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A variation formula for harmonic modules and its application to several complex variables

Dedicated to Professor Fumiyuki MAEDA on his 60th birthday

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Introduction

Let R be a compact or noncompact Riemann surface and let γ be a cycle in R. Then there exists a unique square integrable harmonic differential σ in R such that $\int_\gamma \omega = (\omega, *\sigma)_R$ $(= \iint_R \omega \wedge \sigma)$ for all C^2 square integrable closed differentials ω in R. We call σ the reproducing differential for (R, γ). The norm $\lambda = \|\sigma\|_R^2$ is called the harmonic module for (R, γ). L. V. Ahlfors [2] noted their significance in the theory of functions of one complex variable. In this paper we shall show their usefulness in that of several complex variables.

To a complex parameter t in a disk B, we let correspond a covering surface $R(t)$ over the z-plane $\boldsymbol{C}$ with C^∞ smooth boundary $\partial R(t)$ and with branch points $\xi_i(t)$ $(1 \le i \le q)$, where q does not depend on $t \in B$. Assume that $\partial R(t)$ varies C^∞ smoothly with the parameter $t \in B$ and that $\xi_i(t)$ is a holomorphic function on B. Thus $\mathscr{R} = \bigcup_{t \in B}(t, R(t))$ is a ramified Riemann domain over $B \times \boldsymbol{C}$. We simply denote $\partial\mathscr{R} = \bigcup_{t \in B}(t, \partial R(t))$, and write $\mathscr{R}: t \to R(t)$, $t \in B$. Now let $\gamma(t)$ be a cycle in $R(t)$ which varies continuously with $t \in B$ in $\mathscr{R}$. As a Riemann surface, each $R(t)$ with $\gamma(t)$ carries the reproducing differential $\sigma(t, \cdot)$ and the harmonic module $\lambda(t)$ for $(R(t), \gamma(t))$. We put $\Omega(t, z) = \sigma(t, z) + i * \sigma(t, z) = f(t, z)dz$ for $z \in R(t)$ and $\|\Omega\|(t, z) = |f(t, z)|$. In [15] and [16] we showed that: *If* $\mathscr{R}$ *is pseudoconvex over* $B \times \boldsymbol{C}$, *then* $\dfrac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} \ge \left\| \dfrac{\partial \Omega}{\partial \bar{t}}(t, \cdot) \right\|_{R(t)}^2$ *for* $t \in B$. *Furthermore, the equality holds for all* $t \in B$, *if and only if* $\mathscr{R}$ *is Levi flat*. In this paper, for any $\mathscr{R}: t \to R(t)$, $t \in B$, we shall prove a variation formula for $\lambda(t)$ of the second order, which deduces the above result in the pseudoconvex or Levi flat case. Precisely, let $\varphi(t, z)$ be a C^2 defining function of $\mathscr{R}$, and put, for $(t, z) \in \partial\mathscr{R}$.

$$k_2(t, z) = \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\mathrm{Re} \left\{ \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right\} \Big/ \left| \frac{\partial \varphi}{\partial z} \right|^3$$

which is called the Levi curvature of ∂R at (t, z). Then, we have

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = \frac{1}{2} \int_{\partial R(t)} k_2(t, z) \|\Omega\|^2(t, z) |dz| + \left\| \frac{\partial \Omega}{\partial \bar{t}}(t, \cdot) \right\|^2_{R(t)}.$$

By a triple $(\mathscr{M}, \pi, B)$, we mean that $\mathscr{M}$ is a connected 2-dimensional complex manifold, B a region in the complex plane $\boldsymbol{C}$, and π a holomorphic mapping from $\mathscr{M}$ onto B such that each $\pi^{-1}(t)$, $t \in B$ is a 1-dimensional irreducible non-singular analytic set in $\mathscr{M}$. We put $M(t) = \pi^{-1}(t)$ for $t \in B$, which is a compact or noncompact Riemann surface. A triple $\mathfrak{M} = (\mathscr{M}, \pi, B)$ is said to be *topologically trivial*, if there exist a Riemann surface R and a topological mapping T from $\mathscr{M}$ onto $B \times R$ such that $\pi_B \circ T = \pi$ where π_B is the projection from $B \times R$ to B. If R is of (topological) type (g, n), that is, R is of genus g $(0 \le g \le \infty)$ and has n $(0 \le n \le \infty)$ ideal boundary components, then $\mathfrak{M}$ is said to be *of type* (g, n). If g and n are finite, $\mathfrak{M}$ is said to be *of finite type*. Otherwise, $\mathfrak{M}$ is said to be *of infinite type*. A triple $\mathfrak{M}$ is said to be *holomorphically trivial*, if we can take a biholomorphic mapping T from $\mathscr{M}$ onto $B \times R$ such that $\pi_B \circ T = \pi$. A triple $\mathfrak{M}$ is said to be *of locally Stein*, if for any $t_o \in B$, there exists a disk B_o in B centered at t_o such that $\pi^{-1}(B_o)$ is a Stein manifold. As usual a holomorphic mapping α from B into $\mathscr{M}$ such that $\pi \circ \alpha =$ (identity) is called a *holomorphic section of* $\mathscr{M}$ *defined on* B.

As an application of the variation formula, we shall show

THEOREM. *Let* $\mathfrak{M} = (\mathscr{M}, \pi, \boldsymbol{C})$ *be a topologically trivial triple of finite or infinite type* (g, n). *Then we have the following results* (I) $\sim$ (IV):

(I) *If* $n = 0$, *then* $\mathfrak{M}$ *is holomorphically trivial.*

Let $n \ge 1$ *and assume that* $\mathfrak{M}$ *is of locally Stein. Then*

(II) $\mathfrak{M}$ *is holomorphically trivial except for the following three cases* (i), (ii) *and* (iii):

- (i) $(g, n) = (0, 1)$ *and* $M(t_o)$ *for some* $t_o \in \boldsymbol{C}$ *is conformally equivalent to a unit disk,*
- (ii) $(g, n) = (0, 2)$ *and* $M(t_o)$ *for some* $t_o \in \boldsymbol{C}$ *is conformally equivalent to a punctured unit disk,*
- (iii) $\mathfrak{M}$ *is of infinite type.*

(III) *In case* (i), $\mathfrak{M}$ *is holomorphically trivial, provided that there exist at least two holomorphic sections of* $\mathscr{M}$ *defined on* $\boldsymbol{C}$ *(which may intersect each other).*

(IV) *In cases* (ii) *and* (iii), *the same is true, provided that there exists at least one holomorphic section of* $\mathscr{M}$ *defined on* $\boldsymbol{C}$.

Assertion (I) is proved by a combination of some classical theorems for compact Riemann surfaces. (We shall give its brief proof at the end of this paper.) We recall that any noncompact Riemann surface S of finite type (g, n) is conformally equivalent to the interior R of a compact Riemann surface $R^\wedge$ of genus g excluded n' $(0 \le n' \le n)$ simply connected domains $\{D_i\}$ with C^ω smooth boundary ∂D_i and $n - n'$ points $\{P_j\}$, namely, $R = R^\wedge - \bigcup_{\substack{1 \le i \le n' \\ 1 \le j \le n-n'}} \{D_i \cup \partial D_i \cup P_j\}$. Then we say that S has n' *non-degenerating*, and $n - n'$ *degenerating ideal boundary components*. The special case in (II) such that all ideal boundary components of each $\pi^{-1}(t)$, $t \in \boldsymbol{C}$ are degenerating, is immediately reduced to (I) by Theorem 2 in Nishino [11]. Thus the variation formula will be essentially used in the proof of the general case in (II) such that $\pi^{-1}(t_o)$ for some $t_o \in \boldsymbol{C}$ has at least one non-degenerating ideal boundary component, and in the proofs of (III) and (IV).

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1. Harmonic modules

Let R be a compact or noncompact Riemann surface. Following Ahlfors and Sario [3] we define

$\Gamma(R)$ = the Hilbert space of square integrable differentials in R;
$\Gamma_c^2(R)$ = the space of square integrable closed differentials of class C^2 in R;
$\Gamma_h(R)$ = the space of square integrable harmonic differentials in R.

Let γ be a cycle in R. Then there exists a unique $\sigma_\gamma \in \Gamma_h(R)$ such that

$$(1.1) \qquad \int_\gamma \omega = (\omega, *\sigma_\gamma)_R \quad \text{for all } \omega \in \Gamma_c^2(R).$$

The harmonic differential σ_γ is called the *reproducing differential* (or briefly, *r-diff.*) *for* (R, γ). The norm $\lambda_\gamma = \|\sigma_\gamma\|_R^2$ is called the *harmonic module* (or, *h-mod.*) *for* (R, γ). It is well-known that, for any cycle δ in R,

$$(1.2) \qquad \int_\delta \sigma_\gamma = \gamma \times \delta \qquad (\textit{intersection number}).$$

Assume that R is a compact bordered Riemann surface of type (g, n). That is, R is of genus g and ∂R consists of n smooth curves $\{C_i\}$ $(1 \le i \le n)$ of class C^ω in a larger Riemann surface $R^\wedge \supset\supset R$. We put $\ell = 2g + n - 1$. As a canonical homology base of $R \cup \partial R$, we can take ℓ smooth curves on $R \cup \partial R$: $\{A_j, B_j; C_i\}$ $(1 \le j \le g; 1 \le i \le n - 1)$ such that $A_i \times B_j = \delta_{ij}$ (Kronecker's delta) and $A_i \times A_j = B_i \times B_j = 0$ $(1 \le i, j \le g)$. Let γ be a cycle in R. Then σ_γ is constructed as follows:

Case 1. $\gamma \sim C_i (1 \leq i \leq n-1)$. We consider the harmonic function $u_i(z)$ in R with boundary values 1 on C_i and 0 on $(\partial R) - C_i$. Then $\sigma_\gamma = du_i$ in R.

Case 2. $\gamma \sim A_i$. We cut R along A_i, so that $\partial(R - A_i) = (\partial R) + A_i^+ - A_i^-$. We form a harmonic function $v_i(z)$ on $R - A_i$ such that $v_i(z) = 0$ on ∂R and such that $v_i(z)$ is harmonically extended across A_i^+ and A_i^- to be functions $v_i^+(z)$ and $v_i^-(z)$ with $v_i^+(z) = v_i^-(z) - 1$ for $z \in U_i$ where U_i is an annulus around A_i. Then $\sigma_\gamma = dv_i$ in R.

Case 3. $\gamma \sim B_i$. By replacing A_i and $v_i(z)$ by B_i and $w_i(z)$ such that $w_i^+(z) = w_i^-(z) + 1$ for $z \in U_i$ where U_i is an annulus around B_i, we have $\sigma_\gamma = dw_i$ in R.

General Case. $\gamma \sim \sum_{i=1}^{g} [a_i A_i + b_i B_i] + \sum_{k=1}^{n-1} c_k C_k$, ($a_i$, b_i, c_k are integers). If we set $u_\gamma(z) = \sum [a_i v_i(z) + b_i w_i(z)] + \sum c_k u_k(z)$, then $u_\gamma(z)$ is a harmonic function in $R - \bigcup_{i=1}^{g}(A_i \cup B_i)$ such that (1) $du_\gamma = \sigma_\gamma$ in R; (2) $u_\gamma(z) = 0$ on C_n.

Such $u_\gamma(z)$ being unique, we say that $u_\gamma(z)$ is the *normalized Abelian integral for* (R, γ). We note that

(1.3) $$u_\gamma(z) = const.\ c_k \text{ on each contour } C_k \ (1 \leq k \leq n-1).$$

In particular, $\sigma_\gamma(z)$ is of class C^ω up to ∂D.

New let γ be a Jordan curve in R. Two cases occur:

Case (i). γ is a dividing cycle. Namely, R is divided into two domains R' and R'' by γ where the orientation of γ is negative (resp. positive) with respect to R' (resp. R'').

Case (ii). γ is a non-dividing cycle, so that $R - \gamma$ is connected.

In both cases, for a fixed point $a \in R - \gamma$, we consider the Green function $g(a, z)$ for $\Delta g = 0$ of R with (logarithmic) pole at a. We set

(1.4) $$I(a) = \frac{-1}{2\pi} \int_\gamma *dg(a, z)$$

Then we have

PROPOSITION 1.1. *In Case* (i), $u_\gamma(a) = I(a) + 1$ *for* $a \in R'$; $= I(a)$ *for* $a \in R''$, *while, in Case* (ii), $u_\gamma(a) = I(a)$ *for* $a \in R - \gamma$.

PROOF. Stokes' formula implies that

$$\int_{(\partial R) - \gamma - \delta_\varepsilon(a)} u_\gamma(z) * dg(a, z) = \int_{(\partial R) - \gamma - \delta_\varepsilon(a)} g(a, z) * du_\gamma(z)$$

where $\delta_\varepsilon(a)$ is the circle of center a and radius $\varepsilon > 0$. By letting $\varepsilon \to 0$, we obtain Proposition 1.1. □

REMARK 1.1. In §4, we shall treat the case when R consists of a countable number of Riemann surfaces $\{R_j\}$ $(1 \le j < \infty)$ and when $\gamma \subset R$ consists of cycles γ_i in R_i $(1 \le i \le k; k < \infty)$. By relation (1.1) we define the r-diff. σ_γ and the h-mod. λ_γ for (R, γ). Clearly, $\sigma_\gamma = \sigma_{\gamma_j}$ in $R_j (1 \le j \le k)$; $= 0$ in $R_j (k+1 \le j < \infty)$ and $\lambda_\gamma = \lambda_{\gamma_1} + \cdots + \lambda_{\gamma_k}$, where σ_{γ_j} and λ_{γ_j} denote those for (R_j, γ_j).

2. Smooth variations

Let B be a disk: $= \{t \in \boldsymbol{C} \,|\, |t| < r\}$ and consider an *unramified* covering domain $\mathscr{D}$ spread over $B \times \boldsymbol{C}$. We simply say that $\mathscr{D}$ is a *domain over* $B \times \boldsymbol{C}$. Given $t \in B$, we set $D(t) = \{z \,|\, (t, z) \in \mathscr{D}\}$. Then $D(t)$ consists of an at most countable number of covering Riemann surfaces over $\boldsymbol{C}$ without branch points. We call $D(t)$ the *fiber of* $\mathscr{D}$ *at* t. $\mathscr{D}$ may be regarded as a variation of Riemann surfaces $D(t)$ with the complex parameter $t \in B$. We write $\mathscr{D}: t \to D(t)$, $t \in B$. The following condition is imposed on $\mathscr{D}$:

CONDITION 2.1. There exist another domain $\mathscr{D}^\sim$ over $B \times \boldsymbol{C}$ and a real-valued C^∞ function $\varphi(t, z)$ in $\mathscr{D}^\sim$ such that

(1) $\mathscr{D}^\sim \supset \mathscr{D}$ and $D^\sim(t) \supset\supset D(t) \neq \phi$ for any $t \in B$; We denote by $\partial\mathscr{D}$ the boundary of $\mathscr{D}$ in $\mathscr{D}^\sim$, and by $\partial D(t)$ the boundary of $D(t)$ in $D^\sim(t)$;

(2) $\mathscr{D} = \{(t, z) \in \mathscr{D}^\sim \,|\, \varphi(t, z) < 0\}$; $\partial\mathscr{D} = \{(t, z) \in \mathscr{D}^\sim \,|\, \varphi(t, z) = 0\}$;

(3) For any fixed $t \in B$, $\dfrac{\partial\varphi}{\partial z} \neq 0$ for any $z \in \partial D(t)$.

When Condition 2.1 is satisfied, we say that $\mathscr{D}$ is a C^∞ *smooth variation*, and that the pair $(\mathscr{D}^\sim, \varphi)$ *defines* $\mathscr{D}$. Note that $\partial\mathscr{D} = \bigcup_{t \in B}(t, \partial D(t))$. We put, for $(t, z) \in \partial\mathscr{D}$.

$$(2.1) \quad k_2(t, z) = \left\{\frac{\partial^2\varphi}{\partial t \partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial\varphi}{\partial\bar{t}}\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial t \partial\bar{z}}\right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2 \frac{\partial^2\varphi}{\partial z \partial\bar{z}}\right\} \Bigg/ \left|\frac{\partial\varphi}{\partial z}\right|^3,$$

which is independent of the choice of the pair $(\mathscr{D}^\sim, \varphi)$. $k_2(t, z)$ is called the *Levi curvature of* $\partial\mathscr{D}$ *at* (t, z) (cf. Levenberg and Yamaguchi [6]). By (3) of Condition 2.1, there exists a compact bordered Riemann surface S and a C^∞ diffeomorphism $T: (t, z) \to (t, w) = (t, \psi(t, z))$ of $\mathscr{D} \cup \partial\mathscr{D}$ onto $B \times \bar{S}$ such that $\psi(t, \overline{D(t)}) = \bar{S}$.

3. Variation formulas

Let $\mathscr{D}$ be a domain over $B \times \boldsymbol{C}$ with Condition 2.1. We keep the notations ψ and S at the end of §2. Let S be of type (g, n). Let γ be a cycle in S, and put $\gamma(t) = \psi^{-1}(t, \gamma)$ for $t \in B$. Then $\gamma(t)$ is a cycle in $D(t)$ which varies continuously with $t \in B$ in $\mathscr{D}$. For any fixed $t \in B$, we have the r-diff. $\sigma(t, \cdot)$ and the h-mod. $\lambda(t)$ for $(D(t), \gamma(t))$. If we put $\sigma(t, z) = a(t, z)dx + b(t, z)dy$, then $a(t, z)$ and $b(t, z)$ are harmonic functions for $z \in D(t)$ and of class C^2 with respect to $(t, z) \in \mathscr{D} \cup \partial\mathscr{D}$ from (1.3).

DEFINITION 3.1. For $(t, z) \in \mathscr{D} \cup \partial\mathscr{D}$, we put

$$\frac{\partial\sigma}{\partial t}(t, z) = \frac{\partial a}{\partial t}(t, z)dx + \frac{\partial b}{\partial t}(t, z)dy\,;$$

$$\frac{\partial^2\sigma}{\partial t\partial\bar{t}}(t, z) = \frac{\partial^2 a}{\partial t\partial\bar{t}}(t, z)dx + \frac{\partial^2 b}{\partial t\partial\bar{t}}(t, z)dy\,.$$

They are harmonic differentials in each $D(t)$, $t \in B$. We consider the normalized Abelian integral $u(t, z)$ for $(D(t), \gamma(t))$. Then $\dfrac{\partial u}{\partial t}(t, z)$ and $\dfrac{\partial^2 u}{\partial t\partial\bar{t}}(t, z)$ are *single-valued* for $z \in \overline{D(t)}$. Indeed, we shall prove this in the case $\gamma \sim A_i$, for example. Let $t_o \in B$. We can find a disk B_o of center t_o such that $A_i(t_o) \subset D(t)$ for all $t \in B_o$ and $A_i(t_o) \sim A_i(t)$ in $D(t)$. Since $u^+(t, z) = u^-(t, z) - 1$ for $z \in U_1$ and $t \in B_o$ where $U_i \supset A_i(t_o)$, we have $\partial u^+/\partial t = \partial u^-/\partial t$ and $\partial^2 u^+/\partial t\partial\bar{t} = \partial^2 u^-/\partial t\partial\bar{t}$ for all $(t, z) \in B_o \times U_i$, which proves our claim. We thus have

$$\frac{\partial\sigma}{\partial t}(t, z) = d\left(\frac{\partial u}{\partial t}(t, z)\right); \qquad \frac{\partial^2\sigma}{\partial t\partial\bar{t}}(t, z) = d\left(\frac{\partial^2 u}{\partial t\partial\bar{t}}(t, z)\right) \tag{3.1}$$

for $z \in \overline{D(t)}$. Given $t \in B$, we write

$$\Omega(t, z) = \sigma(t, z) + i*\sigma(t, z) = f(t, z)dz\,; \qquad \|\Omega\|(t, z) = |f(t, z)|\,;$$

$$\frac{\partial\Omega}{\partial\bar{t}}(t, z) = \frac{\partial\sigma}{\partial\bar{t}}(t, z) + i*\frac{\partial\sigma}{\partial\bar{t}}(t, z) = \frac{\partial f}{\partial\bar{t}}(t, z)dz\,.$$

Then $(\partial f/\partial\bar{t})(t, z)$ as well as $f(t, z)$ is a holomorphic function for $z \in D(t)$, and is of class C^2 up to $\partial D(t)$. Clearly, $\Omega(t, z) = 2\dfrac{\partial u}{\partial z}(t, z)$ and $\dfrac{\partial\Omega}{\partial\bar{t}}(t, z) = 2\dfrac{\partial^2 u}{\partial\bar{t}\partial z}(t, z)dz$. We shall show the variation formulas of the h-mod. $\lambda(t)$ for $(D(t), \gamma(t))$.

THEOREM 3.1. *For $t \in B$, we have*

(1) $\displaystyle \frac{\partial\lambda(t)}{\partial t} = \frac{1}{2}\left(\Omega(t, \cdot), \frac{\partial\Omega}{\partial\bar{t}}(t, \cdot)\right)_{D(t)};$

(2) $\displaystyle \frac{\partial\lambda^2(t)}{\partial t\partial\bar{t}} = \frac{1}{2}\int_{\partial D(t)} k_2(t, z)\|\Omega\|^2(t, z)ds_z + \left\|\frac{\partial\Omega}{\partial\bar{t}}(t, \cdot)\right\|^2_{D(t)}$

where ds_z denotes the Euclidean line element of $\partial D(t)$.

PROOF. It suffices to prove these at $t = 0$. First, we prove (1) and

$$\frac{\partial\lambda}{\partial t}(0) = \int_{\partial D(0)} \frac{\partial u}{\partial t}(0, z)\frac{\partial u}{\partial n_z}(0, z)ds_z\,; \tag{3.2}$$

$$\frac{\partial^2\lambda}{\partial t\partial\bar{t}}(0) = \int_{\partial D(0)} \frac{\partial^2 u}{\partial t\partial\bar{t}}(0, z)\frac{\partial u}{\partial n_z}(0)ds_z\,. \tag{3.3}$$

In fact, $\frac{\partial\sigma}{\partial t}(0, z)$ is harmonic in $D(0)$, and is of class C^2 up to $\partial D(0)$. Since $\gamma(t) \sim \gamma(0)$ in $D(t)$ for any t close to 0, we have

$$\frac{\partial\lambda}{\partial t}(0) = \left(\frac{\partial}{\partial t}\int_{\gamma(0)} *\sigma(t, \cdot)\right)_{t=0} = \left(*\frac{\partial\sigma}{\partial t}(0, \cdot), *\sigma(0, \cdot)\right)_{D(0)}.$$

Since the last term is equal to $\frac{1}{2}\left(\Omega(0, \cdot), \frac{\partial\Omega}{\partial\bar{t}}(0, \cdot)\right)_{D(0)}$ from (1.2), we get (1). By (1.3), we have $*\sigma(0, z) = \frac{2}{i}\frac{\partial u}{\partial z}(0, z)dz = \frac{\partial u}{\partial n_z}(0, z)ds_z$ along $\partial D(0)$. Here $\partial/\partial n_z$ denotes the outer normal derivative. It follows from (3.1) that

$$\frac{\partial\lambda}{\partial t}(0) = \iint_{D(0)} \frac{\partial\sigma}{\partial t}(0, z) \wedge *\sigma(0, z) = \int_{\partial D(0)} \frac{\partial u}{\partial t}(0, z)\frac{\partial u}{\partial n_z}(0, z)ds_z\,,$$

which proves (3.2). Analogously, we have (3.3).

Next we shall prove that, for any $z \in \partial D(0)(= \bigcup_{k=1}^{n} C_k(0))$ with $\frac{\partial u}{\partial z}(0, z) \neq 0$,

$$\frac{\partial^2 u}{\partial t\partial\bar{t}}(0, z) = \frac{1}{2}k_2(0, z)\frac{\partial u}{\partial n_z}(0, z) + 2\,\mathrm{Re}\left\{\frac{\dfrac{\partial u}{\partial t}\dfrac{\partial^2 u}{\partial\bar{t}\partial z}}{\dfrac{\partial u}{\partial z}}\right\}(0, z)\,. \tag{3.4}$$

In fact, let $z_o \in C_k(0)$ $(1 \leq k \leq n)$ with $\frac{\partial u}{\partial z}(0, z_o) \neq 0$. In case $\frac{\partial u}{\partial n_z}(0, z_o) \gtrless 0$, (1.3) implies that the function $\pm(u(t, z) - c_k)$ is a defining function of $\partial\mathscr{D}$ near $(0, z_o)$. Hence

$$k_2(0, z_o) = \pm\left\{\frac{\partial^2 u}{\partial t\partial\bar{t}}\left|\frac{\partial u}{\partial z}\right|^2 - 2\,\mathrm{Re}\left\{\frac{\partial u}{\partial t}\frac{\partial u}{\partial\bar{z}}\frac{\partial^2 u}{\partial\bar{t}\partial z}\right\} + \left|\frac{\partial u}{\partial t}\right|^2\frac{\partial^2 u}{\partial z\partial\bar{z}}\right\}\Big/\left|\frac{\partial u}{\partial z}\right|^3,$$

where the right hand side is evaluated at $(0, z_o)$. Since $u(0, z)$ is harmonic on $D(0)$ and continuous on $\overline{D(0)}$, and $\frac{\partial u}{\partial n_z}(0, z_o) = \pm 2\left|\frac{\partial u}{\partial z}(0, z_o)\right|$, we obtain (3.4). If we substitute (3.4) for (3.3), then

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}} = \frac{1}{2} \int_{\partial D(0)} k_2(0, z) \left(\frac{\partial u}{\partial n_z}(0, z) \right)^2 ds_z + 4 \operatorname{Re} \left\{ \frac{1}{i} \int_{\partial D(0)} \left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} \right)(0, z) dz \right\}.$$

Since $\frac{\partial u}{\partial t}(0, z)$ is a single-valued harmonic function for $z \in D(0)$, it follows by Stokes' formula that the second term of the right hand side is equal to

$$4 \operatorname{Re} \left\{ \frac{1}{i} \iint_{D(0)} d\left(\left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} \right)(0, z) dz \right) \right\}$$

$$= 4 \operatorname{Re} \left\{ \frac{1}{i} \iint_{D(0)} \left| \frac{\partial^2 u}{\partial \bar{t} \partial z}(0, z) \right|^2 d\bar{z} \wedge dz \right\} = \left\| \frac{\partial \Omega}{\partial \bar{t}}(0, z) \right\|^2_{D(0)}.$$

(2) of Theorem 3.1 is proved. □

In the Introduction we defined a triple $\mathfrak{M} = (\mathcal{M}, \pi, B)$. We call π a *projection.* We put $M(t) = \pi^{-1}(t)$, $t \in B$, and call it the *fiber* of $\mathcal{M}$ at t. We write $\mathcal{M} = \bigcup_{t \in B}(t, M(t))$. For an open set $B_o \subset B$, we put $\mathcal{M}_{B_o} = \pi^{-1}(B_o)$, and define $\mathfrak{M}_{B_o} = (\mathcal{M}_{B_o}, \pi_{|B_o}, B_o)$, which is called a *subtriple of* $\mathfrak{M}$ *on* B_o. Let $p_o \in \mathcal{M}$ with $\pi(p_o) = t_o$. Then we can take local coordinates $B_o \times U_o$ where $B_o = \{|t - t_o| < r_o\}$ and $U_o = \{|z| < \rho_o\}$ of a neighborhood $\mathcal{U} \subset\subset \mathcal{M}$ of p_o such that p_o corresponds to $(t_o, 0)$, and $M(t) \cap \mathcal{U}$ to $\{t\} \times U_o$. We call $(t, z) \in B_o \times U_o$ *π-local coordinates at* p_o.

DEFINITION 3.2. A triple $\mathfrak{M} = (\mathcal{M}, \pi, B)$ is said to have C^∞(resp. C^ω) *smooth boundary*, if there exists a larger triple $\mathfrak{M}^\sim = (\mathcal{M}^\sim, \pi^\sim, B)$ and a real-valued C^∞(resp. C^ω) function $\varphi(p)$ on $\mathcal{M}^\sim$ such that

(1) $\mathcal{M}^\sim \supset \mathcal{M}$ and $\pi^\sim = \pi$ on $\mathcal{M}$, and $M^\sim(t) \supset\supset M(t)$ for all $t \in B$. We denote by $\partial\mathcal{M}$ the boundary of $\mathcal{M}$ in $\mathcal{M}^\sim$;

(2) $\mathcal{M} = \{p \in \mathcal{M}^\sim | \varphi(p) < 0\}$, $\partial\mathcal{M} = \{p \in \mathcal{M}^\sim | \varphi(p) = 0\}$ and $(\partial\varphi/\partial z)(p_o) \neq 0$ at any $p_o \in \partial\mathcal{M}$ in $\mathcal{M}^\sim$ where (t, z) is $\pi^\sim$-local coordinates at p_o.

We say that the pair $(\mathfrak{M}^\sim, \varphi)$ *defines* $\mathfrak{M}$ *with* C^∞(resp. C^ω) *smooth boundary.* In the rest of this section we assume that a triple $\mathfrak{M}$ have a C^∞ smooth boundary. Let $p_o \in \partial\mathcal{M}$ in $\mathcal{M}^\sim$, and let $(t, z) \in B_o \times U_o$ be $\pi^\sim$-local coordinates at p_o. Using $\varphi(t, z)$ in $B_o \times U_o$, we define $k_2(t, z)$ on $(\partial\mathcal{M}) \cap (B_o \times U_o)$ by (2.1). By simple calculation we see that $k_2(t, z)/|dz|$ is independent of the choice of the pair $(\mathcal{M}^\sim, \varphi)$ and of $\pi^\sim$-local coordinates (t, z) at p_o, and so is $k_2(t, z) > 0$ or $= 0$.

Now let a cycle $\gamma(t)$ in $M(t)$ vary continuously in $\mathcal{M}$ with $t \in B$. We consider the r-diff. $\sigma(t, z)$ and the h-mod. $\lambda(t)$ for $(M(t), \gamma(t))$. We put $\Omega(t, z) = \sigma(t, z) + i * \sigma(t, z) = f(t, z)dz$ on $M(t)$. Let $p \in \mathcal{M}$ and $(t, z) \in B_o \times U_o$ be π-local coordinates at p. Then $(\partial f/\partial \bar{t})dz$ does not depend on the choice of π-local coordinates. It follow that, for a fixed $t \in B$, $(\partial f/\partial \bar{t})(t, z)dz$ defines a holomorphic differential on $M(t)$. We denote it by $(\partial\Omega/\partial\bar{t})(t, z)$. Since $\|\Omega\|(t, z)|dz|$ is invariant on $M(t) \cup \partial M(t)$, $k_2(t, z)\|\Omega\|(t, z)$ is a function on $\partial M(t)$. Then we have

THEOREM 3.2. *For a triple $\mathfrak{M} = (\mathcal{M}, \pi, B)$ with C^∞ smooth boundary, the same variation formulas* (1) *and* (2) (*where $D(t)$ and ds_z are replaced by $M(t)$ and $|dz|$*) *of Theorem* 3.1 *hold.*

PROOF. It suffices to prove these at $t = 0$. By Nishimura [9] there exists a disk B_o of center 0 such that the subtriple $\mathfrak{M}_{B_o}$ of $\mathfrak{M}$ on B_o is biholomorphically mapped onto an unramified domain $R = \bigcup_{t \in B_o}(t, R(t))$ over $B_o \times \boldsymbol{C}$ with Condition 2.1 by a transformation $\Phi: (t, z) \to (t, w) = (t, \phi(t, z))$ where $\phi(t, M(t)) = R(t)$ for all $t \in B_o$. We put $\gamma^\wedge(t) = \phi(t, \gamma(t))$ in $R(t)$, and consider the r-diff. $\sigma^\wedge(t, w)$ and the h-mod. $\lambda^\wedge(t)$ for $(R(t), \gamma^\wedge(t))$. We apply Theorem 3.1 to R and $\lambda^\wedge(t)$, so that formulas (1) and (2) for $\lambda^\wedge(t)$ hold. Since all five terms appeared in (1) and (2) are invariant under the transformation Φ, we have Theorem 3.2. □

We note that the variation formula for $\lambda(t)$ of the second order stated in the Introduction is a special caseof (2) of Theorem 3.2. We recall the definition of pseudoconvexity for $(\mathcal{M}, \pi, B)$: Let $p \in \partial\mathcal{M}$ and let (t, z) be π-local coordinates at p in $\mathcal{M}^\sim$. If $k_2(t, z) \geq 0$ at p, then p is called a *pseudoconvex boundary point of $\mathcal{M}$*. If $\partial\mathcal{M}$ consists of all pseudoconvex boundary points, $\mathcal{M}$ is said to be *pseudoconvex*. If $k_2(t, z) = 0$ on $\partial\mathcal{M}$, $\mathcal{M}$ is said to be *Levi flat*. By Theorem 3.2, we have

COROLLARY 3.1. *Let $(\mathcal{M}, \pi, B)$ be a triple with C^∞ smooth boundary. Suppose that $\mathcal{M}$ is pseudoconvex. Then,* (1) $\dfrac{\partial^2\lambda(t)}{\partial t\partial\bar{t}} \geq \left\|\dfrac{\partial\Omega}{\partial\bar{t}}(t, \cdot)\right\|^2_{M(t)}$ *for $t \in B$. The equality holds for all $t \in B$, if and only if $\mathcal{M}$ is Levi flat;* (2) *If $\lambda(t)$ is a harmonic function on B, then $\mathcal{M}$ is Levi flat and $\Omega(t, z)$ is holomorphic for $(t, z) \in \mathcal{M}$. Conversely, if $\Omega(t, z)$ is holomorphic for $(t, z) \in \mathcal{M}$, then $\lambda(t)$ is a constant on B.*

4. Differentiability condition

In order to study the case of an infinitely many sheeted ramified domain over $B \times \boldsymbol{C}$, we need a differentiability condition for $\sigma(t, z)$. Let $\mathcal{D}$ be an unramified covering domain over $B \times \boldsymbol{C}$ which satisfies

CONDITION 4.1. There exist another unramified domain $\mathcal{D}^\sim$ and a C^ω *plurisubharmonic* function $\varphi(t, z)$ in $\mathcal{D}^\sim$ such that

(1) $\mathscr{D}^{\sim} \supset \mathscr{D}$; $D(t)^{\sim} \supset\supset D(t) \neq \phi$ for any $t \in B$; We denote by $\partial\mathscr{D}$ the boundary of $\mathscr{D}$ in $\mathscr{D}^{\sim}$, and by $\partial D(t)$ the boundary of $D(t)$ in $D^{\sim}(t)$;

(2) $\mathscr{D} = \{(t, z) \in \mathscr{D}^{\sim} \mid \varphi(t, z) < 0\}$; $\partial\mathscr{D} = \{(t, z) \in \mathscr{D}^{\sim} \mid \varphi(t, z) = 0\}$;

(3) $\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial z}\right)(t, z) \neq 0$ for any $(t, z) \in \partial\mathscr{D}$;

(4) The subset $L = \left\{(t, z) \in \mathscr{D}^{\sim} \mid \varphi(t, z) = \frac{\partial\varphi}{\partial z}(t, z) = 0\right\}$ consists of a finite number of real 1-dimensional C^{ω} smooth arcs (which may intersect each other) in $\mathscr{D}^{\sim}$. We denote by ℓ the projection of L to B.

By real analyticity of $\varphi(t, z)$ in $\mathscr{D}^{\sim}$, (*i*) ℓ consist of a finite number of real 1-dimensional C^{ω} smooth arcs; (*ii*) $\partial D(t)$ for $t \in \ell$ has a finite number of singular points; (*iii*) $\partial\mathscr{D} = \bigcup_{t\in B}(t, \partial D(t))$. In general, the variation $\mathscr{D}: t \to D(t)$, $t \in B$ is no longer even topologically trivial.

Fix $t \in B$ and $a \in D(t)$. We denote by $g(t, a, z)$ the Green function for $(D(t), a)$. Precisely, let $D_1(t)$ be a connected component of $D(t)$ containing a, and denote by $g_1(t, a, z)$ the Green function of $D_1(t)$ with pole at a. Then $g(t, a, z) = g_1(t, a, z)$ for $z \in D_1(t)$; $= 0$ for $z \in D(t) - D_1(t)$. We put

$$\Lambda(t, a) = \lim_{z\to a}\left(g(t, a, z) - \log\frac{1}{|z - a|}\right)$$

which is called the *Robin constant for* $(D(t), a)$. In [17] it was shown that under Condition 4.1,

(4.1) $g(t, a, z)$ is continuous for $(t, a, z) \in \bigcup_{t\in B}(t, D(t), D(t))$ with $a \neq z$. Moreover, $\frac{\partial\Lambda}{\partial t}(t, a)$ exists and is continuous for $(t, a) \in \mathscr{D}$.

By the same method we can prove

LEMMA 4.1. *Under Condition* 4.1, $\frac{\partial g}{\partial t}(t, a, z)$ *exists and is continuous for* $(t, a, z) \in \bigcup_{t\in B}(t, D(t), D(t))$ *with* $a \neq z$. *Furthermore, if we set* $\frac{\partial g}{\partial t}(t, a, a) = \frac{\partial\Lambda}{\partial t}(t, a)$, *then* $\frac{\partial g}{\partial t}(t, a, z)$ *is continuous even at* $a = z$.

Now we assume

CONDITION 4.2. To each $t \in B$ we let correspond a cycle $\gamma(t)$ in $D(t)$ in a continuous way in $\mathscr{D}$.

Precisely speaking, for any $t_o \in B$ and any product neighborhood $B_o \times G$ of $(t_o, \gamma(t_o))$ in $\mathscr{D}$, we can find a disk $B_1 \subset B_o$ of center t_o such that $\gamma(t) \subset G$

for all $t \in B_1$ and $\gamma(t) \sim \gamma(t_o)$ in G. Therefore, the following situations may occur: $\gamma(t)$ for some $t \in B_o$ is a dividing cycle in $D(t)$, while $\gamma(t)$ for other $t \in B_o$ is a non-dividing cycle in $D(t)$; the number of components of $\gamma(t)$ varies with $t \in B_o$. For each $t \in B$, we denote by $\sigma(t, \cdot)$ and $\lambda(t)$ the r-diff. and h-mod. for $(D(t), \gamma(t))$. Then we have

THEOREM 4.1. *Suppose that* $\mathscr{D}$ *satisfies Conditions* 4.1 *and* 4.2. *Then,* (1) $\dfrac{\partial \sigma}{\partial t}(t, z)$ *exists and is continuous for* $(t, z) \in \mathscr{D}$; (2) $\lambda(t)$ *is of class* C^1 *in* B.

PROOF. Let $t_o \in B$ and write $\gamma(t_o) = \gamma_o$. It suffices to prove the case when γ_o is a smooth curve. By Condition 4.2 we find a neighborhood $B_o \times G$ of (t_o, γ_o) in $\mathscr{D}$ such that $\gamma(t) \sim \gamma_o$ in G. We thus assume $\gamma(t) = \gamma_o$ for $t \in B_o$. We denote by $D_1(t)$ the connected component of $D(t)$ containing γ_o. Two cases occur:

Case (i). γ_o is a dividing cycle in $D_1(t)$. Then $D_1(t)$ is divided into two domains $D_1'(t)$ and $D_1''(t)$ such that $D_1(t) - \gamma_o \sim D_1'(t) \cup D_2''(t)$; $\partial D_1'(t) = C_1'(t) - \gamma_o$; $\partial D_1''(t) = C_2''(t) + \gamma_o$ where $\partial D_1(t) = C_1'(t) + C_2''(t)$.

Case (ii). γ_o is a non-dividing cycle in $D_1(t)$.

In both cases we take a point $a \in D(t) - \gamma_o$ and consider the integral defined by (1.4): $I(t, a) = \dfrac{1}{2\pi} \int_{\gamma_o} *dg(t, a, z)$. By (4.1), $I(t, a)$ is continuous for $(t, a) \in \mathscr{D}_{B_o} - (B_o \times \gamma_o)$. Since $(\partial g/\partial t)(t, a, z)$ is separately harmonic for a and z, Lemma 4.1 implies that $\dfrac{\partial I}{\partial t}(t, a) = \dfrac{1}{2\pi} \int_{\gamma_o} *d\left(\dfrac{\partial g}{\partial t}(t, a, z)\right)$ exists and is continuous for $(t, a) \in \mathscr{D}_{B_o} - (B_o \times \gamma_o)$. Since the integrand of the right hand side is a harmonic differential for $z \in D(t)$ (even at $z = a$), the integral is invariant under replacing γ_o by another curve $\gamma \sim \gamma_o$ in G. It follows that $\dfrac{\partial I}{\partial t}(t, a)$ defines a continuous function for all $(t, a) \in \mathscr{D}_{B_o}$ and that, for any fixed $t \in B_o$, $\dfrac{\partial I}{\partial t}(t, a)$ is harmonic for $a \in D(t)$.

Now let $u(t, z)$ be the nomralized Abelian integral for $(D(t), \gamma_o)$ such that $\sigma(t, z) = du(t, z)$ for $z \in D(t)$. By Remark 1.1, we have $\sigma(t, z) = 0$ in $D(t) - D_1(t)$ and $u(t, z) = 0$ on $D(t) - D_1(t)$. Proposition 1.1 implies that, for $t \in B_o$ in Case (i), $u(t, z) = I(t, a) + 1$ for $a \in D_1'(t)$; $= I(t, a)$ for $a \in D_1''(t)$; $= 0$ for $a \in D(t) - D_1(t)$, while, for $t \in B_o$ in Case (ii), $u(t, a) = I(t, a)$ for $a \in D_1(t) - \gamma_o$; $= 0$ for $a \in D(t) - D_1(t)$. In both cases, $\dfrac{\partial u}{\partial t}(t, a)$ exists and $\dfrac{\partial u}{\partial t}(t, a) = \dfrac{\partial I}{\partial t}(t, a)$ for $(t, a) \in \mathscr{D}_{B_o} -$

$(B_o \times \gamma_o)$. Again moving γ_o a little in G, we have $\dfrac{\partial u}{\partial t}(t, a) = \dfrac{\partial I}{\partial t}(t, a)$ for all $(t, a) \in \mathscr{D}_{B_o}$. By (3.1), $\dfrac{\partial \sigma}{\partial t}(t, a) = d\left(\dfrac{\partial I}{\partial t}(t, a)\right)$ exists and is continuous for $(t, a) \in \mathscr{D}_{B_o}$, which proves (1) of Theorem 4.1. Since $\dfrac{\partial \lambda(t)}{\partial t} = \int_{\gamma_o} * \dfrac{\partial \sigma}{\partial t}(t, z)$, (2) follows from (1). □

COROLLARY 4.1. *Under the same conditions as in Theorem* 4.1, *either* $1/\lambda(t)(> 0)$ *is a* C^1 *superharmonic function on* B, *or* $1/\lambda(t) \equiv +\infty$ *on* B.

PROOF. We denote by B' (resp. B'') $= \{t \in B | \lambda(t) > 0$ (resp. $= 0$)$\}$. Note that $t \in B''$ iff $\gamma(t) \sim 0$. By Theorem 4.1, B' is open in B. Since $\mathscr{D}$ is unramified over $B \times C$, B'' is open in B. Consequently, $B = B'$ or B''. We assume $B = B'$. Let $t_o \in B - \ell$ where ℓ was defined in (4) of Condition 4.1. We take a disk B_o centered at t_o such that $B_o \subset B - \ell$. Then, the triple $(\mathscr{D}_{B_o}, \pi, B_o)$ satisfies Condition 2.1. Since $\varphi(t, z)$ in Condition 4.1 is plurisubharmonic in $\mathscr{D}^{\sim}$, the domain $\mathscr{D}_{B_o}$ over $B_o \times C$ is pseudoconvex.

By Corollary 3.1, $\partial^2\lambda(t)/\partial t\partial\bar{t} \geq \|\partial\Omega/\partial\bar{t}\|^2_{D(t)}$ for $t \in B_o$. Applying Schwarz's inequality to (1) of Theorem 3.1, we have

$$\left|\frac{\partial \lambda(t)}{\partial t}\right|^2 \leq \frac{1}{4}\|\Omega(t, \cdot)\|^2_{D(t)} \left\|\frac{\partial \Omega}{\partial \bar{t}}(t, \cdot)\right\|^2_{D(t)} \leq \frac{1}{2}\lambda(t)\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}}.$$

Thus, $1/\lambda(t)$ is a C^2 superharmonic function in B_o, and hence in $B - \ell$. On the other hand, by Theorem 4.1, $1/\lambda(t)$ is of class C^1 on B. Since ℓ consists of real 1-dimensional smooth curves in B, it follows from Stokes' formula that $1/\lambda(t)$ is a C^1 superharmonic function on B. □

5. Approximation theorem

Let $\mathscr{D}$ be a ramified domain over $B \times C$ which may be infinitely many sheeted. For $t \in B$, we denote by $(g(t), n(t))$ the topological type of the fiber $D(t)$, and put $k(t) = 2g(t) + n(t) - 1$. In general, $(g(t), n(t))$ depend on $t \in B$, and $k(t)$ may be $+\infty$. Let $\mathscr{S}$ be the set of branch surfaces of $\mathscr{D}$, and, $\mathscr{A}$, the set of singular points of $\mathscr{S}$, so that $\mathscr{A}$ consists of isolated points in $\mathscr{D}$. We put $\mathscr{D} = \bigcup_{t \in B}(t, D(t))$; $\mathscr{S} = \bigcup_{t \in B}(t, S(t))$ and $\mathscr{A} = \bigcup_{t \in B}(t, A(t))$. We assume

CONDITION 5.1. $\mathscr{S}$ contains no surfaces of the form $t = \text{const.}$, and $D(t) \neq \phi$ for any $t \in B$.

Let $(t_o, z_o) \in \mathscr{A}$. We find a bidisk $B_o \times K_o$ centered at (t_o, z_o) such that $[B_o \times (\partial K_o)] \cap \mathscr{S} = \phi$ and $\mathscr{S} \cap [\{t_o\} \times K_o] = (t_o, z_o)$. Each fiber $D(t) \cap K_o$, $t \in$

$B_o - \{t_o\}$ consists of a finite number of components, each a sheeted surface over K_o without relative boundary, say $d_i(t)$ $(1 \le i \le m)$. Note that m is independent of t. Let $t \to t_o$. Then some of these components, say $d_i(t)$ $(1 \le i \le k \le m)$, will be separated into several components $d_i^j(t_o)$ $(m+1 \le j \le h_i)$. Each $d_i^j(t_o)$ or $d_s(t_o)$ $(m+1 \le s \le n)$ is equivalent to the unit disk as Riemann surface.

Now we set $\mathscr{D}^* = \mathscr{D} - \mathscr{S} = \bigcup_{t \in B}(t, D^*(t))$, so that $\mathscr{D}^*$ is an unramified domain over $B \times C$, and each fiber $D^*(t)$, $t \in B$ consists of an at most countable number of unramified domains over C. We assume

CONDITION 5.2. $\mathscr{D}^*$ is a Stein manifold;

CONDITION 5.3. To each $t \in B$ we let correspond a cycle $\gamma(t)$ in $D(t) - A(t)$ in a continuous way in $\mathscr{D} - \mathscr{A}$.

Fix $t_o \in B$. We denote by $\sigma(t, \cdot)$ and $\lambda(t)$ the r-diff. and the h-mod. for $(D(t), \gamma(t))$. We set $\Omega(t, z) = \sigma(t, z) + i * \sigma(t, z)$ for $z \in D(t)$. Contrary to the case of compact bordered Riemann surfaces, it may happen that $\sigma(t, z) = 0$ on $D(t)$ and $\lambda(t) = 0$ for some $t \in B$ even when $\gamma(t)$ is not homologous to 0 in $D(t)$. (Precisely when $\gamma(t)$ is a dividing cycle on $D(t)$ such that the ideal boundary component of $D(t)$ determined by $\gamma(t)$ or $-\gamma(t)$ is of generalized capacity zero. See Marden and Rodin [8], for details.) The following theorem is useful in §§6 ~ 8.

THEOREM 5.1. *Assume that* $\mathscr{D}$ *satisfies Conditions* 5.1 ~ 5.3. *Then* (1) $1/\lambda(t)$ (> 0) *is a superharmonic function on B, which may be identically* $+\infty$; (2) *If* $\lambda(t)$ *is a harmonic function on B, then* $\Omega(t, z)$ *is holomorphic for* $(t, z) \in \mathscr{D} - \mathscr{A}$.

PROOF. Let $t_o \in B$. By Condition 5.3 we find a cycle γ_o near $\gamma(t_o)$ in $D^*(t_o)$ and a neighborhood $B_o \times G \subset\subset \mathscr{D}^*$ of (t_o, γ_o) such that $\gamma(t) \sim \gamma_o$ in $D(t)$ for all $t \in B_o$. By Condition 5.2 there exists a C^ω plurisubharmonic function $\varphi(t, z)$ in $\mathscr{D}^*$ such that, for any $\alpha < \infty$,

$$\mathscr{D}_\alpha = \{(t, z) \in \mathscr{D}^* | t \in B_o \text{ and } \varphi(t, z) < \alpha\} \subset\subset \mathscr{D}^* .$$

We take an α_o such that $\mathscr{D}_{\alpha_o} \supset\supset B_o \times G$. We can choose an increasing sequence $\{\alpha_n\}$ with $\alpha_n \to \infty$ such that $\mathscr{D}_{\alpha_n} = \bigcup_{t \in B_o}(t, D_{\alpha_n}(t))$ is an unramified domain over $B_o \times C$ satisfying Condition 4.1. Note the $\mathscr{D}_{\alpha_n} \to \mathscr{D}^* (n \to \infty)$. We simply put $\alpha_n = n$. Each $(D_n(t), \gamma_o)$ carries the r-diff. $\sigma_n(t, \cdot)$ and the h-mod. $\lambda_n(t)$. By Corollary 4.1, $1/\lambda_n(t)$ (> 0) is either a C^1 superharmonic function on B_o, or identically $+\infty$.

Now fix $t \in B_o$ and let $m > n$. Since $\sigma_m(t, \cdot) \in \Gamma_h(D_n(t))$ and $\gamma(t) \sim \gamma_o$ in $D_m(t)$, we have

$$(5.1)\qquad \|\sigma_m(t,\cdot)\|^2_{D_m(t)} = \int_{\gamma_o} *\sigma_m(t,\cdot) = (\sigma_m(t,\cdot), \sigma_n(t,\cdot))_{D_n(t)},$$

so that $\|\sigma_m(t,\cdot) - \sigma_n(t,\cdot)\|^2_{D_n(t)} \le \lambda_n(t) - \lambda_m(t)$. Hence, the sequence $\{\sigma_n(t,\cdot)\}$ uniformly converges to a harmonic differential $\sigma^\wedge(t,\cdot)$ on any compact set in $D^*(t)$, and $\lambda_n(t) \to \|\sigma^\wedge(t,\cdot)\|^2_{D^*(t)}$ as $n \to \infty$. Since $D(t) - D^*(t) (\subset S(t))$ is an isolated set, $\sigma^\wedge(t,\cdot)$ is harmonically extended to $D(t)$, so that $\sigma^\wedge(t,\cdot) \in \Gamma_h(D(t))$ and $\sigma^\wedge(t,\cdot) = \sigma(t,\cdot)$ in $D(t)$. Hence, $\lambda_n(t) \to \lambda(t)$ decreasingly as $n \to \infty$. This implies (1) of Theorem 5.1.

To prove (2), assume that $\lambda(t)$ is a harmonic function on B. Then, by Dini's theorem, $\lambda_n(t) \to \lambda(t)$ uniformly on B_o. We set, for any $t \in B_o$, $\Omega_n(t,z) = \sigma_n(t,z) + i*\sigma_n(t,z) = f_n(t,z)dz$ on $D_n(t)$; $\Omega(t,z) = \sigma(t,z) + i*\sigma(t,z) = f(t,z)dz$ on $D^*(t)$. If we extend $f_n(t,z)$ to be 0 on $D^*(t) - D_n(t)$, then $f_n(t,z) \to f(t,z)$ uniformly on any compact set on $D^*(t)$. We write $t = t_1 + it_2$ and $dV = dxdydt_1dt_2$ (the volume element of $\boldsymbol{R}^4$) and $\mathscr{D}^*_{B_o} = \bigcup_{t\in B_o}(t, D^*(t))$. It follows from (5.1) and (1.2) that

$$\lim_{n\to\infty} \iint_{\mathscr{D}^*_{B_o}} |f_n(t,z) - f(t,z)|^2 dV \le 2 \lim_{n\to\infty} \int_{B_o} (\lambda_n(t) - \lambda(t)) dt_1 dt_2 = 0\,.$$

We shall show

$$(5.2)\qquad \frac{\partial f}{\partial \bar t}(t,z) = 0 \text{ for } (t,z) \in \mathscr{D}^*_{B_o} \text{ in the sense of distributions}\,.$$

In fact, take any bidisk $B_1 \times V_1 \subset\subset \mathscr{D}^*_{B_o}$ and let $\phi(t,z) \in C^\infty_0(B_1 \times V_1)$. Since $f_n(t,z)$ is of class C^1 in $\mathscr{D}_n$ from (1) of Theorem 4.1, we have

$$J \equiv \iint_{B_1\times V_1} f\frac{\partial\phi}{\partial\bar t}dV = \lim_{n\to\infty} \iint_{B_1\times V_1} f_n\frac{\partial\phi}{\partial\bar t}dV = -\lim_{n\to\infty} \iint_{B_1\times V_1} \frac{\partial f_n}{\partial\bar t}\phi dV.$$

By Schwarz's inequality,

$$\begin{aligned}|J|^2 &\le \left\{\lim_{n\to\infty} \iint_{B_1\times V_1} \left|\frac{\partial f_n}{\partial\bar t}\right|^2 dV\right\}\left\{\iint_{B_1\times V_1} |\phi|^2 dV\right\}\\ &\le \left\{\lim_{n\to\infty} \int_{B_1} \left\|\frac{\partial\Omega_n}{\partial\bar t}(t,\cdot)\right\|^2_{B_n(t)} dt_1dt_2\right\}\left\{\iint_{B_1\times V_1} |\phi|^2 dV\right].\end{aligned}$$

Let $\ell_n = \{t \in B_o | \partial D_n(t)$ has at least one singular point$\}$, which consists of 1-dimensional C^ω smooth arcs. We set $B_o - \ell_n = \bigcup_{k=1}^\infty B_o^{(k)}$ where $B_o^{(k)}$ is a connected component. Then, $\mathscr{D}_n : t \to D_n(t)$, $t \in B_o^{(k)}$ is a C^∞ smooth variation. Since $\mathscr{D}_n$ is pseudoconvex, (1) of Corollary 3.1 implies $\partial^2\lambda_n/\partial t\partial\bar t \ge \|\partial\Omega_n/\partial\bar t\|^2_{D_n(t)}$ in $B_o^{(k)}$, and hence in $B_o - \ell_n$. Let $\psi(t) \in C^\infty_0(B_o)$ such that $\psi(t) \ge 0$ in B_o and $\psi(t) = 1$ on B_1. Then

$$\int_{B_1} \left\| \frac{\partial \Omega_n}{\partial \bar{t}}(t, \cdot) \right\|^2_{D_n(t)} dt_1 dt_2 \le \iint_{B_o - \ell_n} \frac{\partial^2 \lambda_n(t)}{\partial t \partial \bar{t}} \psi(t) dt_1 dt_2 \equiv K_n .$$

By (2) of Theorem 4.1, $\lambda_n(t)$ is of class C^1 on B_o and of class C^2 on $B_o - \ell_n$. Since ℓ_n consists of C^ω smooth arcs, it follows by Stokes' formula that

$$K_n = \iint_{B_o} \lambda_n(t) \frac{\partial^2 \psi(t)}{\partial t \partial \bar{t}} dt_1 dt_2 \to \iint_{B_o} \lambda(t) \frac{\partial^2 \psi}{\partial t \partial \bar{t}} dt_1 dt_2 = 0 .$$

The last equality follows from the assumption $\partial^2 \lambda(t)/\partial t \partial \bar{t} = 0$ on B. Hence $J = 0$ and (5.2) is proved.

On the other hand, $f(t, z)$ is a holomorphic for $z \in D(t)$, so that $f(t, z)$ is holomorphic for $(t, z) \in \mathscr{D}^*_{B_o}$. In other words, $\Omega(t, z)$ is holomorphic for $(t, z) \in \mathscr{D}^*_{B_o}$. Since each $\Omega(t, z)$, $t \in B_o$ is holomorphic for z in $D(t) - \mathscr{A}$, $\Omega(t, z)$ is holomorphic for $(t, z) \in \mathscr{D}_{B_o} - \mathscr{A}$. (2) of Theorem 5.1 is proved □

By a *generalized triple* $(\mathscr{M}, \pi, B)$ we mean that $\mathscr{M}$ is a connected 2-dimensional complex space, B a region in the complex plane $\mathbf{C}$, and π a holomorphic mapping from $\mathscr{M}$ onto B such that each $\pi^{-1}(t)$, $t \in B$ consists of an at most countable number of 1-dimensional irreducible analytic sets. We denote by $\mathscr{A}$ the set of singular points of $\mathscr{M}$. Assume that there exists a cycle $\gamma(t)$ in $M(t) - \mathscr{A}$ varying continuously with $t \in B$. We have the r-diff. $\sigma(t, z)$ and the h-mod. $\lambda(t)$ for $(M(t), \gamma(t))$. We put $\Omega(t, z) = \sigma(t, z) + i * \sigma(t, z)$ on $M(t)$. Then we have

COROLLARY 5.1. *If $\mathscr{M}$ is a Stein space, then* (1) *and* (2) (*where $\mathscr{D}$ is replaced by $\mathscr{M}$*) *of Theorem* 5.1 *hold.*

PROOF. By Bishop's theorem [4], $\mathscr{M}$ is biholomorphically mapped onto a ramified domain $\mathscr{D}$ over $B \times \mathbf{C}$ with Conditions 5.1 and 5.2 by a transformation $\Phi : (t, z) \in \mathscr{M} \to (t, w) = (t, \phi(t, z)) \in \mathscr{D}$. We put $\mathscr{A}^\wedge = \Phi(\mathscr{A})$ and $\gamma^\wedge(t) = \phi(t, \gamma(t))$. Then $\mathscr{A}^\wedge$ is the set of singular points of $\mathscr{D}$ and $\gamma^\wedge(t)$ varies continuously in $\mathscr{D} - \mathscr{A}^\wedge$ with $t \in B$, that is, $\gamma^\wedge(t)$ satisfies Condition 5.3. Hence Theorem 5.1 is applied to $\mathscr{D}$ and $\gamma^\wedge(t)$. Since $\lambda(t)$ and $\Omega(t, z)$ are invariant under the transformation Φ, we have Corollary 5.1. □

6. Proof of (III) and (IV) in the Introduction

Given two triples $\mathfrak{M}_i = (\mathscr{M}_i, \pi_i, B)$ $(i = 1, 2)$, we say that $\mathfrak{M}_1$ is *topologically* (resp. *holomorphically*) *equivalent to* $\mathfrak{M}_2$, if there exists a topological (resp. biholomorphic) mapping T from $\mathscr{M}_1$ onto $\mathscr{M}_2$ such that $\pi_2 \circ T = \pi_1$. In the holomorphic case we write $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. As defined in the Introduction, in the case when $\mathfrak{M}_2 = (B \times R, \pi_B, B)$, where R is a Riemann surface and π_B is the first projection, we say that $\mathfrak{M}_1$ is *topologically* (resp. *holomorphically*) *trivial.* If R is of (topological) type (g, n), $\mathfrak{M}_1$ is said to be of type (g, n).

THEOREM 6.1. *Let* $\mathfrak{M} = (\mathcal{M}, \pi, \boldsymbol{C})$ *be a topologically trivial triple of type* $(0, 1)$. *Assume that (a)* $\mathfrak{M}$ *is of locally Stein; (b)* $M(t_o)$ *for some* $t_o \in \boldsymbol{C}$ *is conformally equivalent to the unit disk* $D = \{|w| < 1\}$; *(c) There exist at least two holomorphic sections* $\alpha_i : t \to \alpha_i(t)$ $(i = 1, 2)$ *of* $\mathcal{M}$ *defined on* $\boldsymbol{C}$. *Then* $\mathfrak{M}$ *is holomorphically trivial:* $\mathfrak{M} \simeq (\boldsymbol{C} \times D, \pi_{\boldsymbol{C}}, \boldsymbol{C})$.

PROOF. By (c) we draw a Jordan curve $\gamma(t)$ on each $M(t)$, $t \in \boldsymbol{C}$ rounding $\alpha_1(t)$ and $\alpha_2(t)$ positively such that $\gamma(t)$ varies continuously with the parameter $t \in \boldsymbol{C}$ in $\mathcal{M}$. We consider the double sheeted domain $\mathcal{M}^\wedge$ over $\mathcal{M}$ with branch surfaces α_1 and α_2 and without relative boundary. Let $J : \mathcal{M}^\wedge \to \mathcal{M}$ be the canonical projection. We put $\pi^\wedge = \pi \circ J$ and $\pi^{\wedge -1}(t) = M^\wedge(t)$, so that $\mathcal{M}^\wedge = \bigcup_{t \in \boldsymbol{C}}(t, M^\wedge(t))$ and $M^\wedge(t)$ is the double sheeted surface over $M(t)$ with branch points $\alpha_1(t)$ and $\alpha_2(t)$ and without relative boundary. Denote by $J_t : M^\wedge(t) \to M(t)$ the restriction J to $M^\wedge(t)$. We find two disjoint curves $\gamma_i(t) \subset M^\wedge(t)$ $(i = 1, 2)$ over $\gamma(t)$, which vary continuously with $t \in \boldsymbol{C}$ in $\mathcal{M}^\wedge$. For any $t \in \boldsymbol{C}$, we consider the r-diff. $\sigma(t, z^\wedge)$ and the h-mod. $\lambda(t)$ for $(M^\wedge(t), \gamma_1(t))$. We write $\Omega(t, z^\wedge) = \sigma(t, z^\wedge) + i * \sigma(t, z^\wedge)$ on $M^\wedge(t)$. Fix $t_o \in \boldsymbol{C}$. By (a), we find a disk $B \subset \boldsymbol{C}$ of center t_o such that $\mathcal{M}_B^\wedge$ is a Stein space. By Corollary 5.1, $1/\lambda(t)$ is a superharmonic function on B. Consequently, $1/\lambda(t)$ is a non-negative superharmonic function on $\boldsymbol{C}$, so that it is a constant $1/c$ on $\boldsymbol{C}$, namely, $0 \leq \lambda(t) = c < \infty$ for all $t \in \boldsymbol{C}$. Theorem in [14, p. 84] says that, under conditions (a) and (b), all $M(t)$ for $t \in \boldsymbol{C} - K$, where K is a closed set of logarithmic capacity 0: *Cap* $K = 0$ in $\boldsymbol{C}$, are conformally equivalent to the unit disk D. If we take a point $t_o \in \boldsymbol{C} - K$ such that $\alpha_1(t_o) \neq \alpha_2(t_o)$, then $0 < \lambda(t_o) = c < \infty$. It conversely follows that $\alpha_1(t) \neq \alpha_2(t)$ for all $t \in \boldsymbol{C}$, and that each $M(t)$, $t \in \boldsymbol{C}$ is conformal to D. By (2) of Corollary 5.1, $\Omega(t, z^\wedge)$ is holomorphic for $(t, z^\wedge) \in \mathcal{M}^\wedge$.

Fix $t \in \boldsymbol{C}$ and let $z^\wedge \in M^\wedge(t)$. We put $\phi(t, z^\wedge) = \exp\left\{\frac{2\pi}{c}\int_{\ell(t)} \Omega(t, z^\wedge)\right\}$, where $\ell(t)$ is an arc connecting $\alpha_1(t)$ and $z^\wedge$ in $M^\wedge(t)$. From the theory of one complex variable, we have (*i*) $M^\wedge(t)$ is conformally equivalent to the annulus $A = \{1/r < |W| < r\}$ by $W = \phi(t, z^\wedge)$ such that $r = e^{\pi/c}$; $\phi(t, \alpha_1(t)) = 1$ and $\phi(t, \alpha_2(t)) = -1$; (*ii*) for $z \in M(t)$, we take two points $z_i^\wedge \in M^\wedge(t)$ $(i = 1, 2)$ such that $J_t(z_i^\wedge) = z$. If we put $W_i = \phi(t, z_i^\wedge)$, then $W_1 W_2 = 1$; (*iii*) if we consider the identification I: $W_1 \sim W_2$ by $W_1 W_2 = 1$ on A, then the quotient space $A/\sim$ is conformally equivalent to the unit disk D. If follows that, for each $t \in \boldsymbol{C}$, $w = I \circ \phi(t, J_t^{-1}(z))$ $(= \psi(t, z))$ is well-defined and one to one conformal mapping from $M(t)$ onto D. Since $\Omega(t, z^\wedge)$ is holomorphic for $(t, z^\wedge) \in$

$\mathscr{M}^\wedge$, $\psi(t, z)$ holomorphically depends on $t \in \boldsymbol{C}$. Hence, $\mathfrak{M} \simeq (\boldsymbol{C} \times D, \pi_{\boldsymbol{C}}, \boldsymbol{C})$ by $\Psi : (t, z) \to (t, \Psi(t, z))$. □

REMARK 6.1. The idea of constructing a double covering $\mathscr{M}^\wedge$ of $\mathscr{M}$ is useful to prove the Picared theorem: *If an entire function $f(t)$ on $\boldsymbol{C}$ attains neither* 0 *nor* 1, *then $f(t)$ is a constant.* Indeed, for each $t \in \boldsymbol{C}$, we construct a double covering $D(t)$ over $\boldsymbol{P}^1$ with 4 distinct branch points $\{0, 1, f(t), \infty\}$. $D(t)$ is a compact Riemann surface of genus 1. We can draw a non-trivial cycle $\gamma(\mathrm{t})$ in $D(t)$ such that $\gamma(t)$ varies continuously with $t \in \boldsymbol{C}$, and consider the h-mod. $\lambda(t)$ for $(D(t), \gamma(t))$. If we put $\mathscr{D} = \bigcup_{t \in \boldsymbol{C}}(t, D(t))$, then $\mathscr{D}$ and $\gamma(t)$ satisfy Conditions 5.1 ~ 5.3. By (1) of Theorem 5.1, $1/\lambda(t)$ is a positive superharmonic function on $\boldsymbol{C}$, so that $\lambda(t)$ is a constant on $\boldsymbol{C}$, and hence $f(t)$ is a constant on $\boldsymbol{C}$.

COROLLARY 6.1. *Let $\mathfrak{M} = (\mathscr{M}, \pi, \boldsymbol{C})$ be a topologically trivial triple of finite or infinite type (g, n). Assume that (a) $n \geq 1$ and $\mathfrak{M}$ is of locally Stein; (b) There exists $t_o \in \boldsymbol{C}$ such that the universal covering surface $M^\sim(t_o)$ of $M(t_o)$ is conformally equivalent to the unit disk D; (c) There exists at least one holomorphic section $\alpha : t \to \alpha(t)$ of $\mathscr{M}$ defined on $\boldsymbol{C}$. Then $\mathfrak{M}$ is holomorphically trivial: $\mathfrak{M} \simeq (\boldsymbol{C} \times R, \pi_{\boldsymbol{C}}, \boldsymbol{C})$.*

PROOF. For any $t \in \boldsymbol{C}$ we construct the universal covering surface $M^\sim(t)$ of $M(t)$ starting from the point $\alpha(t)$. We denote by $G(t) = \{f_n(t, z)\}_{n=0,1,\dots}$ the cover transformation group of $M^\sim(t)$, so that $M^\sim(t)/G(t) = M(t)$. Since $\mathfrak{M}$ is topologically trivial, we canonically obtain the topologically trivial triple $\mathfrak{M}^\sim = (\mathscr{M}^\sim, \pi, \boldsymbol{C})$ of type (0, 1) and a holomorphic cover transformation group $\mathscr{G}^\sim = \{T_n\}_{n=0,1,\dots}$ of $\mathscr{M}^\sim$ such that (i) $\mathscr{M}^\sim/\mathscr{G}^\sim = \mathscr{M}$; (ii) $\mathscr{M}^\sim = \bigcup_{t \in \boldsymbol{C}}(t, M^\sim(t))$ with $\pi^{\sim -1}(t) = M^\sim(t)$ for $t \in \boldsymbol{C}$; (iii) each $T_n \in \mathscr{G}^\sim$ satisfies $\pi^\sim \circ T_n = \pi^\sim$ in $\mathscr{M}^\sim$ and the restriction of T_n to each $M^\sim(t)$, $t \in \boldsymbol{C}$ is identical with $f_n(t, z)$. We note that $f_n(t, z)$ is holomorphic for $(t, z) \in \mathscr{M}$. Since $\mathfrak{M}$ is of locally Stein, so is $\mathfrak{M}^\sim$. By condition (c), $\mathfrak{M}^\sim$ has infinitely many holomorphic sections $\alpha_n^\sim = T_n \circ \alpha$ $(n = 0, 1, \cdots)$ defined on $\boldsymbol{C}$. By (b), Theorem 6.1 implies that $\mathfrak{M}^\sim \simeq (\boldsymbol{C} \times D, \pi_{\boldsymbol{C}}, \boldsymbol{C})$. We denote by $L_n(t, z)$ the cover transformation of D corresponding to $f_n(t, z)$ of $M^\sim(t)$. Since $L_n(t, z)$ is of the form $L_n(t, z) = e^{i\theta_n}(z - a_n(t))/(1 - \overline{a_n(t)}z)$ and since $L_n(t, z)$ is holomorphic for $(t, z) \in \boldsymbol{C} \times D$, $L_n(t, z)$ does not depend on $t \in \boldsymbol{C}$: $L_n(t, z) = L_n(z)$. If we put $D/\{L_n(z)\}_{n=0,1,\dots} = R$, then $\mathfrak{M} \cong (\boldsymbol{C} \times R, \pi_{\boldsymbol{C}}, \boldsymbol{C})$. □

By the proof, we note that the holomorphic section α of $\mathscr{M}$ corresponds to a constant section of $\boldsymbol{C} \times R$. Statement (III) in the Introduction is Theorem 6.1, and (IV) easily follows Corollary 6.1.

In the theory of function algebra, it has been studied when an analytic multivalued function $\mathscr{E}$ in $B \times \boldsymbol{C}$ contains a 1-dimensional analytic set (cf.

Wermer [9, Theorem 1]). It is known that $\mathscr{E}$ is a pseudoconcave set in $B \times \boldsymbol{C}$ in the theory of several complex variables, and the converse is true (see Aupetit [1, Chap. VII]). We put $E(t) = \{z \in \boldsymbol{C} | (t, z) \in \mathscr{E}\}$ for each $t \in B$. By applying the usual normal family method to Theorem 6.1 we immediately have the following result concerning this subject:

COROLLARY 6.2. *Let $\mathscr{E}$ be a nonempty pseudoconcave set in $\boldsymbol{C}^2$ of two complex variables (t, z) such that each $E(t)$, $t \in \boldsymbol{C}$ is bounded in $\boldsymbol{C}$ and $\boldsymbol{P}^1 - E(t)$ is simply connected. Assume that there exists a meromorphic function $f(t)$ on $\boldsymbol{C}$ such that $f(t) \notin E(t)$ for $t \in \boldsymbol{C}$. Then we find an entire function $g(t)$ such that $g(t) \in E(t)$ for $t \in \boldsymbol{C}$.*

We often use the following

NOTATION 6.1. Let $\mathfrak{M} = (\mathscr{M}, \pi, B)$ be a topologically trivial triple of finite type (g, n) with $n \geq 1$. We thus have a topological mapping

$$(\mathrm{N}_1) \qquad T: (t, z) \in \mathscr{M} \to (t, w) = (t, \varphi(t, z)) \in B \times S \text{ with } \pi_B \circ T = \pi\,,$$

where S is a Riemann surface of genus g and with n boundary components. For any $t \in B$ and any $K \subset S$, we define $K(t) = \varphi^{-1}(t, K) \subset M(t)$. Hence, given cycle γ in S, $\gamma(t)$ defines a cycle in $M(t)$ which varies continuously with $t \in B$ in $\mathscr{M}$. For each $t \in B$, we denote by $\sigma_\gamma(t, z)$ and $\lambda_\gamma(t)$ the r-diff. and h-mod. for $(M(t), \gamma(t))$, and write $\Omega_\gamma(t, z) = \sigma_\gamma(t, z) + \sqrt{-1} * \sigma_\gamma(t, z)$ for $z \in M(t)$. We put $\partial S = C_1^{\sim} + \cdots + C_n^{\sim}$. We can draw n smooth cycles C_i in S such that C_i and $C_i^{\sim}$ surround annulus domain E_i of S in the manner that

$$(\mathrm{N}_2) \qquad \partial E_i = C_i^{\sim} - C_i \text{ and } E_i \cap E_j = \phi (i \neq j)\,.$$

We say that E_i is an *end of S with boundary component $C_i^{\sim}$*. So, each $E_i(t)$, $t \in B$ defines a noncompact region in $M(t)$ such that $E_i(t)$ has a relative boundary $\partial E_i(t) = C_i(t)$ and an ideal boundary component of $M(t)$, which we denote by $C_i^{\sim}(t)$. $E_i(t)$ is called an *end of $M(t)$ with ideal boundary component $C_i^{\sim}(t)$*. We write $\delta E_i(t) = C_i^{\sim}(t) - C_i(t)$, and $C_i(t) \sim C_i^{\sim}(t)$ in $M(t)$. In case when $\gamma = C_i (1 \leq i \leq n)$, we simply put

$$(\mathrm{N}_3) \qquad \sigma_{C_i}(t, z) = \sigma_i(t, z)\,; \qquad \lambda_{C_i}(t) = \lambda_i(t) \quad \text{and} \quad \Omega_{C_i}(t, z) = \Omega_i(t, z)\,.$$

As stated in the Introduction, each ideal boundary component $C_i^{\sim}(t)$ of $M(t)$ is either degenerating (to a point) or non-degenerating. We put

$$(\mathrm{N}_4)$$
$$K_{\mathfrak{M}} = \{t \in B | M(t) \text{ has at least one degenerating ideal boundary component}\}\,.$$

Under these notations we have

LEMMA 6.1. *Let $\mathfrak{M} = (\mathcal{M}, \pi, B)$ be a topologically trivial triple of finite type (g, n), and of locally Stein. If Cap $K_{\mathfrak{M}} > 0$, then we find a topologically trivial triple $(\mathcal{M}^{\sim}, \pi^{\sim}, B)$ of type $(g, n-1)$ and a holomorphic section α of $\mathcal{M}^{\sim}$ defined on B such that $\mathfrak{M} \simeq (\mathcal{M}^{\sim} - \alpha(B), \pi^{\sim}, B)$.*

PROOF. Take $t_o \in B$ such that, for any disk B_o of center t_o, $Cap(B_0 \cap K_{\mathfrak{M}}) > 0$. Let $p_o \in M(t_o)$ and let $B_o \times U_o$ be a π-local coordinates at p_o. We put $M'(t) = M(t) - U_o$ and consider a triple $\mathfrak{M}' = (\mathcal{M}', \pi', B_o)$ where $\mathcal{M}' = \mathcal{M} - B_o \times U_o$ ($\subset \mathcal{M}$) and $\pi'^{-1}(t) = M'(t)$ for $t \in B_o$. $\mathfrak{M}'$ is a topologically trivial triple of type $(g, n+1)$. We can take the cycle $C_1(t)(\sim C_i^{\sim}(t)) \subset M'(t)$ $(1 \leq i \leq n)$. Then we have the h-mod. $\mu_i(t)$ for $(M'(t), C_i(t))$ for $t \in B$. It is clear that $\mu_i(t) = 0$, if and only if $C_i^{\sim}(t)$ is a degenerating ideal boundary component of $M(t)$. Since each $1/\mu_i(t)(>0)$ is superharmonic on B_o and since $\sum_{i=1}^{n} 1/\mu_i(t) = +\infty$ on $B_o \cap K_{\mathfrak{M}}$, it follows that one of them, say $1/\mu_1(t)$, is identically $+\infty$ on B_o, and hence on B. Thus, the ideal boundary component $C_1^{\sim}(t)$ of any $M(t)$, $t \in B$ is degenerating. It follows from Fundamental Theorem in Nishino [10] combined with §3 in [13] that, for any $t_o \in B$, we find a disk B_o of center t_o and an end E_1' ($\subset E_1$) of S with ideal boundary $C_1^{\sim}$ such that, if we put $\mathcal{E}_1' = T^{-1}(B_o \times E_1')$ ($\subset \mathcal{M}$), then the triple $(\mathcal{E}_1', \pi, B_o)$ is holomorphically equivalent to a triple $(\mathcal{G}_1, \pi_1, B_o)$ of type $(0, 2)$ such that each fiber $G_1(t)$, $t \in B_o$ is a Jordan domain punctured at 0 which corresponds to $C_1^{\sim}(t)$. Thus the lemma is proved. □

We can now prove that, *under the condition*: *Cap* $K_{\mathfrak{M}} > 0$, *statement* (II) *in the Introduction is true.*

In fact, let $\mathfrak{M} = (\mathcal{M}, \pi, C)$ be a topologically trivial triple of finite type (g, n) with $n \geq 1$, and of locally Stein. Assume that *Cap* $K_{\mathfrak{M}} > 0$. Then Corollary 6.1 combined with Lemma 6.1 readily implies that $\mathfrak{M}$ is holomorphically trivial, if (i) $n \geq 2$ and $M(t_o)$ for some $t_o \in C$ has at least one non-degenerating ideal boundary component, or (ii) $2g + n - 1 \geq 3$. The other case is: $\mathfrak{M}$ is of type (0, 1), (0, 2) or (1, 1) such that all ideal boundary components of each $M(t)$, $t \in C$ are degenerating. This case is reduced to (I) by Nishino [11]. □

Statement (II) under *Cap* $K_{\mathfrak{M}} = 0$ remains to be proved. In order to study this in §8, we prepare local properties in §7.

7. Local properties

Let $(\mathcal{M}, \pi, B)$ be a triple. Let $f_i(t, z)$ $(i = 1, 2)$ be a meromorphic function for $(t, z) \in \mathcal{M}$ such that, for any fixed $t \in B$, (7.1) $$f_i(t, z) \text{ is non-constant for } z \in M(t).$$

We consider the transformation

$$T_i : (t, z) \in \mathcal{M} \to (t, w_i) = (t, f_i(t, z)) \in B \times \boldsymbol{P}^1,$$

and denote by $\mathcal{D}_i = T_i(\mathcal{M})$. Thus $\mathcal{D}_i$ is a ramified domain over $B \times \boldsymbol{P}^1$, and $T_2 \circ T_1^{-1}$ is a biholomorphic mapping from $\mathcal{D}_1$ onto $\mathcal{D}_2$. We write

(7.2) $$T_2 \circ T_1^{-1} : (t, w_1) \to (t, w_2) = (t, \Phi(t, w_1)),$$

where $\Phi(t, w_1) = f_2(t, f_1^{-1}(t, w_1))$ is a meromorphic function on $\mathcal{D}_1$ such that, for each $t \in B$, $\Phi(t, w_1)$ is non-constant for $w_1 \in D_1(t)$. We put

$$\mathcal{S}_i = \{\text{all irreducible components of the branch surfaces of } \mathcal{D}_i\}.$$

Let $\sigma \in \mathcal{S}_i$ and take a non-singular point (t_o, w_o) of σ. Then, σ near (t_o, w_o) in $\mathcal{D}_i$ is written in the form $\sigma: w_i = \xi(t)$ with $\xi(t_o) = w_o$, where $\xi(t)$ is a meromophic function for t. In the case when $\xi(t)$ is constant (resp. non-constant) for t, we say that the component σ in $\mathcal{S}_i$ is *constant* (resp. *non-contant*) *for t*. We put

$$\mathcal{S}_i' (\text{resp. } \mathcal{S}_i'') = \{\sigma \in \mathcal{S}_i | \sigma \text{ is constant (resp. non-constant) for } t\}.$$

We consider the following subset Σ in $B \times \boldsymbol{P}^2$:

$$\Sigma = \{(t, w_1, w_2) \in B \times \boldsymbol{P}^2 | w_i = f_i(t, z) \text{ for } (t, z) \in \mathcal{M}\},$$

which is a 2-dimensional irreducible analytic set in $B \times \boldsymbol{P}^2$ (not always closed in $B \times \boldsymbol{P}^2$). We call Σ the *graph of* $\mathcal{M}$ *by* (f_1, f_2). We say that Σ *realizes* $\mathcal{M}$, if Σ and $\mathcal{M}$ are one to one except for an at most countable 1-dimensional analytic sets. Then we have

LEMMA 7.1. *Assume that there exists a* 3*-dimensional* C^ω *set* $\mathcal{L} = \bigcup_{t \in B}(t, L(t))$ *in an open set* $\mathcal{G}$ $(\subset \mathcal{M})$ *such that* (*a*) *each* $L(t)$, $t \in B$ *is a* 1*-dimensional* C^ω *non-singular arc in* $M(t)$; (*b*) $\mathrm{Im}\{f_i(t, z)\} = 0$ *for* $(t, z) \in \mathcal{L}$ $(i = 1, 2)$. *Then the following results* (1) *and* (2) *hold*: (1) *If* $\sigma_1 \in \mathcal{S}_1''$ *exists, then* $T_2 \circ T_1^{-1}(\sigma_1)$ $(= \sigma_2)$ *belongs to* $\mathcal{S}_2''$, *and the order of ramification of* $\mathcal{D}_1$ *along* σ_1 *is equal to that of* $\mathcal{D}_2$ *along* σ_2; (2) *If* Σ (*defined above*) *realizes* $\mathcal{M}$, *then* $\mathcal{S}_i''$ $(i = 1, 2)$ *is empty.*

PROOF. We first show that

(7.3) $$\Phi(t, w_1) \text{ of (7.2) does not depend on } t \in B.$$

In fact, by (7.1), (a) and (b), we find a point $q_o \in \mathcal{L}$ with $\pi(q_o) = t_o$ (we put $t_o = 0$) and π-local coordinates $B_o \times U_o = (|t| < r_o) \times (|z| < \rho_o)$ of $\mathcal{M}$ at q_o such that (*i*) each arc $L(t)$, $t \in B_o$ divides U_o into two regions; (*ii*) the function

$f_i(t, z)$ $(i = 1, 2)$ is holomorphic in $B_o \times U_o$; *(iii)* for any fixed $t \in B_o$, $f_i(t, z)$ is univalent on U_o; *(iv)* $\operatorname{Im}\{f_i(t, z)\} = 0$ for all $z \in L(t) \cap U_o$, $t \in B_o$. Now we put $(a_1, a_2) = (f_1(q_o), f_2(q_o)) \in \boldsymbol{C}^2$, where $\operatorname{Im} a_i = 0$ $(i = 1, 2)$. Then, $T_i : (t, z) \to (t, f_i(t, z))$ is a biholomorphic mapping from $B_o \times U_o$ onto a (schlicht) neighborhood $\mathscr{V}_i$ of $(0, a_i)$ in $B_o \times \boldsymbol{C}$, so that $\mathscr{V}_1$ and $\mathscr{V}_2$ are biholomorphic by $T_2 \circ T_1^{-1} : (t, w_1) \to (t, w_2) = ((t, \Phi(t, w_1))$. We have

$$w_2 = \Phi(t, w_1) = \sum_{n=0}^{\infty} c_n(t)(w - a_1)^n \text{ with } c_0(0) = a_2$$

in a bidisk $B_1 \times V_1$ $(\subset \mathscr{V}_1)$ of center $(0, a_1)$. Each $c_n(t)$ is holomorphic on B_1. By (iv), $\operatorname{Im}\{\Phi(t, w_1)\} = 0$ for all $(t, w_1) \in B_1 \times V_1$ with $\operatorname{Im} w_1 = 0$. Hence $c_n(t)$ is real-valued on B_1, and $c_n(t) = \text{const. } c_n$ on B_1. So, $\Phi(t, w_1)$ on $B_1 \times V_1$ does not depend on $t \in B_1$. Since $\mathscr{D}_1$ is connected, (7.3) follows by analytic continuation. In order to prove (1) of Lemma 7.1, let $\sigma_1 \in \mathscr{S}_1''$ and let $\ell - 1$ (≥ 2) be the order of ramification of $\mathscr{D}_1$ along σ_1. We take a point $(t_o, a_1) \in \sigma_1$ such that σ_1 near (t_o, a_1) in $\mathscr{D}_1$ is of the form σ_1: $w_1 = \xi_1(t)$ for $t \in B_o$ with $\xi_1(t_o) = a_1$ where $\xi_1(t)$ is a non-constant holomorphic function in a disk $B_o(\subset\subset B)$ of center t_o. We put $p_o = T_1^{-1}(t_o, a_1)$, $\tau = T_1^{-1}(\sigma_1) \subset \mathscr{M}$, $a_2 = f_2(p_o)$, $\sigma_2 = T_2(\tau) \subset \mathscr{D}_2$ and σ_2: $w_2 = \xi_2(t)$ for $t \in B_o$. Then $\xi_2(t)$ is a meromorphic function on B_o with $\xi_2(t_o) = a_2$. For simplicity we assume that $\xi_2(t)$ is holomorphic on B_o. We take π-local coordinates $(t, z) \in B_o \times U_o$ at p_o such that τ corresponds to $B_o \times \{0\}$. Then, $f_i(t, z)$ near $(t_o, 0)$ is of the form

$$(7.4)\qquad \begin{aligned} w_1 &= f_1(t, z) = \xi_1(t) + b_\ell(t)z^\ell + b_{\ell+1}(t)z^{\ell+1} + \cdots; \\ w_2 &= f_2(t, z) = \xi_2(t) + c_1(t)z + c_2(t)z^2 + \cdots, \end{aligned}$$

where (t, z) runs in $B_o \times U_o$. Each coefficient $b_j(t)$, $c_j(t)$ is holomorphic in B_o and $b_\ell(t) \neq 0$ for any $t \in B_o$. It is enough for (1) of Lemma 7.1 to show that *(i)* $\xi_2(t)$ is non-constant for $t \in B_o$; *(ii)* $c_1(t) = \cdots = c_{\ell-1}(t) = 0$ on B_o and $c_\ell(t) \neq 0$ for some $t \in B_o$. To prove these, we consider a set

$$(7.5)\qquad \sigma = \{(w_1, w_2) \in \boldsymbol{P}^2 \,|\, w_i = f_i(t_o, z) \text{ for } z \in U_o\}.$$

If we take a small bidisk $V_1 \times V_2$ of center (a_1, a_2), then $\sigma \cap (V_1 \times V_2)$ $(= \sigma_o)$ is a closed 1-dimensional analytic set in $V_1 \times V_2$. By (7.1), we can write $\sigma_o = \{(w_1, w_2) \in V_1 \times V_2 \,|\, F(w_1, w_2) = 0\}$ where $F(w_1, w_2)$ is a holomorphic function on $V_1 \times V_2$ such that the set $A = \left\{(w_1, w_2) \in \sigma_o \,\middle|\, \frac{\partial F}{\partial w_1}(w_1, w_2) = 0 \text{ or } \frac{\partial F}{\partial w_2}(w_1, w_2) = 0\right\}$ is a finite point set. We take a smaller bidisk $B_1 \times U_1$ $(\subset B_o \times U_o)$ of center $(t_o, 0)$ such that $f_i(B_1, U_1) \subset V_i$ $(i = 1, 2)$. Consider the subgraph Σ_o of Σ: $\Sigma_o = \{(t, w_1, w_2) \in B_1 \times V_1 \times V_2 \,|\, w_i = f_i(t, z) \text{ for } (t, z) \in B_1 \times$

$U_1\}$. Then (7.3) implies that

$$\Sigma_o \subset B_o \times \sigma_o . \tag{7.6}$$

First, we put $z=0$ in (7.4). Then we have $\{(t, \xi_1(t), \xi_2(t)) \in B_1 \times V_1 \times V_2 | t \in B_1\} \subset \Sigma_o \subset B_o \times \sigma_o$. Since $\xi_1(t)$ is non-const. on B_o and since A is a finite set, $\xi_2(t)$ is non-const. on B_o. (*i*) is proved. Next, we put $B_1' = \{t \in B_1 | c_j(t) \neq 0$ for some j $(1 \leq j \leq \ell - 1)\}$ and $B_1'' = \{t \in B_1 | c_\ell(t) = 0\}$. Fix $t \in B_1' \cup B_1''$. Then (7.4), together with $b_\ell(t) \neq 0$, implies that $\{(\xi_1(t), \xi_2(t)) | t \in B_1' \cup B_1''\} \subset A$. Since $\xi_1(t)$ is non-const. on B_1, the set $B_1' \cup B_1''$ is also a finite point set. Hence, $B_1' = \phi$. Since $b_\ell(t) \neq 0$ for any $t \in B_1$, (7.6) implies $B_1'' = \phi$. (*ii*) is proved.

We shall prove (2) of Lemma 7.1 by contradiction. Assume that there exists an $\mathscr{s}_1 \in \mathscr{S}_1''$ with order of ramification $\ell - 1$ (≥ 1). Using the above notations we have

$$\Sigma_o = \{(t, \xi_1(t) + b_\ell(t) z^\ell + \cdots, \xi_2(t) + c_\ell(t) z^\ell + \cdots) | (t, z) \in B_1 \times U_1\} \subset B_o \times \sigma_o .$$

Since $\Sigma \supset \Sigma_o$ and $\mathscr{M} \supset B_1 \times U_1$, this contradicts the hypothesis. $\square$

ACKNOWLEDGEMENT. (1) of Lemma 7.1 was proved in [16] by calculation. The above intuitive proof by use of the graph is due to Professor Tetsuo Ueda.

Let $\mathfrak{M} = (\mathscr{M}, \pi, B)$ be a topologically trivial triple of finite type (g, n). We use $(N_1) \sim (N_4)$ in Notation 6.1 for this $\mathfrak{M}$. Then we have

LEMMA 7.2. *Assume that* (*a*) $\mathfrak{M}$ *is of locally Stein*; (*b*) *Cap* $K_{\mathfrak{M}} = 0$; (*c*) $n \geq 2$ *and at least one of* $\{\lambda_i(t)\}_{1 \leq i \leq n}$, *say* $\lambda_1(t)$, *is a constant* k_1 *in* B. *Then, for any* $t_o \in B$, *we find a disk* B_o *centered at* t_o *such that the subtriple* $\mathfrak{M}_{B_o}$ *of* $\mathfrak{M}$ on B_o *is holomorphically equivalent to a triple* $\mathfrak{M}' = (\mathscr{M}', \pi', B_o)$ *with* C^ω *smooth boundary*: $\mathfrak{M}_{B_o} \simeq \mathfrak{M}'$.

PROOF. By (b), there exists $t^* \in B$ such that the fiber $M(t^*)$ is conformally equivalent to the interior R_o of a compact bordered Riemann surface $\overline{R}_o$ of genus g with n C^ω smooth boundary contours $\{C_{io}\}$ by a conformal mapping

$$\xi : z \in M(t^*) \rightarrow w = \xi(z) \in R_o . \tag{7.7}$$

We let correspond $C_i^\sim(t^*)$ to C_{io} $(1 \leq i \leq n)$ by ξ. We have $\lambda_1(t^*) = k_1 > 0$. By (2) of Corollary 5.1, $\Omega_1(t, z)$ is holomorphic for $(t, z) \in \mathscr{M}$ such that $\Omega_1(t, z) \neq 0$ on $M(t)$ for any $t \in B$. We shall prove

$$\int_{C_i(t)} \Omega_1(t, z) = \begin{cases} \sqrt{-1}\, c_1 \ (i = 1) & \text{where } c_1 > 0 , \\ \sqrt{-1}\, c_i \ (2 \leq i \leq n) & \text{where } c_i < 0 . \end{cases} \tag{7.8}$$

In fact, the integral $I_i(t) = \int_{C_i(t)} \Omega_1(t, z)$ $(1 \leq i \leq n)$ is a holomorphic function for $t \in B$. By (1.2), we have $\mathrm{Re}\,\{I_i(t)\} = C_1(t) \times C_i(t) = 0$, so that $I_i(t)$ is a constant $\sqrt{-1}\; c_i$. If we consider the harmonic function $U(w)$ on R_o with boundary values 1 on C_{1o} and 0 on C_{io} $(2 \le i \le n)$, then we have $\sigma_1(t^*, z) = dU \circ \xi(z)$ on $M(t^*)$, and $c_i = \int_{C_{io}} \frac{\partial U}{\partial n_z} ds_z$ $(1 \le i \le n)$. Hence, $c_1 = \lambda_1(t^*) = k_1 > 0$, while $c_i < 0$ $(2 \le i \le n)$. (7.8) is proved.

Since $c_i \neq 0$ $(1 \le i \le n)$ in (7.8), we see that *each ideal boundary components of all $M(t)$, $t \in B$ is non-degenerating*. Hence, for any $t \in B$, we find a harmonic function $u(t, z)$ on $M(t)$ such that $\sigma_1(t, z) = du(t, z)$ and

$$(7.9) \qquad \lim_{z \to C_i^{\sim}(t)} u(t, z) = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } 2 \le i \le n \end{cases}$$

where $u(t^*, z) = U \circ \xi(z)$ in $M(t^*)$. Let t_1 be any point of B. Take a small disk B_1 of center t_1 and a holomorphic section $\alpha: t \to \alpha(t)$ of $\mathscr{M}_{B_1}$ defined on B_1. For simplicity we write $t_1 = 0$; $B_1 = B$ and $\mathscr{M}_{B_1} = \mathscr{M}$. We put $E_i = T^{-1}(B \times E_i) = \bigcup_{t \in B}(t, E_i(t))$, where E_i is defined in (N_2) in Notation 6.1. Besides the section α of $\mathscr{M}$ on B, we draw holomorphic sections α_i such that $\alpha_i(t) \in E_i(t)$ for all $t \in B$. For any fixed $t \in B$, we connects $\alpha(t)$ and $\alpha_i(t)$ by an arc $\ell_i(t)$ in $M(t)$ such that $\ell_i(t)$ varies continuously in $\mathscr{M}$ with $t \in B$. We consider the function on E_i:

$$(7.10) \qquad f_i(t, z) = \exp\left\{\frac{2\pi}{c_i}\left(\int_{\ell_i(t)} \Omega_1(t, z) + \int_{\alpha_i(t)}^{z} \Omega_1(t, z)\right)\right\}$$

where a path in the second integration lies in $E_i(t)$. We put $H(t) = u(t, \alpha(t))$. Then $|f_i(t, z)| = \exp\{(2\pi/c_i)(u(t, z) - H(t))\}$. We put

$$r_i(t) = \begin{cases} \exp\{(2\pi/c_1)(1 - H(t))\} & \text{for } i = 1 \\ \exp\{(-2\pi/c_i)H(t) & \text{for } 2 \le i \le n\,. \end{cases}$$

By (7.8), each $f_i(t, z)$ $(1 \le i \le n)$ is a single-valued holomorphic function for $(t, z) \in E_i$. By (7.9), $f_i(t, z)$, $t \in B$ is univalent in $E_i(t)$ (if necessary, take a smaller end $E_i(t)$ with ideal boundary component $C_i^{\sim}(t)$). Hence the mapping

$$T_i\colon (t, z) \in E_i \to (t, w) = (t, f_i(t, z)) \in B \times \boldsymbol{C}$$

is a holomorphic injection. We put $\mathscr{D}_i = T_i(E_i) = \bigcup_{t \in B}(t, D_i(t))$, where $D_i(t) = f_i(t, E_i(t))$. It follows that $D_i(t)$, $t \in B$ is a double connected region in $\boldsymbol{C}$ whose outer boundary component (which corresponds to $C_i^{\sim}(t)$) is given by the circle $|w| = r_i(t)$. It thus suffices for Lemma 7.2 to verify that $H(t)$ is a C^{ω} function on B.

In fact, by condition (a), $\mathscr{D}_i$ is pseudoconvex at all outer boundary points. By measuring the outer radius from the origin $w = 0$ of each $D_i(t)$, we see from Hartogs' theorem that all $r_i(t)$ $(1 \leq i \leq n)$ are logarithmic superharmonic function on B, so that $H(t)$ is harmonic, and hence C^ω on B. □

REMARK 7.1. Under the same conditions as in Lemma 7.2, we assume that $(g, n) = (0, 2)$. Then, for any $t_o \in B$, we find a disk B_o centered at t_o such that the subtriple $\mathfrak{M}_{B_o}$ of $\mathfrak{M}$ on B_o is holomorphically trivial: $\mathfrak{M}_{B_o} \simeq (B_o \times R_o, \pi_{B_o}, B_o)$ where $R_o = (\{1 < |w| < r_o\}$ and $r_o = e^{2\pi/k_1}$.

PROOF. We use the same notation as in the proof of Lemma 7.2. When $(g, n) = (0, 2)$, we have $\sigma_1(t, z) = -\sigma_2(t, z)$ on $M(t)$ and $C_1(t) = -C_2(t)$ for $t \in B$. Moreover, the function $w = f_1(t, z)$ of (7.10) for $i = 1$ conformally maps each $M(t)$, $t \in B$ onto the annulus

$$D(t) = \{w \in \boldsymbol{C} | e^{-(2\pi/c_1)H(t)} < |w| < e^{(2\pi/c_1)(1-H(t))}\} .$$

Since $H(t)$ is harmonic on B, we can find a holomorphic function $g(t)$ on B such that $\log|g(t)| = (2\pi/c_1)H(t)$ on B. Since $c_1 = k_1$, $\mathscr{M}$ is biholomorphic to $B \times R_o$ by the transformation $T_1^{\sim}: (t, z) \to (t, W) = (t, g(t)f_1(t, z))$. Remark 7.1 is valid. □

Now let $\mathfrak{M} = (\mathscr{M}, \pi, B)$ be a triple with C^ω smooth boundary. Then, $\mathfrak{M}$ is topologically trivial of finite type (g, n) with $n \geq 1$. We use Notation 6.1. We assume $\ell = 2g - n - 1 \geq 2$. Take ℓ independent cycles $\gamma_i (1 \leq i \leq \ell)$ in S. For $t \in B$, we have a cycle $\gamma_i(t)$ in $M(t)$ which varies continuously with $t \in B$ in $\mathscr{M}$. Assume that

$$\text{(7.11)} \qquad \Omega_{\gamma_1}(t, z) \text{ and } \Omega_{\gamma_2}(t, z) \text{ are holomorphic for } (t, z) \in \mathscr{M} .$$

Then the ratio $\psi(t, z) = \Omega_{\gamma_1}(t, z)/\Omega_{\gamma_2}(t, z)$ is a meromorphic function for $(t, z) \in \mathscr{M}$ such that $\psi(t, z)$ is non-constant on each $M(t)$, $t \in B$. We consider the mapping

$$\Psi: (t, z) \in \mathscr{M} \to (t, w) = (t, \psi(t, z)) \in B \times \boldsymbol{P}^1 ,$$

and put

$$\text{(7.12)} \qquad \Psi(\mathscr{M}) = \mathscr{D} = \bigcup_{t \in B}(t, D(t)) .$$

Then $\mathscr{D}$ is an (at most $2\ell - 2$) sheeted Riemann domain over $B \times \boldsymbol{P}^1$ with some branch surfaces $\mathscr{S}$ such that $(\mathscr{M}, \pi, B) \simeq (\mathscr{D}, \pi_B, B)$ by Ψ where $\pi_B^{-1}(t) = D(t)$. As in Lemma 7.1, we put $\mathscr{S} = \mathscr{S}' \cup \mathscr{S}''$ and $\mathscr{N} = \Psi^{-1}(\mathscr{S}'') \subset \mathscr{M}$.

Under these notations we shall prove

LEMMA 7.3. *Let $(\mathscr{M}, \pi, B)$ be a triple with C^ω smooth boundary, where B is a disk centered at* 0. *Assume that $\ell \geq 2$ and* (7.11). *We construct $\mathscr{D}$ by* (7.12). *Then* (1) *If $\mathscr{S}'' = \phi$, then $\mathscr{D} = B \times D(0)$;* (2) *If $\mathscr{S}'' \neq \phi$, then any $\Omega_{\gamma_j}(t, z)$*

holomorphic for $(t, z) \in \mathcal{M}$ *is zero on* $\mathcal{N}$; (3) *If all* $\Omega_{\gamma_j}(t, z)$ $(1 \le i \le k)$ *are holomorphic for* $(t, z) \in \mathcal{M}$, *then* $\mathcal{D} = B \times D(0)$.

PROOF. By (2) of Corollary 3.1, $\partial\mathcal{M}$ is Levi flat. By (1.3), we have $\operatorname{Im}\{\psi(t, z)\} = 0$ on $\partial\mathcal{M}$. Since $\partial\mathcal{M}$ is C^ω smooth, $\psi(t, z)$ is meromorphic for (t, z) beyond $\partial\mathcal{M}$. Levi's theorem implies that, for a given $Q \in \partial\mathcal{M}(0)$, we find a unique holomorphic section $\beta: t \in B \to \beta(t, Q) \in \partial\mathcal{M}$ such that $\beta(0, Q) = Q$. It follows that, for $t \in B$,

$$\partial\mathcal{M}(t) = \{\beta(t, Q) | Q \in \partial\mathcal{M}(0)\}\,; \qquad \partial D(t) = \{\psi(t, \beta(t, Q)) | Q \in \partial M(0)\}\,.$$

For any fixed $Q \in \partial M(0)$, the function $\psi(t, \beta(t, Q))$ is meromorphic for $t \in B$, so that it is a constant $\psi(0, Q)$ (because of $\operatorname{Im}\psi = 0$ on $\partial\mathcal{M}$). Hence $\partial D(t) = \partial D(0)$ for all $t \in B$, by which (1) of Lemma 7.3 follows. To prove (2), assume that $\mathcal{S}'' \neq \phi$ and $\Omega_{\gamma_j}(t, z)$ is holomorphic for $(t, z) \in \mathcal{M}$. Then the ratio $\psi_j(t, z) = \Omega_{\gamma_j}(t, z)/d\psi(t, z)$ is meromorphic for $(t, z) \in \mathcal{M}$ such that, for any fixed $t \in B$, $\psi_j(t, z)$ is non-constant on $M(t)$ and $\operatorname{Im}\{\psi_j(t, z)\} = 0$ on $\partial\mathcal{M}$. We construct the mapping $\Psi_j: (t, z) \in \mathcal{M} \to (t, w_j) = (t, \psi_j(t, z)) \in B \times \boldsymbol{P}^1$, and put $\Psi_j(\mathcal{M}) = \mathcal{D}_j = \bigcup_{t \in B}(t, D_j(t))$. Thus $\mathcal{D}_j$ is a Riemann domain over $B \times \boldsymbol{P}^1$ with branch surfaces $\mathcal{S}_j (= \mathcal{S}_j' \cup \mathcal{S}_j'')$. Now, take any component $\sigma \in \mathcal{S}''$ and let $\ell - 1$ (≥ 1) be the order of ramification of $\mathcal{D}$ along σ. We can apply Lemma 7.1 for $\mathcal{L} = \partial\mathcal{M}$; $f_1 = \psi$ and $f_2 = \psi_j$. We put $\tau = \psi^{-1}(\sigma) \subset \mathcal{N}$, $\sigma_j = \psi_j(\tau) \subset \mathcal{S}_j''$; $\tau: z = \beta(t)$ for $t \in B$, $\sigma: w = \eta(t)$ for $t \in B$, and $\sigma_j: w_j = \eta_j(t)$ for $t \in B$. By (1) of Lemma 7.1, ψ and ψ_j near $z = \beta(t)$ are of the form

$$w = \psi(t, z) = \eta(t) + a_\ell(t)(z - \beta(t))^\ell + a_{\ell+1}(t)(z - \beta(t))^{\ell+1} + \cdots\,;$$

$$w_j = \psi_j(t, z) = \eta_j(t) + b_\ell(t)(z - \beta(t))^\ell + b_{\ell+1}(t)(z - \beta(t))^{\ell+1} + \cdots\,,$$

where $a_\ell(t)$, $b_\ell(t) \neq 0$. It follows that

$$\begin{aligned}\Omega_j(t, z) &= \psi_j(t, z) d\psi(t, z)\\ &\equiv \{c_{\ell-1}(t)(z - \beta(t))^{\ell-1} + c_\ell(t)(z - \beta(t))^\ell + \cdots\} dz\,,\end{aligned}$$

where $c_{\ell-1}(t) = \ell\ \eta(t) a_\ell(t)$. Since $\ell \ge 2$, $\Omega_j(t, z)$ is zero on $\beta(t)$. We thus have (2) of Lemma 7.3. To prove (3), fix $t \in B$. Then, each $\Omega_i(t, z)$ $(1 \le i \le k)$ can be holomorphically extended to the double $M^*(t)$ of $M(t)$, which is a compact Riemann surface of genus k. Consequently, $\bigcap_{i=1}^{k} \{z \in M^*(t) | \Omega_i(t, z) = 0\} = \phi$. Hence, (3) follows by (1) and (2). □

8. Proof of (II) in the Introduction

We shall give the proof of statement (II) for the triple $\mathfrak{M}$ with *Cap* $K_{\mathfrak{M}} = 0$.

THEOREM 8.1. *Let $\mathfrak{M} = (\mathcal{M}, \pi, \boldsymbol{C})$ be a topologically trivial triple of finite type (g, n) except for $(g, n) = (0, 1)$. If $\mathfrak{M}$ is of locally Stein and Cap $K_{\mathfrak{M}} = 0$, then $\mathfrak{M}$ is holomorphically trivial.*

PROOF. Throughout the proof we use Notation 6.1 for our $\mathfrak{M}$ for $B = \boldsymbol{C}$. By *Cap* $K_{\mathfrak{M}} = 0$, we take a point $t^* \in \boldsymbol{C}$ which satisfies (7.7). The proof of Theorem 8.1 is divided into four short steps:

1st step. (1) For any nontrivial cycle γ in S (defined in (N_1)). $\lambda_\gamma(t)$ is constant on $\boldsymbol{C}$; (2) $\Omega_\gamma(t, z)$ is holomorphic for $(t, z) \in \mathcal{M}$ such that $\Omega_\gamma(t, z) \neq 0$ on each $M(t)$, $t \in \boldsymbol{C}$.

In fact, (1) of Corollary 5.1 implies that $1/\lambda_\gamma(t)$ (> 0) is a superharmonic function on $\boldsymbol{C}$ (which may be $\equiv +\infty$ on $\boldsymbol{C}$). Thus, $\lambda_\gamma(t)$ is a constant c_γ (≥ 0) on $\boldsymbol{C}$. It follows from (2) of Corollary 5.1 that $\Omega_\gamma(t, z)$ is holomorphic for $(t, z) \in \mathcal{M}$. Since $c_\gamma = \lambda_\gamma(t^*) > 0$, we see that $\Omega_\gamma(t, z) \neq 0$ on each $M(t)$, $t \in \boldsymbol{C}$.

2nd step. Theorem 8.1 is true in the case $(g, n) = (0, 2)$.

In fact, we can take $\gamma = C_1$ in the 1st step. Then, $\lambda_1(t)$ (defined in (N_3)) is a constant $k_1 > 0$, so that Remark 7.1 is applied to our triple $\mathfrak{M}$. The rest of the proof of the 2nd step is standard: We choose a family of disks $(B_i\}$ (i $= 1, 2, \ldots$) in $\boldsymbol{C}$ such that $\bigcup_{i=1}^{\infty} B_i = \boldsymbol{C}$ and $(\mathcal{M}_{B_i}, \pi, B_i) \simeq (B_i \times R_o, \pi_{B_i}, R_o)$ by a holomorphic T_i: $(t, z) \in \mathcal{M}_{B_i} \to (t, w_i) = (t, f_i(t, z)) \in B_i \times S$. Here R_o was defined in Remark 7.1 (independent of $i = 1, 2, \ldots$). Assume $B_i \cap B_j \neq \phi$ and fix $t \in B_i \cap B_j$. Then $w_j = f_j \circ f_i^{-1}(t, w_i) \equiv f_{ij}(t, w_i)$ gives a holomorphic automorphism of the annulus R_o. Since $f_{ij}(t, w_i)$ holomorphically depend on $t \in B_i \cap B_j$ and since $f_{ij}(t, C_1) \sim C_1$ in S, it follows that $w_j = f_{ij}(t, w_i) = e^{\sqrt{-1}\theta_{ij}} w_i$, where θ_{ij} is a real constant on $B_i \cap B_j$. Since $\theta_{ij} + \theta_{jk} + \theta_{ki} \equiv 0 \pmod{2\pi}$ on $B_i \cap B_j \cap B_k \neq \phi$, we find a real constant θ_i on B_i $(i = 1, 2, \ldots)$ such that $\theta_{ij} \equiv \theta_i - \theta_j \pmod{2\pi}$ on $B_i \cap B_j$. Then, the mapping

$$(t, z) \in \mathcal{M}_{B_i} \to (t, e^{\sqrt{-1}\theta_i} f_i(t, z)) \in B_i \times R_o$$

is a well-defined holomorphic transformation from $\mathcal{M}$ onto $\boldsymbol{C} \times R_o$.

From now on we may assume that $k = 2g + n - 1 \geq 2$. Our triple $\mathfrak{M}$ is not assumed to have a C^ω smooth boundary. However we make

3rd step. For any $t_o \in \boldsymbol{C}$, there exists a disk B_o centered at t_o such that the subtriple $\mathfrak{M}_{B_o}$ of $\mathfrak{M}$ on B_o is holomorphically equivalent to a triple $\mathfrak{M}' = (\mathcal{M}', \pi', B_o)$ with C^ω smooth boundary: $\mathfrak{M}_{B_o} \simeq \mathfrak{M}'$.

In fact, we first assume that $n \geq 2$. Then we can take $\gamma = C_1$ in the 1st step, so that $\lambda_1(t)$ is constant $k_1 > 0$. From Lemma 7.2 we obtain the 3rd step for $n \geq 2$. We next assume that $n = 1$. Since $g \geq 1$, we can construct a two-sheeted covering surface S_1 over S with neither relative boundary point nor branch point such that S_1 is of type $(g, 2)$. Since $\mathfrak{M}$ is of locally Stein and is topologically equivalent to $(\boldsymbol{C} \times S, \pi_{\boldsymbol{C}}, \boldsymbol{C})$, we have the triple $\mathfrak{M}_1 = (\mathcal{M}_1, \pi_1, \boldsymbol{C})$ where $\mathcal{M}_1$ is a double covering of $\mathcal{M}$ with neither branch surface nor relative boundary point such that $\mathfrak{M}_1$ is also of locally Stein and is topologically equivalent to $(\boldsymbol{C} \times S_1, \pi_{\boldsymbol{C}}, \boldsymbol{C})$. Since $n \geq 2$ for $\mathfrak{M}_1$, the 3rd step is true for $\mathfrak{M}_1$, and hence for $\mathfrak{M}$.

4^{th} step. Theorem 8.1 holds.

In fact, we have $\ell (\geq 2)$ independent cycles $\{\gamma_j\}$ on S. By the 1st step, we make ℓ holomorphic $\Omega_{\gamma_j}(t, z)$ in $\mathcal{M}$. Then $\psi(t, z) = \Omega_{\gamma_2}(t, z)/\Omega_{\gamma_1}(t, z)$ is a meromorphic function on $\mathcal{M}$. We consider the mapping $\Psi: (t, z) \in \mathcal{M} \rightarrow (t, w) = (t, \psi(t, z)) \in B \times \boldsymbol{P}^1$, and put $\Psi(\mathcal{M}) = \mathcal{D} = \bigcup_{t \in \boldsymbol{C}}(t, D(t))$ like (7.12). Hence $\mathcal{D}$ is a (at most $2\ell - 2$) sheeted Riemann domain over $\boldsymbol{C} \times \boldsymbol{P}^1$ such that $(\mathcal{M}, \pi, \boldsymbol{C}) \simeq (\mathcal{D}, \pi_{\boldsymbol{C}}, \boldsymbol{C})$ by Ψ, where $\pi_{\boldsymbol{C}}^{-1}(t) = D(t) = \psi(t, M(t))$ for $t \in C$. It is enough for the 4th step to prove $\mathcal{D} = \boldsymbol{C} \times D(0)$. By the 3rd step we find a family of disks B_j $(j = 1, 2, \cdots)$ of center t_j such that $\boldsymbol{C} = \bigcup_{j=1}^{\infty} B_j$ and $\mathfrak{M}_{B_j} \simeq \mathfrak{M}'_j = (\mathcal{M}'_j, \pi'_j, B_j)$, where $\mathfrak{M}'_j$ has a C^{ω} smooth boundary. Note that, for any fixed $t \in C$, $\Omega_i(t, z)$ $(1 \leq i \leq \ell)$ is invariant under the holomorphic mappings for z. Since all $\Omega_j(t, z)$ $(1 \leq j \leq \ell)$ are holomorphic for $(t, z) \in \mathcal{M}$, it follows from (3) of Lemma 7.3 that $\mathcal{D}_{B_j} = B_j \times D(t_j)$ for each j, where $\mathcal{D}_{B_j} = \pi_{\boldsymbol{C}}^{-1}(B_j)$. Consequently, $D(t_j) = D(0)$ for $j = 1, 2, \cdots$, so that $\mathcal{D} = \boldsymbol{C} \times D(0)$. □

Proof of (I) in Introduction. Since $\mathfrak{M}$ is topological trivial, we draw a canonical homology basis $\{A_i(t), B_i(t)\}_{i=1}^{g}$ of each compact Riemann surface $M(t)$ (of genus g independent of $t \in \boldsymbol{C}$), where $A_i(t)$ and $B_i(t)$ vary continuously in $\mathcal{M}$ with $t \in \boldsymbol{C}$. For any i $(1 \leq i \leq g)$, we have a unique analytic differential $\omega_i(t, \cdot)$ on $M(t)$ such that $\int_{A_j(t)} \omega_i(t, \cdot) = \delta_{ij}$ $(1 \leq j \leq g)$. If we put $b_{ij}(t) = \int_{B_j(t)} \omega_i(t, \cdot)$, then $\mathrm{Im}\,\{(b_{ij}(t)\}_{1 \leq i, j \leq g}$ is a positive definite matrix. Since $\mathfrak{M}$ is a triple, each $b_{ij}(t)$ is a holomorphic function on $\boldsymbol{C}$. Hence, $b_{ij}(t)$ must be a constant on $\boldsymbol{C}$. By Torelli's theorem each $M(t)$ is thus conformal equivalent to $M(0)$. Then Fischer-Grauert's theorem [5] (even in the case when $M(t)$ is higher dimensional) implies that the triple $\mathfrak{M}$ is locally holomorphically trivial. By the standard argument in the cohomology theory like the 2nd step in the proof of Theorem 8.1, we see that $\mathfrak{M}$ is holomorphically trivial. □

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