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On 4-dimensional closed manifolds with free fundamental groups

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Abstract. Let M be a 4-dimensional connected closed manifold whose fundamental group is a free group of rank m. We will show that the punctured manifold $M - pt$ has the homotopy type of a bouquet $\vee_m S^1 \vee_m S^3 \vee_n S^2$ of spheres for some n.

1. Introduction

Let M be a 4-dimensional connected closed manifold whose fundamental group is a free group $F_m = *_m \mathbf{Z}$ of rank m.

PROPOSITION 1. *$\#_\ell S^2 \times S^2 \# M$ is homeomorphic to $\#_m S^1 \times S^3 \# M_1$ or $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_1$ for some ℓ and some simply connected closed 4-dimensional manifold M_1 according as M is orientable or not. If M has a smooth structure, then the same statement holds for a diffeomorphism.*

With the help of an algebraic argument Proposition 1 would imply

PROPOSITION 2. *The punctured manifold $M - pt$ has the homotopy type of a bouquet $\vee_m S^1 \vee_m S^3 \vee_n S^2$ of spheres for some n.*

Here, we may conjecture that M has the homotopy type of $\#_m S^1 \times S^3 \# M_0$ or $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_0$ for some simply connected closed 4-dimensional manifold M_0 according as M is orientable or not. In the case that M is orientable and $m = 1$ this conjecture is true; in fact, Kawauchi [3] proved that M is homeomorphic to $S^1 \times S^3 \# M_0$.

As a corollary of Proposition 2 we have

PROPOSITION 3. *For a connected closed 4-dimensional manifold M the following statements are equivalent: (1) The Lusternik-Schnirelmann category of the punctured manifold $M - pt$* is one. *(2) The fundamental group $\pi_1(M)$ is a free group. (3) The punctured manifold $M - pt$ has the homotopy type of a bouquet of spheres.*

In fact, since $\pi_1(M) = \pi_1(M - pt)$, (1) implies (2) and follows from (3); (2) implies (3) by Proposition 2.

We may ask whether the conditions are equivalent also to the following statement: the Lusternik-Schnirelmann category of M is two. We refer the reader to [7] for a quick review of Lusternik-Schnirelmann category.

2. Proof of Proposition 1

By attaching higher dimensional cells to M we get an Eilenberg-Maclane space $K(F_m, 1)$. If we realize the generators $x_1,\dots,x_m$ of $\pi_1(M)$ by $\vee_{i=1}^m S_i^1$ in M, this is a deformation retract of $K(F_m, 1)$. So, the composed map $f: M \subset K(F_m, 1) \xrightarrow{r} \vee_{i=1}^m S_i^1$ is a retraction. Even if there is no smooth structure on M we have a smooth structure on $M - pt$ by [8] and we may assume that f is regular at m points $p_1,\dots,p_m$ one from each component S^1 of $\vee_{i=1}^m S_i^1$. We see that the submanifolds $f^{-1}(p_1),\dots,f^{-1}(p_m)$ are orientable because they can be considered codimension one bilateral submanifolds in the universal covering of M which is orientable.

Let N_i be the connected component of $f^{-1}(p_i)$ which contains $p_i \in \vee_{i=1}^m S_i^1 \subset M$. Then, $N_1,\dots,N_m$ are clearly dual to the generators of $\pi_1(M)$. By the same technique as was used in Matumoto [6] we can modify the submanifolds $N_1,\dots,N_m$ so that they are diffeomorphic to S^3 in the connected sum $\#_\ell S^2 \times S^2 \# M$ of M with ℓ copies of $S^2 \times S^2$ for some ℓ. In fact, first we take N_i and the spin cobordism connecting with S^3 which consist of only 1-handles and 2-handles; Second we embed each elementary cobordism of this cobordism in the surgered manifold of M surgered at the embedded circles parallel to the feet of the 2-handles; The framings should be compatible with the spin structure on the universal covering of M if it exists; Then, the surgered manifold is diffeomorphic to the connected sum $\#_\ell S^2 \times S^2 \# M$. Do surgery on this manifold at m numbers of S^3 and we get a simply connected manifold M_1. The backward surgery would give $\#_m S^1 \times S^3 \# M_1$ or $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_1$ according as M is orientable or not. We have to remark here that once $\# S^1 \tilde{\times} S^3$ occurs the other $\# S^1 \tilde{\times} S^3$ can be changed to $\# S^1 \times S^3$ without changing the homeomorphism (or diffeomorphism) type of the manifold. The detailed proof of Proposition 1 is given by Katanaga in her Master thesis [2].

3. Proof of Proposition 2

We start with lemmas:

LEMMA 3.1. *If the punctured manifolds $N_1 - pt$ and $N_2 - pt$ have the homotopy type of K_1 and K_2, then the puncture manifold $N_1 \# N_2 - pt$ has the homotopy type of $K_1 \vee K_2$.*

LEMMA 3.2. *If M_0 is a simply connected closed 4-manifold, then $M_0 - pt$ has the homotopy type of the bouquet $\vee_\ell S^2$ of 2-spheres.*

The proof of Lemma 3.1 is elementary and Lemma 3.2 is an easy exercise of homotopy theory (cf. [9]).

Now $S^2 \times S^2 - pt$ has the homotopy type of $S^2 \vee S^2$. Also $S^1 \times S^3 - pt$ as well as $S^1 \tilde{\times} S^3 - pt$ has the homotopy type of $S^1 \vee S^3$. So, by Lemmas 3.1 and 3.2 both of $\#_m S^1 \times S^3 \# M_1 - pt$ and $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_1 - pt$ have the homotopy type of a bouquet $\vee_m S^1 \vee_m S^3 \vee_{2\ell+n} S^2$ of spheres for some ℓ, where n is the second betti number of M.

We may assume that $M - pt$ has the homotopy type of a finite CW complex K [4, III, §5]. So, $\vee_{2\ell} S^2 \vee K$ and $\vee_m S^1 \vee_m S^3 \vee_{2\ell+n} S^2$ have the same homotopy type by Proposition 1. Then, the $\mathbf{Z}[F_m]$-module $H_i(\tilde{K}; \mathbf{Z})$ is a direct summand of a free module and hence a projective module, where $\tilde{K}$ denotes the universal covering of K. The following lemma implies that $H_i(\tilde{K}; \mathbf{Z}) = H_i(K; \mathbf{Z}[F_m])$ itself is a free module.

LEMMA 3.3 [1]. *Any finitely generated projective $\mathbf{Z}[F_m]$-module is a free $\mathbf{Z}[F_m]$-module.*

So, $H_2(\tilde{K}; \mathbf{Z})$ is a free $\mathbf{Z}[F_m]$-module of rank n. By Hurewicz theorem $\pi_2(K)$ is isomorphic to $H_2(\tilde{K}; \mathbf{Z})$ and we get a map $g\colon \vee_m S^1 \vee_n S^2 \to K$ which induces an isomorphism on π_1 and π_2.

Moreover by Hurewicz theorem ([5, Th. 7.1.6]) again the Hurewicz map h: $\pi_3(K) = \pi_3(\tilde{K}) \to H_3(\tilde{K}; \mathbf{Z})$ is a surjection, because $\tilde{K}$ is simply connected. Since $H_3(\tilde{K}; \mathbf{Z})$ is a free $\mathbf{Z}[F_m]$-module, we get a splitting $j\colon H_3(\tilde{K}; \mathbf{Z}) \to \pi_3(K)$ and get an extension $f\colon K_0 = \vee_m S^1 \vee_n S^2 \vee_m S^3 \to K$ of g such that $f_*\colon H_3(\tilde{K}_0; \mathbf{Z}) \to H_3(\tilde{K}; \mathbf{Z})$ is an isomorphism. We know that $H_i(\tilde{K}; \mathbf{Z}) = 0$ for $i \geq 4$ because K has the homotopy type of a connected punctured 4-dimensional manifold. So, $f_*\colon H_i(\tilde{K}_0; \mathbf{Z}) \to H_i(\tilde{K}; \mathbf{Z})$ are isomorphisms of zero modules for $i \geq 4$. Now by the theorem of J.H.C. Whitehead $f_*\colon \pi_i(K_0) \to \pi_i(K)$ are isomorphisms for any i and we see that f is a homotopy equivalence. This completes a proof of Proposition 2.

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