int_s^\infty r^{-3}dr\right)^{-1/2} ds = 2^{-1/2}\ln\theta.$$

From Theorem 2.5 it follows that, for such a $\theta$, (2.18) possesses a nonoscillatory solution tending to zero as $t \to \infty$.

(iii) Let $1 < \lambda < 3$. Then the condition (2.17) is satisfied for (2.18) since $P_{t_0} = 1$ and

$$\int_{g_1(t)}^t p_1(s)ds = \int_{t/\theta}^t s^{-\lambda}ds = (\lambda-1)^{-1}(\theta^{\lambda-1}-1)t^{1-\lambda} \to 0 \qquad \text{as } t \to \infty.$$

Therefore there exists a decaying nonoscillatory solution of (2.18) by Theorem 2.6.

EXAMPLE 2.2. Consider again the equation (1.14) given in Example 1.1. The condition (2.17) applied to (1.4) reduces to

$$(2.19)\quad k\sigma < \frac{\alpha+1}{e}\left(\frac{k}{\alpha}\right)^{\alpha/(\alpha+1)} \qquad \text{or} \qquad \sigma < \frac{(\alpha+1)^{1/(\alpha+1)}}{e}\left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)} k^{-1/(\alpha+1)}$$

It would be of interest to compare (2.19) with the condition (1.17) rewritten as

$$(2.20)\qquad k > (\alpha+1)^{1/(\alpha+1)}k^{-1/(\alpha+1)} \qquad \text{or} \qquad k > \left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)} k^{-1/(\alpha+1)}$$

which guarantees the nonexistence of bounded nonoscillatory solutions for (1.14). It is not an easy task to bridge the gap between (2.19) and (2.20).

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