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An approach by difference to the porous medium equation with convection

Michiaki WATANABE

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Introduction

In this paper, we shall discuss the existence and then the uniqueness of the solution to the Cauchy problem for the porous medium equation with "convection" terms:

$$(1)\qquad (\partial/\partial t)u = \Delta\varphi(u) + \textstyle\sum_{i=1}^{N} F_i(u)_{x_i}, \qquad x \in \boldsymbol{R}^N,\ t > 0;$$

$$(2)\qquad u(0, x) = u_0(x), \qquad x \in \boldsymbol{R}^N.$$

Here, $(\cdot)_{x_i} = \partial/\partial x_i$ $(i=1,\cdots,N)$ and $\Delta = \sum_{i=1}^{N}(\partial/\partial x_i)^2$, and φ and $F_i (i=1,\cdots,N)$ are assumed to satisfy the conditions below:

(C1) The function φ is strictly increasing, locally Lipschitz continuous on R^1 and satisfies $\varphi(0) = 0$;

(C2) The functions $F_i, i = 1,\cdots,N$, are defined on $\boldsymbol{R}^1, F_i(0) = 0$, and $|F_i(r) - F_i(s)|/|\varphi(r) - \varphi(s)|$ are bounded for r, s in every bounded subinterval of $\boldsymbol{R}^1$.

First, we shall provide a direct method for solving the problem (1)–(2) via the method of difference approximation:

$$\begin{cases} h^{-1}(u(t+h, x) - u(t, x)) \\ = \sum_{i=1}^{N} k^{-2}(T_i(k) - 2I + T_i(-k))\varphi(u(t, x)) \\ \qquad + \sum_{i=1}^{N} (2k)^{-1}(T_i(k) - T_i(-k))F_i(u(t, x)), \\ T_i(k)u(x) = u(x + ke_i),\ e_i = (0,\cdots,0,\overset{i}{1},0,\cdots,0), \\ i = 1,\cdots,N. \end{cases}$$

We shall explain in Section 1 that this scheme itself converges as $h, k \downarrow 0$ and the limits give rise to a semigroup $\{S(t): t \geqq 0\}$ of contractions on $L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ associated with the problem (1)–(2).

We have once tried an approach similar to that described in the above, to a simpler equation without "convection" terms:

$$(1)_0 \qquad (\partial/\partial t)u = \Delta\varphi(u), \qquad x \in \boldsymbol{R}^N,\ t > 0$$

(see [8, 9]) to obtain a semigroup $\{S_0(t): t \geqq 0\}$ associated with the problem $(1)_0$–(2). We have then found that the function $u(t, x) = S_0(t)u_0(x)$ satisfies the equality:

$$(3)_0 \qquad u(t, \cdot) - u_0 = \Delta \int_0^t \varphi(u(r, \cdot))\,dr, \qquad t > 0$$

in $L^1(\boldsymbol{R}^N)$ (see [9, Lemma 3.3]). With the aid of this property we constructed Trotter's products of semigroups generated by such operators as $\Delta\varphi$ in $L^1(\boldsymbol{R}^N)$ which is not reflexive. In our approach to the problem (1)–(2) of this paper, there is difficulty caused by the presence of the "convection" terms. We are indeed forced to deal with the Laplacian Δ in some distribution sense. But we can show that the function $u(t, x) = S(t)u_0(x)$, constructed by means of the above semigroup associated with the problem (1)–(2), satisfies an equality of analogous type:

$$(3) \qquad \begin{aligned} &u(t, \cdot) - u_0 \\ &= \textstyle\sum_{i=1}^N (\partial/\partial x_i) \displaystyle\int_0^t \{\varphi(u(r, \cdot))_{x_i} + F_i(u(r, \cdot))\}\,dr, \qquad t > 0 \end{aligned}$$

in $\mathscr{D}'(\boldsymbol{R}^N)$ together with the smoothness $\varphi(u)_{x_i} \in L^2((0, t) \times \boldsymbol{R}^N)$ $(i = 1, \cdots, N)$ (see LEMMA 6).

Secondly, we shall use (3) as an equation to give a weak formulation of the Cauchy problem (1)–(2) which admits a unique solution in a certain sense. In Section 2, we show that $u(t, x) = S(t)u_0(x)$ is a unique solution of (3) under appropriate additional conditions, and moreover that it becomes a solution of (1)–(2) in $\mathscr{D}'((0, \infty) \times \boldsymbol{R}^N)$ (see LEMMA 7). Our method uses the inequality for pairs of solutions u and v of (3):

$$\begin{aligned} &\int_0^T \int_{\boldsymbol{R}^N} e^{-\varepsilon t}(u - v)(\varphi(u) - \varphi(v))\,dx\,dt \\ &\leqq (2\varepsilon)^{-1} \textstyle\sum_{i=1}^N \displaystyle\int_0^T \int_{\boldsymbol{R}^N} e^{-\varepsilon t}(F_i(u) - F_i(v))^2\,dx\,dt, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. This inequality together with the conditions (C1) and (C2) implies

$$\int_0^T \int_{\boldsymbol{R}^N} (u - v)(\varphi(u) - \varphi(v))\,dx\,dt \leqq 0.$$

From this the uniqueness of solution of (3) follows. In the case that "convection" terms are not contained in (3), this inequality always holds, and therefore the solution $u(t, x)$ of $(3)_0$ with $\varphi(u)_{x_i} \in L^2((0, t) \times \boldsymbol{R}^N)$, $i = 1, \cdots, N$, among others, coincides with $S_0(t)u_0(x)$ under condition (C1). It is of interest to compare this fact with the very general uniqueness theorem due to Brézis and Crandall [2, Theorem 1].

As is well-known, the Cauchy problem (1)–(2) arises in many physical and biological contexts (see e.g. [5] and its references). Our conditions (C1) and (C2) appear to be somewhat strong, but the method is simple and elementary. The present author desires that this research will contribute not only to the mathematical study but also to the numerical analysis for the problem (1)–(2). It should be noted that if φ does not degenerate, condition (C2) always holds true for locally Lipschitz continuous functions F_i, $i = 1, \cdots, N$ on R^1 with $F_i(0) = 0$.

Our method used for the problem (1)–(2) is applicable to the Cauchy problem for the equation containing "convection" and "absorption":

$$(\partial/\partial t)u = \Delta\varphi(u) + \sum_{i=1}^N F_i(u)_{x_i} - \psi(u), \qquad x \in \boldsymbol{R}^N,\ t > 0,$$

where ψ is a nondecreasing function. It is indeed an easy exercise to construct the associated semigroup in a similar way. But, it seems to be difficult to discuss analogously the uniqueness for the Cauchy problem unless some additional assumptions are made. We shall describe only the outline at the end of this paper (see REMARKS 1 and 2).

1. Construction of the associated semigroup

In this section we shall give a method for construction of the solution u of the problem (1)–(2) by the formula:

$$u(t, \cdot) = \lim_{h \downarrow 0} C_{h,m}^{[t/h]} u_0 \qquad \text{in } L^1(\boldsymbol{R}^N).$$

Here, $C_{h,m}$ is an operator in $L^1(\boldsymbol{R}^N)$ suggested by the above difference scheme and defined for each fixed $m > 0$ by

$$\begin{aligned} h^{-1}(C_{h,m} - I)u = {} & \sum_{i=1}^N k^{-2}(T_i(k) - 2I + T_i(-k))\varphi(u) \\ & + \sum_{i=1}^N (2k)^{-1}(T_i(k) - T_i(-k))F_i(u) \end{aligned}$$

under the condition:

(D) $$h > 0 \quad \text{and} \quad (2NM_m h)^{1/2} \leqq k \leqq 2/N_m$$

where

$$\begin{cases} M_m = \sup_{r,s\in I_m}(\varphi(r)-\varphi(s))/(r-s); \\ N_m = \max_{1\le i\le N}\sup_{r,s\in I_m}|F_i(r)-F_i(s)|/|\varphi(r)-\varphi(s)|; \\ I_m = [-m, m]. \end{cases}$$

The most important step of this section is to show the convergence in $L^1(\boldsymbol{R}^N)$ of

$$(I-\lambda h^{-1}(C_{h,m}-I))^{-1}u, \qquad \lambda>0 \quad \text{as} \quad h\downarrow 0.$$

Let us discuss the properties of $C_{h,m}$:

$$\begin{aligned} C_{h,m}u = u - 2Nhk^{-2}\varphi(u) + hk^{-2}\textstyle\sum_{i=1}^N T_i(k)(\varphi(u)+(k/2)F_i(u)) \\ + hk^{-2}\textstyle\sum_{i=1}^N T_i(-k)(\varphi(u)-(k/2)F_i(u)) \end{aligned} \tag{4}$$

as an operator on the set L_m of all $u\in L^1\cap L^\infty$ such that $\|u\|_\infty \leqq m$, where L^p denotes the real Banach space $(L^p(\boldsymbol{R}^N), \|\cdot\|_p)$ ($p=1$, 2 or ∞).

The following is our key lemma.

LEMMA 1. *Under the conditions* (C1) *and* (C2) *with* (D) *the functions*:

$$r\to r-2Nhk^{-2}\varphi(r);$$
$$r\to\varphi(r)\pm(k/2)F_i(r) \qquad (i=1,\cdots,N)$$

are nondecreasing on I_m, *and satisfy, for* $r, s\in I_m$,

$$\begin{aligned} |r-s| = |r-2Nhk^{-2}\varphi(r)-s+2Nhk^{-2}\varphi(s)| \\ + hk^{-2}\textstyle\sum_{i=1}^N|\varphi(r)+(k/2)F_i(r)-\varphi(s)-(k/2)F_i(s)| \\ + hk^{-2}\textstyle\sum_{i=1}^N|\varphi(r)-(k/2)F_i(r)-\varphi(s)+(k/2)F_i(s)|. \end{aligned}$$

PROOF. If $r\geqq s$ with $r, s\in I_m$, then

$$r-2Nhk^{-2}\varphi(r)-s+2Nhk^{-2}\varphi(s)\geqq(1-2NM_mhk^{-2})(r-s);$$

$$\begin{aligned} &\varphi(r)\pm(k/2)F_i(r)-\varphi(s)\mp(k/2)F_i(s) \\ &\quad\geqq\varphi(r)-\varphi(s)-(k/2)|F_i(r)-F_i(s)| \\ &\quad\geqq(1-N_mk/2)(\varphi(r)-\varphi(s)) \qquad (i=1,\cdots,N). \end{aligned}$$

Thus, we obtain the former, and hence the latter. Q.E.D.

In the following we always assume (C1) and (C2).

LEMMA 2. $C_h=C_{h,m}$ *maps* L_m *into itself and satisfies, for* $u, v\in L_m$,

(i) $\|C_hu-C_hv\|_1\leqq\|u-v\|_1$;

(ii) $\|C_hu\|_p\leqq\|u\|_p \qquad (p=1,\infty)$;

(iii) $(C_hu)_y=C_hu_y$ for $y\in R^N$ *where* $u_y(x)=u(x+y)$;

(iv) $$\int_{R^N} \operatorname{sgn}(u)\cdot h^{-1}(C_h - I)uf(x)\,dx$$
$$\leqq \|\varphi(u)\|_1 \|\Delta f\|_\infty + \textstyle\sum_{i=1}^N \|F_i(u)\|_1 \|f_{x_i}\|_\infty$$
for $f \in L^\infty$ *with* $f(x) \geqq 0$ *and* $f_{x_i}, f_{x_i x_i} \in L^\infty$ $(i = 1, \cdots, N)$;

(v) $$\int_{R^N} \{h^{-1}(C_h - I)uf + \textstyle\sum_{i=1}^N D_i(k)\varphi(u)D_i(k)f$$
$$+ \textstyle\sum_{i=1}^N F_i(u)2^{-1}(D_i(k) + D_i(-k))f\}\,dx = 0$$
for $f \in L^2 \cap L^\infty$, *where* $D_i(k) = (T_i(k) - I)/k$ $(i = 1, \cdots, N)$.

Proof. Using LEMMA 1, we obtain (i) and (ii):

$$\|C_h u - C_h v\|_1 \leqq \int_{R^N} \{|u - 2Nhk^{-2}\varphi(u) - v + 2Nhk^{-2}\varphi(v)|$$
$$+ hk^{-2}\textstyle\sum_{i=1}^N |\varphi(u) + (k/2)F_i(u) - \varphi(v) - (k/2)F_i(v)|$$
$$+ hk^{-2}\textstyle\sum_{i=1}^N |\varphi(u) - (k/2)F_i(u) - \varphi(v) + (k/2)F_i(v)|\}\,dx;$$

$$|C_h u| \leqq \|u\|_\infty - 2Nhk^{-2}\varphi(\|u\|_\infty)$$
$$+ hk^{-2}\textstyle\sum_{i=1}^N (\varphi(\|u\|_\infty) + (k/2)F_i(\|u\|_\infty))$$
$$+ hk^{-2}\textstyle\sum_{i=1}^N (\varphi(\|u\|_\infty) - (k/2)F_i(\|u\|_\infty)).$$

The proof of (iii) and (v) is easy and may be omitted. It remains to prove (iv). Noting the expression of $h^{-1}(C_h - I)$ with (4), we find

$$\operatorname{sgn}(u)\cdot h^{-1}(C_h - I)u \leqq -2Nk^{-2}|\varphi(u)|$$
$$+ k^{-2}\textstyle\sum_{i=1}^N T_i(k)|\varphi(u) + (k/2)F_i(u)|$$
$$+ k^{-2}\textstyle\sum_{i=1}^N T_i(-k)|\varphi(u) - (k/2)F_i(u)|.$$

Multiplication by $f(x)$ and integration over $\boldsymbol{R}^N$ gives

$$\int_{R^N} \operatorname{sgn}(u)\cdot h^{-1}(C_h - I)uf(x)\,dx \leqq -\int_{R^N} |\varphi(u)|\,2Nk^{-2}f(x)\,dx$$
$$+ \int_{R^N} \textstyle\sum_{i=1}^N |\varphi(u) + (k/2)F_i(u)|\,k^{-2}T_i(-k)f(x)\,dx$$
$$+ \int_{R^N} \textstyle\sum_{i=1}^N |\varphi(u) - (k/2)F_i(u)|\,k^{-2}T_i(k)f(x)\,dx.$$

Here, again using LEMMA 1, we obtain the inequality:

$$\int_{R^N} \operatorname{sgn}(u)\cdot h^{-1}(C_h - I)uf(x)\,dx$$
$$\leqq \int_{R^N} |\varphi(u)| \textstyle\sum_{i=1}^N k^{-2}(T_i(k) - 2I + T_i(-k))f(x)\,dx$$
$$- \int_{R^N} \operatorname{sgn}(u)\cdot \textstyle\sum_{i=1}^N F_i(u)(2k)^{-1}(T_i(k) - T_i(-k))f(x)\,dx,$$

implying (iv). Q.E.D.

REMARK. The last inequality in the above proof suggests that the inequality of Kato-type:

$$\operatorname{sgn}(u)\cdot(\Delta\varphi(u) + \textstyle\sum_{i=1}^N F_i(u)_{x_i})$$
$$\leqq \Delta|\varphi(u)| + \textstyle\sum_{i=1}^N (\operatorname{sgn}(u)\cdot F_i(u))_{x_i} \qquad \text{in } D'(\boldsymbol{R}^N)$$

holds true which is essential for our approach.

Using LEMMA 2, (i) and (ii), we obtain that for every $u \in L_m$ there exists a unique fixed point $J_\lambda^h u$ in L_m of the mapping:

$$v \to h(\lambda + h)^{-1}u + \lambda(\lambda + h)^{-1}C_h v$$

for each fixed $\lambda > 0$, which implies $J_\lambda^h = (I - \lambda h^{-1}(C_h - I))^{-1}$.

LEMMA 3. *J_λ^h maps L_m into itself and satisfies, for $u, v \in L_m$,*

(i) $\|J_\lambda^h u\|_p \leqq \|u\|_p \qquad (p = 1, \infty)$;

(ii) $\|J_\lambda^h u - J_\lambda^h v\|_1 \leqq \|u - v\|_1$;

(iii) $\|J_\lambda^h u - (J_\lambda^h u)_y\|_1 \leqq \|u - u_y\|_1 \qquad$ *for* $y \in \boldsymbol{R}^N$;

(iv) $\displaystyle\int_{|x|>2\rho} |J_\lambda^h u|\,dx \leqq \int_{|x|>\rho} |u|\,dx + \lambda\rho^{-1}C_m\|u\|_1$

for $\rho > 1$, where C_m is a positive constant independent of h, λ and ρ;

(v) $\displaystyle\int_{R^N} \{(J_\lambda^h u - u)f + \lambda\textstyle\sum_{i=1}^N D_i(k)\varphi(J_\lambda^h u)D_i(k)f$

$\qquad + \lambda\sum_{i=1}^N F_i(J_\lambda^h u)2^{-1}(D_i(k) + D_i(-k))f\}\,dx = 0$

for $f \in L^2 \cap L^\infty$;

(vi) $\displaystyle 2\int_{R^N} H(J_\lambda^h u)\,dx + \lambda\textstyle\sum_{i=1}^N \|D_i(k)\varphi(J_\lambda^h u)\|_2^2$

$$\leqq 2\int_{R^N} H(u)\,dx + \lambda \sum_{i=1}^N \|F_i(J_\lambda^h u)\|_2^2,$$

where $H(r) = \int_0^r \varphi(s)\,ds$ *for* $r \in \boldsymbol{R}^1$.

PROOF. The statements (i)–(iii) immediately follow from LEMMA 2, (i)–(iii).

Let us prove (iv). Replacing u in LEMMA 2, (iv) by $J_\lambda^h u$, we have

$$\int_{R^N} |J_\lambda^h u| f(x)\,dx \leqq \int_{R^N} |u| f(x)\,dx$$
$$+ \lambda \|\varphi(J_\lambda^h u)\|_1 \|\Delta f\|_\infty + \lambda \sum_{i=1}^N \|F_i(J_\lambda^h u)\|_1 \|f_{x_i}\|_\infty.$$

Here, let us choose a function $g \in C^2(\boldsymbol{R}^1)$ with values in [0, 1] which equals 0 on $(-\infty, 0)$ and 1 on $(1, \infty)$, and set $f(x) = g(|x|/\rho - 1)$ for $\rho > 0$. Then, we have for $x \in \boldsymbol{R}^N$

$$\begin{cases} |f(x)_{x_i}| \leqq \rho^{-1} \|g'\|_\infty & (i = 1, \cdots, N); \\ |\Delta f(x)| \leqq \rho^{-2} \|g''\|_\infty + (N-1)\rho^{-2} \|g'\|_\infty \end{cases}$$

and therefore

$$\int_{|x|>2\rho} |J_\lambda^h u|\,dx \leqq \int_{|x|>\rho} |u|\,dx$$
$$+ \lambda M_m \|u\|_1 \rho^{-2}(\|g''\|_\infty + (N-1)\|g'\|_\infty)$$
$$+ \lambda N N_m M_m \|u\|_1 \rho^{-1} \|g'\|_\infty.$$

We thus obtain (iv) with $C_m = NM_m(\|g'\|_\infty + \|g''\|_\infty) + NN_m M_m \|g'\|_\infty$.

Next, replacing u in LEMMA 2, (v) by $J_\lambda^h u$, we have (v). Finally, setting $f = \varphi(J_\lambda^h u)$ in (v) and using the inequality $H(v) - H(u) \leqq \varphi(v)(v-u)$, we obtain (vi). Q.E.D.

We are now in a position to discuss the convergence in L^1 of $J_\lambda^h u$ as $h \downarrow 0$. For the sake of simplicity we assume

(D′) $$h > 0 \quad \text{and} \quad k = (2NM_m h)^{1/2},$$

a special case of (D), in what follows.

PROPOSITION 4. *Let* $\lambda > 0$ *be fixed and* $u \in L_m$ *be given. Then, under* (C1) *and* (C2) *the following hold*:

(i) $(I - \lambda h^{-1}(C_{h,m} - I))^{-1} u \to w$ *in* L^1 *as* $h \downarrow 0$;

(ii) w *belongs to* L_m *with* $\|w\|_p \leqq \|u\|_p$ $(p = 1, \infty)$ *and* $\varphi(w)$ *to* H^1 *satisfying*

$$(5)\qquad \int_{R^N} \{(w-u)f + \lambda \sum_{i=1}^N (\varphi(w)_{x_i} + F_i(w))f_{x_i}\}\, dx = 0$$

for all $f \in H^1 \cap L^\infty$, *where* H^1 *denotes the usual Sobolev space* $H^1(\boldsymbol{R}^N)$.

PROOF. From LEMMA 3, (i), (iii) and (iv) we obtain by the Fréchet-Kolmogorov theorem that the set $\{J_\lambda^h u: h > 0\}$ is precompact in L^1 (see [9, Lemma 2.1]). Next, noting the inequalities $0 \leqq H(r) \leqq r\varphi(r)$ for $r \in \boldsymbol{R}^1$, we obtain

$$\sum_{i=1}^N \| D_i(k)\varphi(J_\lambda^h u)\|_2^2 \leqq (2/\lambda) M_m \|u\|_1 \|u\|_\infty + NM_m^2 N_m^2 \|u\|_1 \|u\|_\infty$$

from LEMMA 3, (vi) with (i). Therefore, for each $i = 1, \cdots, N$ the set $\{D_i(k)\varphi(J_\lambda^h u): h > 0\}$ is bounded in L^2.

Thus, for any sequence $\{h_n\}$ with $h_n \downarrow 0$, we can choose a subsequence $\{h_{n(j)}\}$ such that as $j \to \infty$ with $h = h_{n(j)}$ and $k = (2NM_m h_{n(j)})^{1/2}$

$$J_\lambda^h u \to w \quad \text{in} \quad L^1;$$

$$D_i(k)\varphi(J_\lambda^h u) \to v_i \quad \text{weakly in} \quad L^2 \ (i = 1, \cdots, N).$$

Here, LEMMA 3, (i) implies $\|w\|_p \leqq \|u\|_p$ $(p = 1, \infty)$ and

$$\varphi(J_\lambda^h u) \to \varphi(w) \quad \text{in} \quad L^2.$$

Therefore, $\varphi(w)$ belongs to $W^{1,2}(\boldsymbol{R}^N)$ which equals H^1, v_i for each $i = 1, \cdots, N$ coincides with $\varphi(w)_{x_i}$ and

$$D_i(k)\varphi(J_\lambda^h u) \to \varphi(w)_{x_i} \quad \text{weakly in} \quad L^2.$$

Going to the limit as $j \to \infty$ in LEMMA 3, (v) with $h = h_{n(j)}$ and $k = (2NM_m h_{n(j)})^{1/2}$ for $f \in H^1 \cap L^\infty$, we obtain (5).

Thus, we have proved (ii). We can finish the proof of (i) and this proposition by showing the uiqueness of the solution w of (5).

LEMMA 5. *Let* $\lambda > 0$ *be fixed and* $u_j \in L^1 \cap L^\infty$ $(j = 1, 2)$ *be given. Then, the corresponding solutions* w_j *of* (5) *with* $w_j \in L^1 \cap L^\infty$ *and* $\varphi(w_j) \in H^1$ $(j = 1, 2)$ *satisfy*

$$(6)\qquad \|w_1 - w_2\|_1 \leqq \|u_1 - u_2\|_1.$$

PROOF. Since $L^1 \cap L^\infty = \bigcup_{m>0} L_m$, we may assume that u_1 and u_2 belong to L_m for some $m > 0$. From (5) we obtain

$$\int_{R^N} (w_1 - w_2 - u_1 + u_2) f\, dx + \lambda \sum_{i=1}^N \int_{R^N} \{(\varphi(w_1) - \varphi(w_2))_{x_i} + F_i(w_1) - F_i(w_2)\} f_{x_i}\, dx = 0.$$

Here, taking a function $p \in C(\boldsymbol{R}^1)$ with values -1 for $r < -1$, r for $-1 \leqq r \leqq 1$ and 1 for $r > 1$, and setting $f(x) = p(n(\varphi(w_1) - \varphi(w_2)))$ for a positive integer n, we have

$$\int_{\boldsymbol{R}^N} (w_1 - w_2) p(n(\varphi(w_1) - \varphi(w_2)))\,dx - \|u_1 - u_2\|_1$$
$$\leqq (n\lambda/2) \textstyle\sum_{i=1}^N \displaystyle\int_{E_n} (F_i(w_1) - F_i(w_2))^2\,dx,$$

where $E_n = \{x \in \boldsymbol{R}^N : |\varphi(w_1) - \varphi(w_2)| \leqq 1/n\}$. In fact, we have

$$\int_{\boldsymbol{R}^N} (w_1 - w_2 - u_1 + u_2) p(n(\varphi(w_1) - \varphi(w_2)))\,dx$$
$$= -\lambda \textstyle\sum_{i=1}^N \displaystyle\int_{\boldsymbol{R}^N} \{(\varphi(w_1) - \varphi(x_2))_{x_i} + F_i(w_1) - F_i(w_2)\} \cdot p(n(\varphi(w_1) - \varphi(w_2)))_{x_i}\,dx.$$

Since $|p(r)| \leqq 1$ on $\boldsymbol{R}^1$, the left-hand side is bounded below by

$$\int_{\boldsymbol{R}^N} (w_1 - w_2) p(n(\varphi(w_1) - \varphi(w_2)))\,dx - \|u_1 - u_2\|_1.$$

On the other hand, since $p'(r)$ equals 1 for $-1 \leqq r \leqq 1$ and 0 for $r < -1$ or $r > 1$, the right-hand side is written as

$$-n\lambda \int_{\boldsymbol{R}^N} p'(n(\varphi(w_1) - \varphi(w_2))) |\nabla\varphi(w_1) - \nabla\varphi(w_2)|^2\,dx$$
$$-n\lambda \textstyle\sum_{i=1}^N \displaystyle\int_{\boldsymbol{R}^N} (F_i(w_1) - F_i(w_2)) p'(n(\varphi(w_1) - \varphi(w_2))) \cdot (\varphi(w_1) - \varphi(w_2))_{x_i}\,dx$$
$$= -n\lambda \int_{E_n} |\nabla\varphi(w_1) - \nabla\varphi(w_2)|^2\,dx$$
$$-n\lambda \textstyle\sum_{i=1}^N \displaystyle\int_{E_n} (F_i(w_1) - F_i(w_2))(\varphi(w_1) - \varphi(w_2))_{x_i}\,dx,$$

and this is bounded above by

$$-n\lambda \int_{E_n} |\nabla\varphi(w_1) - \nabla\varphi(w_2)|^2\,dx$$
$$+ n\lambda \textstyle\sum_{i=1}^N \displaystyle\int_{E_n} 2^{-1}\{(F_i(w_1) - F_i(w_2))^2 + (\varphi(w_1)_{x_i} - \varphi(w_2)_{x_i})^2\}\,dx$$
$$= -(n\lambda/2) \int_{E_n} |\nabla\varphi(w_1) - \nabla\varphi(w_2)|^2\,dx$$

$$+ (n\lambda/2)\sum_{i=1}^{N}\int_{E_n}(F_i(w_1) - F_i(w_2))^2\,dx.$$

Thus the desired estimate is obtained.

The right-hand side of it is, by (C2), dominated by

$$(N_m\lambda/2)\sum_{i=1}^{N}\int_{E_n}|F_i(w_1) - F_i(w_2)|\,dx,$$

and therefore vanishes as $n \to \infty$. Consequently we obtain

$$\int_{R^N}\operatorname{sgn}(\varphi(w_1) - \varphi(w_2))\cdot(w_1 - w_2)\,dx \leqq \|u_1 - u_2\|_1,$$

and hence, by (C1), the inequality (6). Q.E.D.

This lemma suggests that there is an acretive operator $-A$ in L^1 such that the solution w of (5) equals $(I - \lambda A)^{-1}u$ with $A = \Delta\varphi + \sum_{i=1}^{N}F_i(\cdot)_{x_i}$ in the following sense.

DEFINITION. We say that u belongs to $D(A)$ and $Au = v \in L^1$ if

$$\begin{cases} u \text{ belongs to } L^1 \cap L^\infty \text{ and } \varphi(u) \text{ to } H^1; \\ \displaystyle\int_{R^N}\{vf + \textstyle\sum_{i=1}^{N}(\varphi(u)_{x_i} + F_i(u))f_{x_i}\}\,dx = 0 \\ \qquad \text{for all } f \in H^1 \cap L^\infty. \end{cases}$$

PROPOSITION 4 implies the range condition:

$$R(I - \lambda A) \supset L^1 \cap L^\infty \qquad \text{for } \lambda > 0,$$

and (6) means that $-A$ is accretive in L^1. Thus, we have that the solution w of (5) coincides with $(I - \lambda A)^{-1}u$.

THEOREM I. *Under the conditions* (C1) *and* (C2), *the operator* $J_\lambda = (I - \lambda A)^{-1}$ *for* $\lambda > 0$ *has the following properties*:

(i) J_λ *maps* $L^1 \cap L^\infty$ *into itself satisfying*

$$\|J_\lambda u\|_p \leqq \|u\|_p \ (p = 1, \infty) \text{ for } u \in L^1 \cap L^\infty;$$

(ii) $\varphi(J_\lambda u)$ *belongs to* H^1 *for* $u \in L^1 \cap L^\infty$ *satisfying*

$$2\int_{R^N}H(J_\lambda u)\,dx + \lambda\sum_{i=1}^{N}\|(\partial/\partial x_i)\varphi(J_\lambda u)\|_2^2$$
$$\leqq 2\int_{R^N}H(u)\,dx + \lambda\sum_{i=1}^{N}\|F_i(J_\lambda u)\|_2^2;$$

(iii) *For* $u \in L_m$, *as* $h \downarrow 0$ *with* $k = (2NM_m h)^{1/2}$

$$(I - \lambda h^{-1}(C_{h,m} - I))^{-1}u \to J_\lambda u \qquad in\ L^1. \tag{7}$$

PROOF. The theorem is a simple consequence of PROPOSITION 4. The assertion (ii) is derived from (5) with $f = \varphi(w)$ and $w = J_\lambda u$. Q.E.D.

REMARK. The operator A is shown to be densely defined in L^1. The proof is carried out by showing that

$$J_\lambda u \to u \qquad \text{in } L^1 \text{ as } \lambda \downarrow 0$$

for $u \in L^1 \cap L^\infty$, which belongs to L_m for some $m > 0$. The convergence (7) implies that LEMMA 3, (i), (iii) and (iv) remain true with $J_\lambda^h u$ replaced by $J_\lambda u$. Again using the Fréchet-Kolmogorov theorem, we obtain that the set $\{J_\lambda u: 0 < \lambda \leqq 1\}$ is precompact in L^1. Therefore, for any sequence $\{\lambda_n\}$ with $\lambda_n \downarrow 0$ we can choose a subsequence $\{\lambda_{n(j)}\}$ such that as $j \to \infty$ with $\lambda = \lambda_{n(j)}$, $J_\lambda u$ converges in L^1 to some $v \in L^1 \cap L^\infty$. But, from (5) we obtain

$$\int_{R^N} \{(J_\lambda u - u)f - \lambda\varphi(J_\lambda u)\Delta f + \lambda \textstyle\sum_{i=1}^N F_i(J_\lambda u) f_{x_i}\}\, dx = 0$$

$$\text{for all } f \in C_0^\infty(\boldsymbol{R}^N).$$

Here, going to the limit as $j \to \infty$ with $\lambda = \lambda_{n(j)}$, we have

$$\int_{R^N} (v - u) f(x)\, dx = 0,$$

which implies $v = u$. Q.E.D.

We are now ready to construct the associated semigroup $\{S(t): t \geqq 0\}$ in L^1 in terms of A with the aid of the theory of nonlinear semigroup generation (see [4]):

$$(I - \lambda A)^{-[t/\lambda]} u \to S(t)u \qquad \text{in } L^1 \text{ as } \lambda \downarrow 0 \tag{8}$$

for $u \in L^1 \cap L^\infty$. To conclude this section we shall describe below the results obtained from THEOREM I: The properties of $\{S(t): t \geqq 0\}$; the approximation of $S(t)$ by means of $C_{h,m}$.

THEOREM II. *Under the conditions* (C1) *and* (C2), *the semigroup* $\{S(t): t \geqq 0\}$ *consisting of contractions on* $L^1 \cap L^\infty$ *has the following properties*:

(i) *For every* $u \in L^1 \cap L^\infty$, $S(\cdot)u$ *belongs to* $C([0, \infty); L^1)$ *with* $\|S(t)u\|_p \leqq \|u\|_p$ $(p = 1, \infty)$ for $t > 0$;

(ii) *For every* $t > 0$ *and* $u \in L^1 \cap L^\infty$, $(\partial/\partial x_i)\varphi(S(\cdot)u)$ *belongs to* $L^2((0, t) \times \boldsymbol{R}^N)$ $(i = 1, \cdots, N)$ *satisfying*

$$2\int_{R^N} H(S(t)u)\,dx + \sum_{i=1}^N \int_0^t \|(\partial/\partial x_i)\varphi(S(r)u)\|_2^2\,dr$$
$$\leqq 2\int_{R^N} H(u)\,dx + \sum_{i=1}^N \int_0^t \|F_i(S(r)u)\|_2^2\,dr;$$

(iii) *For* $u \in L_m$, *as* $h \downarrow 0$ *with* $k = (2NM_m h)^{1/2}$

$$C_{h,m}^{[t/h]}u \;\rightarrow\; S(t)u \quad \text{in } L^1 \tag{9}$$

uniformly on every bounded subinterval of $[0, \infty)$.

PROOF. THEOREM I, (i) with (8) gives (i). Next, by Brézis-Pazy's theorem on nonlinear Chernoff's formula [3, Theorem 3.2], the convergence (7) implies (9). We have therefore only to prove (ii). To this end we shall use Pazy's theory on Lyapunov function for the accretive operator $-A$ (see [7], and also [10]).

Replacing u in THEOREM I, (ii) by $J_\lambda^{j-1}u$ and summing up for $j = 1, \cdots, n$, we have

$$2\int_{R^N} H(J_\lambda^n u)\,dx + \lambda\sum_{j=1}^n \sum_{i=1}^N \|(\partial/\partial x_i)\varphi(J_\lambda^j u)\|_2^2$$
$$\leqq 2\int_{R^N} H(u)\,dx + \lambda\sum_{j=1}^n \sum_{i=1}^N \|F_i(J_\lambda^j u)\|_2^2.$$

Setting here $u_n(r) = J_\lambda^j u$ for $\lambda(j-1) < r \leqq \lambda j$ with $\lambda = t/n$, we obtain

$$2\int_{R^N} H(u_n(t))\,dx + \sum_{i=1}^N \int_0^t \|(\partial/\partial x_i)\varphi(u_n(r))\|_2^2\,dr$$
$$\leqq 2\int_{R^N} H(u)\,dx + \sum_{i=1}^N \int_0^t \|F_i(u_n(r))\|_2^2\,dr.$$

Let us note that $G: L^2 \rightarrow [0, \infty]$ defined by

$$G(u) = \begin{cases} \sum_{i=1}^N \|(\partial/\partial x_i)u\|_2^2 & \text{for } u \in H^1 \\ +\infty & \text{otherwise,} \end{cases}$$

is lower semicontinuous and, by (i), that $u_n(t) \rightarrow S(t)u$, $\varphi(u_n(t)) \rightarrow \varphi(S(t)u)$ and $F_i(u_n(t)) \rightarrow F_i(S(t)u)$ $(i = 1, \cdots, N)$ in L^2 as $n \rightarrow \infty$. Thus, letting $n \rightarrow \infty$ in the above and using Fatou's lemma, we obtain the desired inequality. Q.E.D.

REMARK. The product formula (9) provides a simple proof of the comparison principle: If for $u, v \in L^1 \cap L^\infty$

$$u(x) \leqq v(x), \text{ then } S(t)u(x) \leqq S(t)v(x)$$

for every $t>0$ and a.a. $x \in \boldsymbol{R}^N$.

Indeed, from (4) we see $C_h u(x) \leqq C_h v(x)$ a.e. on $\boldsymbol{R}^N$ and hence for every $t>0$ and a.a. $x \in \boldsymbol{R}^N$

$$(C_{t/n})^n u(x) \leqq (C_{t/n})^n v(x), \qquad n = 1, 2, \cdots$$

since u and v belong to L_m for some $m>0$. Thus, (9) gives the desired inequality.

This principle can also be derived from the inequality

$$\|(S(t)u - S(t)v)^+\|_1 \leqq \|(u-v)^+\|_1 \qquad \text{for } t>0,$$

where $r^+ = \max(0, r)$. Indeed, dealing with $h^{-1}(C_h - I)$ as in the proof of LEMMA 2, (iv), we obtain

$$\int_{R^N} \operatorname{sgn}^+(u-v)\cdot\{h^{-1}(C_h - I)u - h^{-1}(C_h - I)v\}\,dx \leqq 0,$$

where $\operatorname{sgn}^+ r$ equals 1 for $r>0$ and 0 for $r \leqq 0$. Here, replacing u and v by $J_\lambda^h u$ and $J_\lambda^h v$, respectively, we have

$$\|(J_\lambda^h u - J_\lambda^h v)^+\|_1 \leqq \|(u-v)^+\|_1 \qquad \text{for } \lambda>0,$$

which together with (7) implies

$$\|(J_\lambda u - J_\lambda v)^+\|_1 \leqq \|(u-v)^+\|_1 \qquad \text{for } \lambda>0.$$

Thus, the formula (8) gives the desired inequality.

2. Uniqueness of solutions of the Cauchy problem

In this section, we shall show that the function $u(t, x) = S(t)u_0(x)$ constructed by means of the associated semigroup, becomes a unique solution of our Cauchy problem (1)–(2) in the sense mentioned in the introduction. We still assume the conditions (C1) and (C2) throughout.

To begin with we shall describe a precise statement of (3).

LEMMA 6. *For* $u_0 \in L^1 \cap L^\infty$, $u(t, x) = S(t)u_0(x)$ *satisfies the equality*:

$$\int_{R^N} (u(t,x) - u_0(x)) f(x)\,dx + \sum_{i=1}^N \int_{R^N} \int_0^t \{\varphi(u(r,x))_{x_i} + F_i(u(r,x))\}\,dr\, f(x)_{x_i}\,dx = 0 \tag{10}$$

for all $f \in H^1 \cap L^\infty$ *and* $t>0$.

PROOF. From (5) we see

$$\int_{R^N}(J_\lambda u-u)f(x)\,dx-\lambda\int_{R^N}\{\varphi(J_\lambda u)\Delta f(x)-\textstyle\sum_{i=1}^N F_i(J_\lambda u)f(x)_{x_i}\}\,dx=0$$

for all $f\in C_0^\infty(R^N)$.

Here, replacing u by $J_\lambda^{j-1}u_0$ and summing for $j=1,\cdots,n$, we have

$$\int_{R^N}(J_\lambda^n u_0-u_0)f(x)\,dx-\lambda\textstyle\sum_{j=1}^n\displaystyle\int_{R^N}\{\varphi(J_\lambda^j u_0)\Delta f(x)-\textstyle\sum_{i=1}^N F_i(J_\lambda^j u_0)f(x)_{x_i}\}\,dx=0$$

and therefore

$$\int_{R^N}(u_n(t,x)-u_0(x))f(x)\,dx-\int_{R^N}\int_0^t\{\varphi(u_n(r,x))\Delta f(x)-\textstyle\sum_{i=1}^N F_i(u_n(r,x))f(x)_{x_i}\}\,dr\,dx=0,$$

where $u_n(r,x)=J_\lambda^j u_0(x)$ for $\lambda(j-1)<r\leqq\lambda j$ with $\lambda=t/n$. Going to the limit as $n\to\infty$, w obtain

$$\int_{R^N}(u(t,x)-u_0(x))f(x)\,dx-\int_{R^N}\int_0^t\{\varphi(u(r,x))\Delta f(x)-\textstyle\sum_{i=1}^N F_i(u(r,x))f(x)_{x_i}\}\,dr\,dx=0.$$

Thus, the smoothness $\varphi(u)_{x_i}\in L^2((0,t)\times\boldsymbol{R}^N)$ $(i=1,\cdots,N)$ which is a consequence of THEOREM II, (ii), gives the desired equality (10). Q.E.D.

The following lemma shows that the solution $u(t,x)=S(t)u_0(x)$ of the equation (10) becomes a solution of the problem (1)–(2) in $\mathscr{D}'((0,\infty)\times\boldsymbol{R}^N)$, as mentioned in the introduction.

LEMMA 7. *For $u_0\in L^1\cap L^\infty$, $u(t,x)=S(t)u_0(x)$ satisfies the equality:*

$$\int_{R^N}u(T,x)f(T,x)\,dx-\int_{R^N}u_0(x)f(0,x)\,dx$$

$$= \int_0^T \int_{R^N} \{u f_t - \sum_{i=1}^N (\varphi(u)_{x_i} + F_i(u)) f_{x_i}\} \, dx \, dt$$

for all $T > 0$ *and* $f \in H^1(Q_T) \cap L^\infty(Q_T)$, $Q_T = (0, T) \times \boldsymbol{R}^N$.

PROOF. Replacement by $f(t, x)_t = (\partial/\partial t) f(t, x)$ of $f(x)$ in (10) and integration over $(0, T)$ with respect to t gives the equality. Q.E.D.

The following proposition suggests that the solution of the equation (10) is uniquely determined by u_0.

PROPOSITION 8. *Let* (C1) *and* (C2) *hold. Let* u *and* v *satisfy, for* $T > 0$,

$$u, v \in L^1(Q_T) \cap L^\infty(Q_T),$$
$$\varphi(u)_{x_i}, \varphi(v)_{x_i} \in L^2(Q_T) \qquad (i = 1, \cdots, N)$$

and

$$\int_{R^N} \left\{ u(t, x) f + \sum_{i=1}^N \int_0^t \{\varphi(u(r, x))_{x_i} + F_i(u(r, x))\} \, dr \, f_{x_i} \right\} dx$$
$$= \int_{R^N} \left\{ v(t, x) f + \sum_{i=1}^N \int_0^t \{\varphi(v(r, x))_{x_i} + F_i(v(r, x))\} \, dr \, f_{x_i} \right\} dx$$

for all $f \in H^1 \cap L^\infty$ *and* $t \in (0, T)$.

Then, the following hold:

(i) $$\int_0^T \int_{R^N} e^{-\varepsilon t} (u - v)(\varphi(u) - \varphi(v)) \, dx \, dt$$
$$\leqq (2\varepsilon)^{-1} \sum_{i=1}^N \int_0^T \int_{R^N} e^{-\varepsilon t} (F_i(u) - F_i(v))^2 \, dx \, dt$$

for an arbitrary $\varepsilon > 0$;

(ii) $u(t, x) = v(t, x)$ *a.e. on* Q_T.

PROOF. Let us set for the sake of simplicity

$$z(t, x) = u(t, x) - v(t, x),$$
$$h(t, x) = \varphi(u(t, x)) - \varphi(v(t, x)),$$
$$h_i(t, x) = h(t, x)_{x_i}$$

and

$$g_i(t, x) = F_i(u(t, x)) - F_i(v(t, x)) \qquad (i = 1, \cdots, N).$$

Then, the given equality becomes

$$\int_{R^N} z(t, x) f\, dx + \sum_{i=1}^N \int_{R^N} \int_0^t (h_i(r, x) + g_i(r, x))\, dr\, f_{x_i}\, dx = 0.$$

Replacing $f(x)$ by $h(t, x)$, we have

$$\int_{R^N} z(t, x) h(t, x)\, dx$$
$$+ \sum_{i=1}^N \int_{R^N} \int_0^t (h_i(r, x) + g_i(r, x))\, dr\, (h_i(t, x) + g_i(t, x))\, dx$$
$$= \sum_{i=1}^N \int_{R^N} \int_0^t (h_i(r, x) + g_i(r, x))\, dr\, g_i(t, x)\, dx.$$

Therefore, we obtain for a.a. $t \in (0, T)$

$$\int_{R^N} z(t, x) h(t, x)\, dx + 2^{-1}(d/dt) K(t)$$
$$\leqq K(t)^{1/2} \left\{ \sum_{i=1}^N \int_{R^N} g_i(t, x)^2\, dx \right\}^{1/2},$$

where

$$K(t) = \sum_{i=1}^N \int_{R^N} \left\{ \int_0^t (h_i(r, x) + g_i(r, x))\, dr \right\}^2 dx.$$

The right-hand side is written as

$$(\varepsilon K(t))^{1/2} \left\{ \varepsilon^{-1} \sum_{i=1}^N \int_{R^N} g_i(t, x)^2\, dx \right\}^{1/2},$$

which is not larger than

$$(\varepsilon/2) K(t) + (2\varepsilon)^{-1} \sum_{i=1}^N \int_{R^N} g_i(t, x)^2\, dx,$$

and hence this inequality becomes

$$\int_{R^N} z(t, x) h(t, x)\, dx + ((d/dt) K(t) - \varepsilon K(t))/2$$
$$\leqq (2\varepsilon)^{-1} \sum_{i=1}^N \int_{R^N} g_i(t, x)^2\, dx.$$

Multiplication of both sides by $e^{-\varepsilon t}$ therefore gives

$$e^{-\varepsilon t} \int_{R^N} z(t, x) h(t, x)\, dx + 2^{-1}(d/dt)(e^{-\varepsilon t} K(t))$$

$$\leqq (2\varepsilon)^{-1} e^{-\varepsilon t} \sum_{i=1}^{N} \int_{R^N} g_i(t, x)^2\, dx.$$

Thus, integration of both sides of this inequality over $(0, T)$ with respect to t gives

$$\int_0^T \int_{R^N} e^{-\varepsilon t} z(t, x) h(t, x)\, dx\, dt + 2^{-1} e^{-\varepsilon T} K(T)$$

$$\leqq (2\varepsilon)^{-1} \sum_{i=1}^{N} \int_0^T \int_{R^N} e^{-\varepsilon t} g_i(t, x)^2\, dx\, dt,$$

which implies (i).

Let us next prove (ii). Assuming $|u|, |v| \leqq m$ a.e. on Q_T with some $m > 0$ and recalling the definition of the constants M_m and N_m, we find

$$\sum_{i=1}^{N} g_i(t, x)^2 \leqq N N_m^2 M_m z(t, x) h(t, x)$$

and hence, by (i),

$$(1 - N N_m^2 M_m/(2\varepsilon)) \int_0^T \int_{R^N} e^{-\varepsilon t} z(t, x) h(t, x)\, dx\, dt \leqq 0.$$

Choosing $\varepsilon = N N_m^2 M_m$, we obtain

$$\int_0^T \int_{R^N} e^{-N N_m^2 M_m t} z(t, x) h(t, x)\, dx\, dt \leqq 0,$$

and therefore

$$\int_0^T \int_{R^N} z(t, x) h(t, x)\, dx\, dt \leqq 0.$$

This inequality with (C1) implies $z(t, x)h(t, x) = 0$ and hence the desired result $u = v$ a.e. on Q_T. Q.E.D.

REMARK. In the case that "convection" terms are not contained, the given equality with $f(x)$ replaced by $h(t, x)$ becomes

$$\int_{R^N} z(t, x) h(t, x)\, dx\, dt + 2^{-1} (d/dt) \sum_{i=1}^{N} \int_{R^N} \left\{ \int_0^t h_i(r, x)\, dr \right\}^2 dx = 0$$

for a.a. $t \in (0, T)$, and therefore

$$\int_0^T \int_{R^N} z(t, x) h(t, x)\, dx\, dt + 2^{-1} \sum_{i=1}^{N} \int_{R^N} \left\{ \int_0^T h_i(r, x)\, dr \right\}^2 dx = 0.$$

From this the uniqueness of solution of $(3)_0$ immediately follows under (C1), as mentioned in the introduction.

Thus, we can conclude that the function $u(t, x) = S(t)u_0(x)$ for $u_0 \in L^1 \cap L^\infty$, constructed by the formula (9), becomes a unique solution of the equation (10) which implies our Cauchy problem (1)–(2) in $\mathscr{D}'((0, \infty) \times \boldsymbol{R}^N)$.

THEOREM III. *Let* (C1) *and* (C2) *hold. Then, for given* $u_0 \in L^1 \cap L^\infty$, *there exists exactly one solution* $u(t, x)$ *of the equation* (10) (*the precise statement of* (3)) *satisfying*

$$\begin{cases} u \in C([0, \infty); L^1) \cap L^\infty((0, \infty) \times \boldsymbol{R}^N); \\ \varphi(u)_{x_i} \in L^2((0, t) \times \boldsymbol{R}^N) \text{ for every } t > 0, \\ i = 1, \cdots, N. \end{cases}$$

PROOF. Let u be any solution of (10) in the above sense, and set $v(t, x) = S(t)u_0(x)$. Then, for any $T > 0$, u and v belong to $L^1(Q_T) \cap L^\infty(Q_T)$, and $\varphi(u)_{x_i}$ and $\varphi(v)_{x_i}$ to $L^2(Q_T)$ $(i = 1, \cdots, N)$. Using PROPOSITION 8, we obtain $u(t, x) = S(t)u_0(x)$ a.e. on Q_T and hence for $t > 0$ and a.a. $x \in \boldsymbol{R}^N$. Q.E.D.

REMARK 1. Our method for generation and construction of the semigroup associated with (1)–(2) is still available to the problem for the equation containing "absorption":

$$(\partial/\partial t)u = \Delta\varphi(u) + \textstyle\sum_{i=1}^N F_i(u)_{x_i} - \psi(u), \quad x \in \boldsymbol{R}^N, \ t > 0 \tag{11}$$

under the condition:

(C3) The function ψ is nondecreasing, locally Lipschitz continuous on $\boldsymbol{R}^1$ and satisfies $\psi(0) = 0$,

in addition to (C1) and (C2). In this case, we have only to deal with the operator $D_{h,m}$ defined on L_m by

$$\begin{aligned} D_{h,m}u = u &- 2Nhk^{-2}\varphi(u) - h\psi(u) \\ &+ hk^{-2}\textstyle\sum_{i=1}^N T_i(k)(\varphi(u) + (k/2)F_i(u)) \\ &+ hk^{-2}\textstyle\sum_{i=1}^N T_i(-k)(\varphi(u) - (k/2)F_i(u)) \end{aligned}$$

under the condition instead of (D):

$$0 < h < 1/H_m \quad \text{and} \quad (2NM_m h/(1 - H_m h))^{1/2} \leqq k \leqq 2/N_m,$$

where

$$H_m = \sup_{r, s \in I_m} (\psi(r) - \psi(s))/(r - s).$$

Noting $D_{h,m} = C_{h,m} - h\psi$ and $h^{-1}(D_{h,m} - I) = h^{-1}(C_{h,m} - I) - \psi$, we obtain quite easily an analogue of PROPOSITION 4 as follows.

PROPOSITION 9. *Under the conditions* (C1)-(C3), *the following hold for* $\lambda > 0$ *and* $u \in L_m$:

(i) $(I - \lambda h^{-1}(D_{h,m} - I))^{-1}u \to w$ *in* L^1 *as* $h, k \downarrow 0$;

(ii) w *belongs to* L_m *with* $\|w\|_p \leqq \|u\|_p$ $(p = 1, \infty)$ *and* $\varphi(w)$ *to* H^1 *satisfying*

$$\int_{R^N} \{(w - u)f + \lambda \textstyle\sum_{i=1}^N (\varphi(w)_{x_i} + F_i(w))f_{x_i} + \lambda\psi(w)f\}\,dx = 0$$

for all $f \in H^1 \cap L^\infty$.

Thus, we obtain as before the operator $(I - \lambda A_\psi)^{-1}$ such that $w = (I - \lambda A_\psi)^{-1}u$, and the associated semigroup $\{S_\psi(t) : t \geqq 0\}$ with generator $A_\psi = \Delta\varphi + \sum_{i=1}^N F_i(\cdot)_{x_i} - \psi$:

$$u \in D(A_\psi) \quad \text{and} \quad A_\psi u = v \in L^1$$

if

$$\begin{cases} u \text{ belongs to } L^1 \cap L^\infty \text{ and } \varphi(u) \text{ to } H^1; \\ \displaystyle\int_{R^N} \{vf + \textstyle\sum_{i=1}^N (\varphi(u)_{x_i} + F_i(u))f_{x_i} + \psi(u)f\}\,dx = 0 \\ \qquad \text{for all } f \in H^1 \cap L^\infty. \end{cases}$$

Moreover, we find that for $u_0 \in L^1 \cap L^\infty$, the function $u(t, x) = S_\psi(t)u_0(x)$ is a solution of the equation

$$\begin{aligned} &\int_{R^N} (u(t, x) - u_0(x))f\,dx + \int_{R^N}\int_0^t \psi(u(r, x))\,dr\,f\,dx \\ (12) \qquad &+ \textstyle\sum_{i=1}^N \displaystyle\int_{R^N}\int_0^t \{\varphi(u(r, x))_{x_i} + F_i(u(r, x))\}\,dr\,f_{x_i}\,dx = 0 \\ &\qquad \text{for all } f \in H^1 \cap L^\infty \text{ and } t > 0, \end{aligned}$$

satisfying

$$\begin{cases} u \in C([0, \infty); L^1) \cap L^\infty((0, \infty) \times \boldsymbol{R}^N); \\ \varphi(u)_{x_i} \in L^2((0, t) \times \boldsymbol{R}^N) \text{ for every } t > 0, \\ i = 1, \cdots, N. \end{cases}$$

REMARK 2. Uniqueness for the Cauchy problem (11)–(2) can also be discussed through the equation (12), however, under the condition instead of (C3):

(C4) The function ψ is nondecreasing on $\boldsymbol{R}^1$, satisfies $\psi(0)=0$, and

$$(\psi(r)-\psi(s))^2/((r-s)(\varphi(r)-\varphi(s)))$$

is bounded for r, s in every bounded subinterval of $\boldsymbol{R}^1$.

Dealing with the equality (12) by a quite similar method, we obtain an analogue of PROPOSITION 8 as follows.

PROPOSITION 10. *Let* (C1), (C2) *and* (C4) *hold. Let* u *and* v *satisfy, for* $T>0$,

$$u,\ v \in L^1(Q_T) \cap L^\infty(Q_T);$$

$$\varphi(u)_{x_i},\ \varphi(v)_{x_i} \in L^2(Q_T) \qquad (i=1,\cdots,N)$$

and

$$\int_{\boldsymbol{R}^N}\left\{u(t,x)+\int_0^t \psi(u(r,x))\,dr\right\}f(x)\,dx$$

$$+\sum_{i=1}^N \int_{\boldsymbol{R}^N}\int_0^t \{\varphi(u(r,x))_{x_i}+F_i(u(r,x))\}\,dr\,f_{x_i}\,dx$$

$$=\int_{\boldsymbol{R}^N}\left\{v(r,x)+\int_0^t \psi(v(r,x))\,dr\right\}f(x)\,dx$$

$$+\sum_{i=1}^N \int_{\boldsymbol{R}^N}\int_0^t \{\varphi(v(r,x))_{x_i}+F_i(v(r,x))\}\,dr\,f_{x_i}\,dx$$

for all $f \in H^1 \cap L^\infty$ *and* $t \in (0, T)$.

Then, the following hold:

(i)
$$\int_0^T\int_{\boldsymbol{R}^N} e^{-\varepsilon t}(u-v)(\varphi(u)-\varphi(v))\,dx\,dt$$

$$\leqq (2\varepsilon)^{-1}\sum_{i=1}^N \int_0^T\int_{\boldsymbol{R}^N} e^{-\varepsilon t}(F_i(u)-F_i(v))^2\,dx\,dt$$

$$+(2\varepsilon)^{-1}\int_0^T\int_{\boldsymbol{R}^N} e^{-\varepsilon t}\{(\psi(u)-\psi(v))^2+(\varphi(u)-\varphi(v))^2\}\,dx\,dt$$

for an arbitrary $\varepsilon>0$;

(ii)
$$u(t,x)=v(t,x) \qquad \text{a.e. } on\ Q_T.$$

PROOF. Recalling the proof of PROPOSITION 8, we have

$$\int_{R^N} z(t, x)h(t, x)\,dx + 2^{-1}(d/dt)K(t) + 2^{-1}(d/dt)Q(t)$$

$$\leqq K(t)^{1/2}\left\{\sum_{i=1}^{N}\int_{R^N} g_i(t, x)^2\,dx\right\}^{1/2}$$

$$+ Q(t)^{1/2}\left\{\int_{R^N} (p(t, x) - h(t, x))^2\,dx\right\}^{1/2},$$

where

$$p(t, x) = \psi(u(t, x)) - \psi(v(t, x)),$$

$$Q(t) = \int_{R^N}\left\{\int_0^t p(r, x)\,dr\right\}^2 dx.$$

Thus, an elementary calculus gives (i), which together with (C4) gives (ii). Q.E.D.

The condition (C4) trivially holds if ψ satisfies (C3) and if $(\psi(r) - \psi(s))/(\varphi(r) - \varphi(s))$ is bounded for r, s in every bounded subinterval of $\boldsymbol{R}^1$: $\varphi(r) = r^m$ and $\psi(r) = r^p$ with $p \geqq 1$ and $p \geqq m$, for example. The condition (C4) is also satisfied by $\varphi(r) = r^m$ and $\psi(r) = r^p$ with $p < m \leqq 2p - 1$. In fact, $(\psi(r) - \psi(s))^2/((r - s)(\varphi(r) - \varphi(s)))$ converges as $r \to s$ to $(\psi'(s))^2/\varphi'(s)$, which is equal to $(p^2/m)s^{2p-1-m}$ and hence bounded for s in every bounded subinterval of $\boldsymbol{R}^1$ if $m \leqq 2p - 1$.

Note added in proof

Recently we became aware that all the results of this paper can be derived through a slight modification of the operator $C_{h,m}$ even if condition (C2) is replaced by a weaker condition as below:

($\bar{\mathrm{C}}$2) The functions F_i, $i = 1, \cdots, N$, are defined on R^1, $F_i(0) = 0$, and

$$(F_i(r) - F_i(s))^2/|(r - s)(\varphi(r) - \varphi(s))|$$

are bounded for r, s in every bounded subinterval of R^1.

As long as condition (C1) is supposed to hold, (C2) implies ($\bar{\mathrm{C}}$2). Moreover, it should be noted that this condition ($\bar{\mathrm{C}}$2) enables us to deal with the equation

$$(\partial/\partial t)u = \Delta(u^m) + \sum_{i=1}^{N}(\partial/\partial x_i)(u^{n_i})$$

in the cases where $n_i \geq m$ and $m > n_i \geq (m+1)/2$, $i = 1, \cdots, N$, for example. The previous condition (C2) fails to include the latter case. This is easily seen from the fact that the last statement of ($\bar{C}$2) (resp. (C2)) is satisfied by differentiable functions φ and F_i, $i = 1, \cdots, N$, if and only if

$$(F_i'(r))^2/|\varphi'(r)| \text{ (resp. } |F_i'(r)/\varphi'(r)|)$$

are bounded for r in every bounded subinterval of R^1.

We have already shown in effect that both of the uniqueness results for the equations (5) and (10) are obtained under ($\bar{C}$2) with (C1). Indeed, the inequality used in the proof of LEMMA 5:

$$\int_{R^N} \operatorname{sgn}(\varphi(w_1) - \varphi(w_2)) \cdot (w_1 - w_2)\, dx - \|u_1 - u_2\|_1$$

$$\leq (\lambda/2) \sum_{i=1}^{N} \int_{\varphi(w_1) = \varphi(w_2)} (F_i(w_1) - F_i(w_2)^2/|\varphi(w_1 - \varphi(w_2)|\, dx$$

and the estimate exhibited in PROPOSITION 8, (i):

$$\int_0^T \int_{R^N} e^{-\varepsilon t}(u - v)(\varphi(u) - \varphi(v))\, dx\, dt$$

$$\leq (2\varepsilon)^{-1} \sum_{i=1}^{N} \int_0^T \int_{R^N} e^{-\varepsilon t}(F_i(u) - F_i(v))^2\, dx\, dt$$

indicate that condition ($\bar{C}$2) is more natural than condition (C2). To improve this paper, therefore, we have only to show that all the existence results given in Section 1 still hold true with $C_{h,m}$ replaced by a slightly modified operator $\bar{C}_{h,m}$:

$$(\bar{4}) \qquad \begin{aligned} \bar{C}_{h,m} u = u &+ h\textstyle\sum_{i=1}^{N} k^{-2}(T_i(k) - 2I + T_i(-k))(\varphi(u) + ku) \\ &+ h\textstyle\sum_{i=1}^{N} (2k)^{-1}(T_i(k) - T_i(-k))F_i(u), \end{aligned}$$

where

$$(\bar{\mathrm{D}}) \qquad h > 0 \text{ and } Nh + (N^2h^2 + 2NM_m h)^{1/2} \leq k \leq 16/\bar{N}_m^2,$$

and $\bar{N}_m$ denotes

$$\max_{1 \leq i \leq N} \sup_{r, s \in I_m} |F_i(r) - F_i(s)|/((r - s)(\varphi(r) - \varphi(s)))^{1/2}.$$

As is easily seen, our operator $\bar{C}_{h,m}$ is defined by the replacement of $\varphi(r)$ in $C_{h,m}$ with $\varphi_k(r) = \varphi(r) + kr$. Let us show that LEMMA 1 is true for this φ_k. If $r \geq s$ with $r, s \in I_m$, then, thanks to ($\bar{\mathrm{D}}$),

$$r - 2Nhk^{-2}\varphi_k(r) - s + 2Nhk^{-2}\varphi_k(s)$$

$$\geq (1 - 2Nhk^{-1} - 2NM_m hk^{-2})(r - s) \geq 0;$$

$$\varphi_k(r) \pm (k/2)F_i(r) - \varphi_k(s) \mp (k/2)F_i(s)$$
$$= \varphi(r) - \varphi(s) + k(r - s) \pm (k/2)(F_i(r) - F_i(s))$$
$$\geq 2(k(r - s)(\varphi(r) - \varphi(s)))^{1/2} - (k/2)|F_i(r) - F_i(s)|$$
$$\geq (2k^{1/2} - \bar{N}_m k/2)((r - s)(\varphi(r) - \varphi(s)))^{1/2} \geq 0.$$

This implies that $r \to r - 2Nhk^{-2}\varphi_k(r)$ and $r \to \varphi_k(r) \pm (k/2)F_i(r)$ $(i = 1, \cdots, N)$ are nondecreasing on I_m, and that the proof is obtained as before. Consequently, LEMMAS 2 and 3 hold for φ_k. Therefore PROPOSITION 4 is also obtained for $\bar{C}_{h,m}$ since LEMMA 5 holds true under ($\bar{\text{C}}$2) with (C1), as verified above. Thus, THEOREMS I and II with $C_{h,m}$ replaced by $\bar{C}_{h,m}$ are established under the conditions (C1) and ($\bar{\text{C}}$2).

Finally, LEMMAS 6 and 7, PROPOSITION 8, and THEOREM III given in Section 2 are all valid under the weaker condition ($\bar{\text{C}}$2) with (C1).

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Faculty of Engineering
Niigata University
Niigata 950–21, *Japan*