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A note on the existence of nonoscillatory solutions of neutral differential equations

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1. Introduction and the statement of results

In this note we consider the neutral differential equation

$$\frac{d^n}{dt^n}[y(t) - p(t)y(t-\tau)] + f(t, y(\sigma(t))) = 0, \tag{1.1}$$

where $n \geq 1$ and the following conditions are assumed:

(a) $p \in C[a, \infty)$, $p(t) \geq 0$ for $t \geq a > 0$ and τ is a positive constant;

(b) $f \in C([a, \infty) \times R)$, and

$$|f(t, u)| \leq F(t, |u|), \qquad (t, u) \in [a, \infty) \times R,$$

for some continuous function $F(t, u)$ on $[a, \infty) \times [0, \infty)$ which is nondecreasing in u for each fixed $t \geq a$;

(c) $\sigma \in C[a, \infty)$, $\lim_{t\to\infty} \sigma(t) = \infty$.

By a solution of (1.1) we mean a function $y \in C[T_y, \infty)$ for some $T_y \geq a$ such that $y(t) - p(t)y(t-\tau)$ is n-times continuously defferentiable on $[T_y, \infty)$ and that (1.1) is satisfied for $t \geq T_y$. A solution of (1.1) is called nonoscillatory if it is eventually positive or eventually negative.

Recently there has been a lot of study concerning the existence of nonoscillatory solutions of neutral differential equations. For the case where $p(t)$ is a constant coefficient we refer to [2, 4, 5, 9–12, 16, 18, 20, 21]. For the case where $p(t)$ is a variable coefficient, we refer to [1, 3, 6–8, 13–15, 17, 19, 22]. Most of the existence results obtained so far, however, are established by imposing restrictive conditions on the variable coefficient $p(t)$ in (1.1) such as

$$0 \leq p(t) \leq p_0 < 1 \qquad \text{for } t \geq a, \text{ where } p_0 \text{ is a constant.} \tag{1.2}$$

In this note we investigate the existence and asymptotic behavior of nonoscillatory solutions of (1.1) with the variable coefficient $p(t)$ satisfying

$$0 \leq p_0 \leq p(t) \leq p_1 \qquad \text{for } t \geq a, \text{ where } p_0 \text{ and } p_1 \text{ are constants.} \tag{1.3}$$

Our result is the following:

THEOREM. *Assume that* (1.3) *is satisfied. Let* k *be an integer with* $0 \le k \le n-1$. *Define* $u(t; k, p_i)$, $i = 0, 1$, *by*

$$u(t; k, p_i) = \begin{cases} t^k & \text{if } 0 \le p_i < 1, \\ t^{k+1} & \text{if } \quad p_i = 1, \\ p_i^{t/\tau} & \text{if } \quad p_i > 1. \end{cases} \tag{1.4}$$

If

$$\int^{\infty} s^{n-k-1} F(s, cu(\sigma(s); k, p_1))\, ds < \infty \tag{1.5}$$

for some $c > 0$, *then* (1.1) *has an eventually positive solution* y *satisfying*

$$\liminf_{t \to \infty} \frac{y(t)}{u(t; k, p_0)} > 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{y(t)}{u(t; k, p_1)} < \infty. \tag{1.6}$$

Consider the case where the coefficient $p(t)$ satisfies (1.2). We obtain the following corollary.

COROLLARY 1. *Assume that* (1.2) *is satisfied. Let* k *be an integer with* $0 \le k \le n-1$. *If*

$$\int^{\infty} s^{n-k-1} F(s, c[\sigma(s)]^k) ds < \infty$$

for some $c > 0$, *then* (1.1) *has an eventually positive solution* y *satisfying*

$$0 < \liminf_{t \to \infty} \frac{y(t)}{t^k} \le \limsup_{t \to \infty} \frac{y(t)}{t^k} < \infty.$$

REMARK 1. Similar results are obtained in [6, Theorem 3.1], [7, Theorem 1], [15, Theorems 4.1 and 4.2], [18, Theorems 2 and 4] and [20, Theorem 2].

Next we consider the case $p(t) \equiv p_0$, where p_0 is a positive constant, that is, we consider the following neutral differential equation

$$\frac{d^n}{dt^n}[y(t) - p_0 y(t-\tau)] + f(t, y(\sigma(t))) = 0. \tag{1.7}$$

We have the following corollary.

COROLLARY 2: *Consider equation* (1.7). *Let* k *be an integer with* $0 \le k \le n-1$. *Define* $u(t; k, p_0)$ *by* (1.4). *If*

$$\int^{\infty} s^{n-k-1} F(s, cu(\sigma(s); k, p_0)) ds < \infty$$

for some $c > 0$, *then* (1.7) *has an eventually positive solution* y *satisfying*

$$0 < \liminf_{t\to\infty} \frac{y(t)}{u(t; k, p_0)} \le \limsup_{t\to\infty} \frac{y(t)}{u(t; k, p_0)} < \infty.$$

REMARK 2. A similar result is obtained in [17, Corollary 2.2]. Recently, Kitamura and Kusano [10] and Kitamure, Kusano and Lalli [11, 12] have obtained some sufficient conditions for equation (1.7) to have nonoscillatory solutions satisfying more precise asymptotic behavior.

2. Proof of Theorem

The proof of Theorem is based on the following lemma which can be easily derived through a slight modification of Theorem 2.1 of Naito [17].

LEMMA 1. *Let* k *be an integer with* $0 \le k \le n-1$ *and let* u_k *be a positive continuous function satisfying*

$$(2.1)\qquad 0 < \liminf_{t\to\infty} \frac{u_k(t) - p(t)u_k(t-\tau)}{t^k} \le \limsup_{t\to\infty} \frac{u_k(t) - p(t)u_k(t-r)}{t^k} < \infty.$$

If

$$(2.2)\qquad \int^{\infty} s^{n-k-1} F(s, cu_k(\sigma(s)))\,ds < \infty$$

for some $c > 0$, *then* (1.1) *has an eventually positive solution* y *satisfying*

$$(2.3)\qquad 0 < \liminf_{t\to\infty} \frac{y(t)}{u_k(t)} \le \limsup_{t\to\infty} \frac{y(t)}{u_k(t)} < \infty.$$

We notice that there always exists a positive continuous function u_k satisfying (2.1). (See [17, Remark 2.1].)

For the proof of Theorem we investigate the asymptotic properties of the function u_k in the statement of Lemma 1 under condition (1.3). We have the following lemma.

LEMMA 2. *Assume that* $p(t)$ *satisfies* (1.3). *Let* k *be an integer with* $0 \le k \le n-1$, *and define* $u(t; k, p_i)$, $i = 0, 1$, *by* (1.4). *Let* u_k *be a positive continuous function satisfying* (2.1). *Then we have*

$$(2.4)\qquad \liminf_{t\to\infty} \frac{u_k(t)}{u(t; k, p_0)} > 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{u_k(t)}{u(t; k, p_1)} < \infty.$$

Before we prove Lemma 2, we prove Theorem.

PROOF OF THEOREM. Let u_k be a positive continuous function satisfying (2.1). From Lemma 2, we have (2.4). Then, the integral condition (1.5) implies (2.2) for some $c > 0$. From Lemma 1, there exists a positive solution y of (1.1) satisfying (2.3). By virtue of (2.4), we obtain (1.6). This completes the proof of Theorem.

To prove Lemma 2, we prepare the following lemma.

LEMMA 3. *Assume that* $u, v \in C[a - \tau, \infty)$ *satisfy*

$$\begin{cases} u(t) - p(t)u(t-\tau) \geq v(t) - p(t)v(t-\tau), & t \geq a, \\ u(t) \geq v(t), & a - \tau \leq t \leq a. \end{cases}$$

Then

$$u(t) \geq v(t), \qquad t \geq a - \tau.$$

PROOF. Let $w(t) \equiv u(t) - v(t)$ for $t \geq a - \tau$. We have

$$\begin{cases} w(t) - p(t)w(t-\tau) \geq 0, & t \geq a, \\ w(t) \geq 0, & a - \tau \leq t \leq a. \end{cases}$$

Define $\{T_i\}_{i=0}^{\infty}$ by $T_i = a + (i-1)\tau$ for $i = 0, 1, 2, \ldots$. Then we see that $[a - \tau, \infty) = \bigcup_{i=0}^{\infty} [T_i, T_{i+1})$. By the assumption, if $t \in [T_0, T_1) = [a - \tau, a)$, then $w(t) \geq 0$. If $w(t) \geq 0$ is true on the interval $[T_i, T_{i+1})$ for some $i = 0, 1, 2, \ldots$, then

$$w(t) \geq p(t)w(t-\tau) \geq 0$$

on the next interval $[T_{i+1}, T_{i+2})$. By induction on i, we conclude that $w(t) \geq 0$ for $t \geq a - \tau$, which implies $u(t) \geq v(t)$ for $t \geq a - \tau$. This completes the proof.

PROOF OF LEMMA 2. Define $v_{i(k)}$, $i = 0, 1$, by

$$v_{i(k)}(t) = \begin{cases} t^k & \text{if } 0 \leq p_i < 1, \\ t^{k+1} & \text{if } \quad p_i = 1, \\ p_i^{t/\tau} - t^k & \text{if } \quad p_i > 1. \end{cases}$$

We observe that, for $i = 0, 1$,

$$\lim_{t \to \infty} \frac{v_{i(k)}(t) - p_i v_{i(k)}(t-\tau)}{t^k} = \begin{cases} 1 - p_i > 0 & \text{if } 0 \leq p_i < 1, \\ (k+1)\tau > 0 & \text{if } \quad p_i = 1, \\ p_i - 1 > 0 & \text{if } \quad p_i > 1. \end{cases}$$

Then we can choose a sufficiently large $T \geq a$, a sufficiently small $c_* > 0$ and a sufficiently large $c^* > 0$ such that

$$c_* \frac{v_{0(k)}(t) - p_0 v_{0(k)}(t-\tau)}{t^k} \le \frac{u_k(t) - p(t)u_k(t-\tau)}{t^k}, \qquad t \ge T,$$

$$\frac{u_k(t) - p(t)u_k(t-\tau)}{t^k} \le c^* \frac{v_{1(k)}(t) - p_1 v_{1(k)}(t-\tau)}{t^k}, \qquad t \ge T,$$

and

$$c_* v_{0(k)}(t) \le u_k(t) \le c^* v_{1(k)}(t), \qquad T - \tau \le t \le T.$$

It follows that

$$c_* v_{0(k)}(t) - p_0 c_* v_{0(k)}(t-\tau) \le u_k(t) - p(t)u_k(t-\tau), \qquad t \ge T, \tag{2.5}$$

and

$$u_k(t) - p(t)u_k(t-\tau) \le c^* v_{1(k)}(t) - p_1 c^* v_{1(k)}(t-\tau), \qquad t \ge T. \tag{2.6}$$

Because u_k is positive, we obtain

$$u_k(t) - p_1 u_k(t-\tau) \le u_k(t) - p(t)u_k(t-\tau) \le u_k(t) - p_0 u_k(t-\tau), \qquad t \ge T. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have

$$c_* v_{0(k)}(t) - p_0 c_* v_{0(k)}(t-\tau) \le u_k(t) - p_0 u_k(t-\tau), \qquad t \ge T,$$

and

$$u_k(t) - p_1 u_k(t-\tau) \le c^* v_{1(k)}(t) - p_1 c^* v_{1(k)}(t-\tau), \qquad t \ge T.$$

Applying Lemma 3 with u and v replaced by u_k and $c_* v_{0(k)}$, respectively, we obtain $c_* v_{0(k)}(t) \le u_k(t)$ for $t \ge T - \tau$. In a similar fashion, we get $u_k(t) \le c^* v_{1(k)}(t)$ for $t \ge T - \tau$. Therefore we have

$$\liminf_{t\to\infty} \frac{u_k(t)}{v_{0(k)}(t)} > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{u_k(t)}{v_{1(k)}(t)} < \infty.$$

Since

$$\lim_{t\to\infty} \frac{u(t;\, k,\, p_i)}{v_{i(k)}(t)} = 1,$$

we conclude that (2.4) holds. This completes the proof.

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