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The explicit expression of the Harish–Chandra C-function of $SU(n, 1)$ associated to the fundamental representations of K

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ABSTRACT. The Harish–Chandra C-function for $SU(n, 1)$ is explicitly computed in the case of the fundamental representation. As an application, by using the asymptotic expansion of the Eisenstein integral, the conditions for the square-integrability of the Eisenstein integral are given.

1. Introduction

Let G be a semisimple Lie group with finite center, K a maximal compact subgroup of G. Let θ be the Cartan involution of G fixing K. Let $G = KAN$ be an Iwasawa decomposition of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ the corresponding decomposition of Lie algebra $\mathfrak{g}$ of G. Then each element g of G can be uniquely written as $g = \kappa(g) \exp H(g)n(g)$ $(\kappa(g) \in K, H(g) \in \mathfrak{a}, n(g) \in N)$. Put $\overline{N} = \theta N$ and let M be the centralizer of A in K. Let τ be a finite dimensional irreducible unitary representation of K and denote its representation space by V. Then the following operator given by the integral

$$C_\tau(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)(H(\bar{n}))}\tau(\kappa(\bar{n}))^{-1}d\bar{n}\,, \qquad (\lambda \in \mathfrak{a}_C^*)\,,$$

is called Harish–Chandra's C-function associated to τ (see Harish–Chandra [7]). It is well known that the operators $C_\tau(\sigma : \lambda)$ obtained by restricting $C_\tau(\lambda)$ to the irreducible M-components V_σ $(\subset V)$, are closely related to the intertwining operators between induced representations (see Harish–Chandra [7], [8]), and also in some special cases they can be represented by a diagonal matrix having diagonal elements in the form of quotients of products of gamma factors with respect to a certain orthogonal basis (cf. Cohn [2], Wallach [15]). It has been believed for a long time that these phenomena would occur for more general cases. In the previous paper Eguchi–Miyamoto–

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Wada [5] we gave an explicit expression as a diagonal matrix of the C-function for $G = SU(n, 1)$, $K = S(U(n) \times U(1)) \subset G$ and $\tau = Ad$.

In this paper we show that, for G and K above and fundamental representations $\tau_{m,r}$ of K, the C-function can be expressed as a diagonal matrix with entries consisting of quotients of products of Γ-factors with respect to a certain basis.

2. Notation and preliminaries

Let n $(n \geq 2)$ be an integer and

$$G = SU(n, 1) = \{A \in GL(n+1, \boldsymbol{C}); {}^t\bar{A}I_{n,1}A = I_{n,1} \text{ and } \det A = 1\},$$

where

$$I_{n,1} = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix} \in GL(n+1, \boldsymbol{C})$$

and I_n is the unit matrix of order n. Let

$$\mathfrak{g} = \mathfrak{su}(n, 1) = \{X \in \mathfrak{gl}(n+1, \boldsymbol{C}); {}^t\bar{X}I_{n,1} + I_{n,1}X = 0 \text{ and } \operatorname{tr} X = 0\},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} A & 0 \\ 0 & \sqrt{-1}t \end{pmatrix}; A \in \mathfrak{u}(n), t \in \boldsymbol{R} \text{ and } \operatorname{tr} A = -\sqrt{-1}t\right\}.$$

Let

$$\mathfrak{a} = \{tH; t \in \boldsymbol{R}\}, \qquad H = \begin{bmatrix} & & & 1 \\ & & 0 & \\ & & \cdot^{\cdot^{\cdot}} & \\ & 0 & & \\ 1 & & & \end{bmatrix},$$

$$\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha} \qquad \text{and} \qquad \bar{\mathfrak{n}} = \theta\mathfrak{n},$$

where α is the simple root of $(\mathfrak{g}, \mathfrak{a})$ which satisfies $\alpha(H) = 1$, $\mathfrak{g}_\beta$ denotes the root subspace of $\mathfrak{g}$ corresponding to the root β and θ is the Cartan involution of $\mathfrak{g}$ defined by

$$\theta X = I_{n,1}XI_{n,1} = -{}^t\bar{X} \qquad (X \in \mathfrak{g}).$$

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. Let K, A, N and $\bar{N}$ denote the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Then $G = KAN$ is the Iwasawa decomposition of G corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and we have

$$K = \left\{ \begin{pmatrix} X & 0 \\ 0 & (\det X)^{-1} \end{pmatrix}; X \in U(n) \right\},$$

$$A = \left\{ \begin{bmatrix} \cosh t & & \sinh t \\ & I_{n-1} & \\ \sinh t & & \cosh t \end{bmatrix}; t \in \boldsymbol{R} \right\},$$

$$\bar{N} = \left\{ P \begin{bmatrix} 1 & z_1 & \dots & z_{n-1} & 1-F \\ & & & & -\bar{z}_1 \\ & & I_{n-1} & & \vdots \\ & & & & -\bar{z}_{n-1} \\ & & & & 1 \end{bmatrix} P^{-1}; \begin{matrix} F = 1 + \frac{1}{2}\sum_{i=1}^{n-1} |z_i|^2 + \sqrt{-1}u \\ u \in \boldsymbol{R}, z_1, \dots, z_{n-1} \in \boldsymbol{C} \end{matrix} \right\},$$

where

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & 1 \\ & \sqrt{2}I_{n-1} & \\ -1 & & 1 \end{bmatrix}.$$

Moreover, $\bar{N}$ can be identified with $\boldsymbol{C}^{n-1} \times \boldsymbol{R}$ by the mapping $(z_1, \dots, z_{n-1}, u) \to \bar{n}(z, u)$, where

$$\bar{n}(z, u) = P \begin{bmatrix} 1 & z_1 & \dots & z_{n-1} & 1-F \\ & & & & -\bar{z}_1 \\ & & I_{n-1} & & \vdots \\ & & & & -\bar{z}_{n-1} \\ & & & & 1 \end{bmatrix} P^{-1} \in \bar{N}.$$

Let $\bar{n}(z, u) = \kappa(\bar{n}(z, u))a(\bar{n}(z, u))n(\bar{n}(z, u))$ be the Iwasawa decomposition of $\bar{n}(z, u)$. Then we can see that

$$a(\bar{n}(z, u)) = P \operatorname{diag}(|F|^{-1}, 1, \dots, 1, |F|)P^{-1}, \tag{2.1}$$

$$\kappa(\bar{n}(z, u)) = \begin{bmatrix} (2-F)/|F| & \sqrt{2}z_1/F & \dots & \sqrt{2}z_{n-1}/F & 0 \\ -\sqrt{2}\bar{z}_1/|F| & 1-|z_1|^2/F & \dots & -\bar{z}_1 z_{n-1}/F & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -\sqrt{2}\bar{z}_{n-1}/|F| & -z_1\bar{z}_{n-1}/F & \dots & 1-|z_{n-1}|^2/F & 0 \\ 0 & 0 & \dots & 0 & F/|F| \end{bmatrix} \tag{2.2}$$

(cf. Sekiguchi [13]). Let M be the centralizer of A in K, that is

$$M = Z_K(A) = \left\{ \begin{bmatrix} d & & \\ & X & \\ & & d \end{bmatrix} ; d^2 \det X = 1, X \in U(n-1) \right\}.$$

Let $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_\alpha)\alpha = n\alpha$ be the rho function and $d\bar{n}$ the invariant measure on $\bar{N}$ normalized so that

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1 .$$

Since $\lambda \in \mathfrak{a}_C^*$ can be written in the form $\lambda = \mu_\lambda \alpha$ ($\mu_\lambda \in C$), we identify λ with the complex number μ_λ. Thus ρ is identified with n and (2.1) implies

$$e^{-(\lambda+\rho)(H(\bar{n}))} = |F(z, u)|^{-\lambda-n} .$$

3. The Harish–Chandra expansions

We will first make some general statements concerning the Harish–Chandra expansions of the Eisenstein integrals. We use the notation and the definitions introduced in §1. Moreover we assume that G has split rank one and the multiplicity of the M-irreducible components which occur in any irreducible unitary representation of K equals 0 or 1. This is the case if $G = \mathrm{Spin}(n, 1)$ or $SU(n, 1)$.

For $\tau \in \hat{K}$, we put $\hat{M}(\tau) = \{\sigma \in \hat{M}; [\tau : \sigma] \neq 0\}$. Let $\tau \in \hat{K}$ and $\sigma \in \hat{M}(\tau)$. We denote by $\mathrm{Hom}_M(V_\tau, H_\sigma)$ the space of all linear mappings P from V_τ into H_σ satisfying $\sigma(m)P = P\tau(m)$ for all $m \in M$, where V_τ and H_σ are the representation spaces of τ and σ, respectively. By our assumption, there exists a unique element $P_\sigma(\tau)$ of $\mathrm{Hom}_M(V_\tau, H_\sigma)$ such that $P_\sigma(\tau)P_\sigma(\tau)^* = I_{H_\sigma}$, where $P_\sigma(\tau)^*$ denotes the adjoint operator of $P_\sigma(\tau)$ and I_{H_σ} is the identity mapping of H_σ.

Let (τ_1, V_1) and (τ_2, V_2) be two irreducible unitary representations of K. We write E for the space of all linear mappings T from V_2 into V_1 and write E^M for the subspace of E consisting of all elements T satisfying $\tau_1(m)T = T\tau_2(m)$ for any $m \in M$. Then the double unitary representation $(\tau = (\tau_1, \tau_2), E)$ of K is defined as follows:

$$\tau(k_1, k_2)(T) = \tau_1(k_1)T\tau_2(k_2)^{-1} , \qquad (k_1, k_2 \in K, T \in E) .$$

Let α denote the unique simple root and ρ denote the rho function. Then for $T \in E^M$ and $\lambda \in \mathfrak{a}_C^*$, the Eisenstein integral on G is defined by the following integral:

$$E_\tau(T, \lambda, x) = \int_K \tau_1(\kappa(xk))T\tau_2(k)^{-1}e^{(\lambda-\rho)(H(xk))}dk .$$

The following theorem has been proved by Harish–Chandra.

THEOREM 3.1 (Harish–Chandra (cf. Harish–Chandra [9], Wallach [15], Warner [16])). *There exist an open connected dense subset Υ_0 in $\mathfrak{a}_C^*$ and meromorphic functions C_+, C_- on $\mathfrak{a}_C^*$ with values in $\mathrm{Hom}_C(E^M, E^M)$ and rational functions Γ_k $(k = 0, 1, \ldots)$ on Υ_0 with values in $\mathrm{Hom}_C(E^M, E^M)$ satisfying the following properties*:

(1) *Let $\Upsilon = \{\lambda \in \mathfrak{a}_C^*; \lambda - \rho \in \Upsilon_0$ and $-\lambda - \rho \in \Upsilon_0\}$. Then the complement of Υ in $\mathfrak{a}_C^*$ is a discrete set and the functions $\lambda \to \Gamma_k(\lambda - \rho)$ are holomorphic on Υ.*

(2) *For $\lambda \in \Upsilon$ and $a \in A^+$, we put*

$$\Phi(\lambda : a) = \sum_{k=0}^{\infty} \Gamma_k(\lambda - \rho)e^{(\lambda-\rho-k\alpha)(\log a)} . \tag{3.1}$$

Then the Eisenstein integral $E_\tau(T, \lambda, a)$ is expanded as follows:

$$E_\tau(T, \lambda, a) = \Phi(\lambda : a)(C_+(\lambda)(T)) + \Phi(-\lambda : a)(C_-(\lambda)(T)) . \tag{3.2}$$

(3) *For any $T \in E^M$, the following equalities are valid*:

$$C_+(\lambda)(T) = TC_{\tau_2}(\lambda) , \tag{3.3}$$

$$C_-(\lambda)(T) = \tau_1(w)^{-1}C_{\tau_1}(-\bar{\lambda})^* T\tau_2(w) , \tag{3.4}$$

where w denotes the nontrivial element of the Weyl group.

REMARK. The expansion (3.2) is called the Harish–Chandra expansion of the Eisenstein integral.

Let $\hat{M}(\tau_1, \tau_2) = \hat{M}(\tau_1) \cap \hat{M}(\tau_2)$ and assume that $\hat{M}(\tau_1, \tau_2)$ is not empty. For $\sigma \in \hat{M}(\tau_1, \tau_2)$, we put $T_\sigma = P_\sigma(\tau_1)^* P_\sigma(\tau_2)$. Then $\{T_\sigma; \sigma \in \hat{M}(\tau_1, \tau_2)\}$ forms a basis of E^M (cf. Mamiuda [11]). The following proposition tells us that it is sufficient for computing Harish–Chandra's C-function to consider the diagonal component with respect to the M-highest weight vector.

PROPOSITION 3.2 (cf. Sekiguchi [13]). *Let $\tau \in \hat{K}$ and $\sigma \in \hat{M}(\tau)$. Then under the assumptions of this section, there exists a meromorphic function $C_\tau(\sigma : \lambda)$ such that*

$$P_\sigma(\tau)C_\tau(\lambda) = C_\tau(\sigma : \lambda)P_\sigma(\tau) .$$

Combining the above results and the definitions of C_+ and C_-, we obtain

$$C_+(\lambda)(T_\sigma) = C_{\tau_2}(\sigma : \lambda)T_\sigma , \tag{3.5}$$

$$C_-(\lambda)(T_\sigma) = \overline{C_{\tau_1}(\sigma : -\bar{\lambda})}\tau_1(w)^{-1}T_\sigma\tau_2(w) . \tag{3.6}$$

4. Diagonalization of the C-function

In this section, we shall return to the $SU(n, 1)$ case. Recall the notation and definitions introduced in §2. Put $V_r^n = C^n \wedge \cdots \wedge C^n$ (exterior products of r times, $r \geq 2$) and $v_{i_1 \dots i_r} = e_{i_1} \wedge \cdots \wedge e_{i_r}$ ($i_1 < \cdots < i_r$), where $\{e_j\}$ is the standard basis of C^n. Identifying C^{n-1} with the subspace of C^n by the mapping $z \to \begin{pmatrix} 0 \\ z \end{pmatrix}$ ($z \in C^{n-1}$), we regard V_r^{n-1} as the subspace of V_r^n. We denote by Φ_r^n the usual representation of the unitary group $U(n)$ on the space V_r^n. For $m \in Z$ and $r \in N$, we define the irreducible unitary representations of K and M as follows:

$$\tau_{m,r}\left(\begin{bmatrix} X & \\ & u \end{bmatrix}\right) = u^m \Phi_r^n(X) \qquad \left(\begin{bmatrix} X & \\ & u \end{bmatrix} \in K\right),$$

$$\sigma_{m,r}\left(\begin{bmatrix} u & & \\ & X & \\ & & u \end{bmatrix}\right) = u^m \Phi_r^{n-1}(X) \qquad \left(\begin{bmatrix} u & & \\ & X & \\ & & u \end{bmatrix} \in M\right).$$

Then it is known that $\tau_{m,r}|_M = \sigma_{m,r} + \sigma_{m+1,r-1}$ and the projection mappings $P_{\sigma_{m,r}}(\sigma_{m,r})$ and $P_{\sigma_{m+1,r-1}}(\tau_{m,r})$ defined in §3 are given as follows:

$$P_{\sigma_{m,r}}(\tau_{m,r})(v_{i_1 \dots i_r}) = (1 - \delta_{i_1 1}) v_{i_1 \dots i_r},$$

$$P_{\sigma_{m+1,r-1}}(\tau_{m,r})(v_{i_1 \dots i_r}) = \delta_{i_1 1} v_{i_2 \dots i_r},$$

where δ_{ij} is Kronecker's delta.

In this case, Harish-Chandra's C-function is given as follows:

$$C_{\tau_{m,r}}(\lambda) = \int_{\bar{N}} e^{-(\lambda+\rho)(H(\bar{n}))} \tau_{m,r}(\kappa(\bar{n}))^{-1} d\bar{n}, \qquad (\lambda \in \mathfrak{a}_C^*). \tag{4.1}$$

It is known that the integral converges absolutely for $\lambda = \mu_\lambda \alpha \in \mathfrak{a}_C^*$ such that $\operatorname{Re} \mu_\lambda > 0$ (see Wallach [15, §8.10.16]). By the mapping $(z, u) \to \bar{n}(z, u)$ $(C^{n-1} \times R \to \bar{N})$, the measure $(n-1)!/\pi^n \cdot dz_1 d\bar{z}_1 \cdots dz_{n-1} d\bar{z}_{n-1} du$ on $C^{n-1} \times R$ induces an invariant measure on $\bar{N}$. Then the above C-function is written in the following form:

$$C_{\tau_{m,r}}(\lambda) = c \int_{C^{n-1} \times R} |F(z, u)|^{-\lambda-n} \tau_{m,r}(\kappa(\bar{n}(z, u)))^{-1} dz d\bar{z} du, \tag{4.2}$$

where $c = (n-1)!/\pi^n$. For simplicity we write $c_1(\lambda)$ for $C_{\tau_{m,r}}(\sigma_{m+1,r-1} : \lambda)$ and $c_2(\lambda)$ for $C_{\tau_{m,r}}(\sigma_{m,r} : \lambda)$. Put $v_1 = e_1 \wedge \cdots \wedge e_r$ and $v_2 = e_2 \wedge \cdots \wedge e_{r+1}$. Then $P_{\sigma_{m+1,r-1}}(\tau_{m,r})(v_1)$ and $P_{\sigma_{m,r}}(\tau_{m,r})(v_2)$ are M-highest weight vectors of $\sigma_{m+1,r-1}$ and $\sigma_{m,r}$, respectively. Hence we have the following expressions:

$$(4.3)\qquad \tau_{m,r}(\kappa(\bar{n}))^{-1}v_1 = \frac{F - |z_r|^2 - \cdots - |Z_{n-1}|^2}{|F|}\left(\frac{\bar{F}}{|F|}\right)^m v_1 + \text{other}\,,$$

$$(4.4)\qquad \tau_{m,r}(\kappa(\bar{n}))^{-1}v_2 = \frac{\bar{F} - |z_1|^2 - \cdots - |z_r|^2}{\bar{F}}\left(\frac{\bar{F}}{|F|}\right)^m v_2 + \text{other}\,.$$

Therefore, we have

$$(4.5)\quad c_1(\lambda) = c\int_{C^{n-1}\times R} |F|^{-\lambda-n}\left(\frac{\bar{F}}{|F|}\right)^m |F|^{-1}(F - |z_r|^2 - \cdots - |z_{n-1}|^2)dzd\bar{z}du\,,$$

$$(4.6)\qquad c_2(\lambda) = c\int_{C^{n-1}\times R} |F|^{-\lambda-n}\left(\frac{\bar{F}}{|F|}\right)^m \bar{F}^{-1}(\bar{F} - |z_1|^2 - \cdots - |z_r|^2)dzd\bar{z}du\,,$$

respectively.

THEOREM 4.1. *The matrix elements $c_1(\lambda)$, $c_2(\lambda)$ of $C_{\tau_{m,r}}(\lambda)$ are represented as follows*:

$$(4.7)\qquad c_1(\lambda) = \frac{(n-1)!\,2^{-\lambda+n-1}\Gamma(\lambda)(\lambda - n + m + 2r - 1)}{\Gamma\left(\frac{\lambda+n-m+1}{2}\right)\Gamma\left(\frac{\lambda+n+m+1}{2}\right)}\,,$$

$$(4.8)\qquad c_2(\lambda) = \frac{(n-1)!\,2^{-\lambda+n-1}\Gamma(\lambda)(\lambda + n - m - 2r)}{\Gamma\left(\frac{\lambda+n-m+2}{2}\right)\Gamma\left(\frac{\lambda+n+m}{2}\right)}\,.$$

REMARK. In Sekiguchi [13], $c_1(\lambda)$ is computed in another way.

To prove the theorem we need the following lemma.

LEMMA 4.2 (Sekiguchi [13]). *For $\lambda \in C$, $l \in Z$, $p_i \in Z$, $p_i \geq 0$ $(i = 1, \ldots, n-1)$, we put*

(4.9)

$$I_n(\lambda, l; p_1, \ldots, p_{n-1}) = \int_{C^{n-1}\times R} F^{(\lambda+l)/2}\bar{F}^{(\lambda-l)/2}\prod_{i=1}^{n-1}\left(\bar{F} - \sum_{j=1}^{i}|z_{n-j}|^2\right)^{p_i} dzd\bar{z}du\,.$$

Then the following formula holds:

(4.10) $I_n(\lambda, l; p_1, \ldots, p_{n-1})$

$$= \frac{(2\pi)^n\cdot 2^{\lambda+p_1+\cdots+p_{n-1}+n}\Gamma(-\lambda - p_1 - \cdots - p_{n-1} - n)}{\Gamma\left(-\frac{\lambda+l}{2}\right)\Gamma\left(-\frac{\lambda-l}{2} - p_1 - \cdots - p_{n-1} - n + 1\right)\prod_{i=1}^{n-1}\left(-\frac{\lambda-l}{2} - p_1 - \cdots - p_{i-1} - i\right)}\,.$$

The proof of the theorem 4.1. Since

$$\bar{F} - |z_1|^2 - \cdots - |z_r|^2 = 2 - (F - |z_{r+1}|^2 - \cdots - |z_{n-1}|^2),$$

we obtain the following formulae.

$$c_1(\lambda) = c \int_{C^{n-1}\times R} F^{(-\lambda-n-m-1)/2}\bar{F}^{(-\lambda-n+m-1)/2}(F - |z_r|^2 - \cdots - |z_{n-1}|^2)dzd\bar{z}du$$

$$= cI_n(-\lambda - n - 1, m; 0, \dots, 1, \dots, 0),$$

$$c_2(\lambda) = c \int_{C^{n-1}\times R} F^{(-\lambda-n-m)/2}\bar{F}^{(-\lambda-n+m-2)/2}(2-(F-|z_{r+1}|^2-\cdots-|z_{n-1}|^2))dzd\bar{z}du$$

$$= c\{2I_n(-\lambda-n-1, m-1; 0, \dots, 0) - I_n(-\lambda-n-1, m-1; 0, \dots, 1, \dots, 0)\}.$$

The theorem now follows from the last expressions and the above lemma.

REMARK. In $r = n$ case, we see that V_r^n is one dimensional space implying the K-highest weight vector $v = e_1 \wedge \cdots \wedge e_n$. Then we have

$$\tau_m(\kappa(\bar{n}))^{-1}v = \frac{|F|}{\bar{F}}\left(\frac{\bar{F}}{|F|}\right)^m v$$

and

$$c_{\tau_m}(\lambda) = c \int_{C^{n-1}\times R} |F|^{-\lambda-n}\left(\frac{\bar{F}}{|F|}\right)^{m-1} dzd\bar{z}du.$$

Thus

$$c_{\tau_m}(\lambda) = \frac{(n-1)!\,2^{-\lambda+n}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+n-m+1}{2}\right)\Gamma\left(\frac{\lambda+n+m-1}{2}\right)}.$$

(cf. Muta [12]).

5. Square-integrability of the Eisenstein integrals

In this section, as an application of the results of §4, we write down the condition for the norm of the Eisenstein integrals to be square-integrable. For this purpose we need the following theorem.

THEOREM 5.1 (Casselman and Miličić (cf. Casselman–Miličić [1], Knapp [10])). *Retain the notation defined in* §1. *Then the following conditions are mutually equivalent*:

(1) *Every leading character* ν *of the spherical function* F *has*

$$|\nu| < \delta^{-1/2}, \tag{5.1}$$

where $\delta(a) = \det(\mathrm{Ad}(a))|_{\mathfrak{n}}$ $(a \in A)$

(2) F *is square-integrable on* G.

REMARK. A simple calculation yields $\delta(a) = e^{2\rho(\log a)}$.

Our main results of this section can be stated.

THEOREM 5.2. *We use the notation introduced in* §3 *and* §4. *Let* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ *and assume that* $\mathrm{Re}\,\lambda > 0$. *Then we have the following statements.*

(1) *Let* $\tau_1 = \tau_{m-1,r+1}$ *or* $\tau_{m,r}$, $\tau_2 = \tau_{m,r}$, $\sigma = \sigma_{m,r}$ *and* $\tau = (\tau_1, \tau_2)$. *Then the norm of the Eisenstein integral* $E_\tau(T_\sigma, \lambda, x)$ *is square-integrable if* λ *satisfies the following conditions* (i) *or* (ii) *or* (iii).

(i) $\lambda + n - m$ *is a negative even integer,*

(ii) $\lambda + n + m$ *is a non-positive even integer,*

(iii) $\lambda = -n + m + 2r > 0$.

Moreover, the Eisenstein integral $E_\tau(T_\sigma, \lambda, x)$ *vanishes at* $\lambda = -n + m + 2r > 0$ *if* $\tau_1 = \tau_{m-1,r+1}$.

(2) *Let* $\tau_1 = \tau_{m+1,r-1}$ *or* $\tau_{m,r}$, $\tau_2 = \tau_{m,r}$, $\sigma = \sigma_{m+1,r-1}$ *and* $\tau = (\tau_1, \tau_2)$. *Then the norm of the Eisenstein integral* $E_\tau(T_\sigma, \lambda, x)$ *is square-integrable if* λ *satisfies the following conditions* (i) *or* (ii) *or* (iii).

(i) $\lambda + n - m$ *is a negative odd integer,*

(ii) $\lambda + n + m$ *is a negative odd integer,*

(iii) $\lambda = n - m - 2r + 1 > 0$.

Moreover, the Eisenstein integral $E_\tau(T_\sigma, \lambda, x)$ *vanishes at* $\lambda = n - m - 2r + 1 > 0$ *if* $\tau_1 = \tau_{m+1,r-1}$.

PROOF. Since the proof of (2) is the same as that of (1), we shall prove the case of (1).

We write $a_+(\lambda)$ for $C_{\tau_2}(\sigma : \lambda)$ and $a_-(\lambda)$ for $\overline{C_{\tau_1}(\sigma : -\bar{\lambda})}$, respectively. For $\lambda \in \Upsilon$ and $a \in A^+$, we put

$$\Phi(\sigma : \lambda : a) = \sum_{k=0}^{\infty} \Gamma_k(\lambda - \rho)(T_\sigma) e^{(\lambda - \rho - k\alpha)(\log a)}. \tag{5.1}$$

Because $\tau_1(w)^{-1} T_\sigma \tau_2(w) = \varepsilon T_\sigma$, where $\varepsilon = 1$ if $\tau_1 = \tau_{m,r}$ and $\varepsilon = -1$ if $\tau_1 = \tau_{m-1,r+1}$, the Harish–Chandra expansion of $E_\tau(T_\sigma, \lambda, x)$ $(\tau = (\tau_1, \tau_2))$ can be written as follows:

$$E_\tau(T_\sigma, \lambda, a) = a_+(\lambda)\Phi(\sigma : \lambda : a) + \varepsilon a_-(\lambda)\Phi(\sigma : -\lambda : a). \tag{5.2}$$

Using the estimate of Γ_k (cf. Eugchi–Hashizume–Koizumi [4]), we see that the function $\lambda \to \Phi(\sigma : \lambda : a)$ can be extended to a meromorhic function on $\mathfrak{a}_{\mathbf{C}}^*$.

Let $\lambda \in \mathfrak{a}_C^*$ be such that $\operatorname{Re}\lambda > 0$. If $a_+(\lambda) \neq 0$, then the character $e^{(\lambda-\rho)(\log a)}$ is contained in the leading characters of $E_\tau(T_\sigma, \lambda, x)$. Thus (5.1) implies that $E_\tau(T_\sigma, \lambda, x)$ is not square-integrable. Therefore, if $E_\tau(T_\sigma, \lambda, x)$ is square-integrable at λ then $a_+(\lambda) = 0$. We denote by S_+ the zeros of $a_+(\lambda)$. From (4.8), we have the following:

$$S_+ = \{\lambda > 0; \lambda = -n + m + 2r \text{ or } -n + m - 2 - 2l \text{ or } -n - m - 2l \ (l = 0, 1, 2, \dots)\}.$$

We first consider the case $\tau_1 = \tau_{m-1,r+1}$. It is clear that $\hat{M}(\tau_1, \tau_2) = \{\sigma\}$ (i.e. $\dim E^M = 1$) and there exists a rational function $\lambda \to A_k(\lambda)$ such that $\Gamma_k(\lambda - \rho)(T_\sigma) = A_k(\lambda)T_\sigma$. Hence from the definition of Γ_k (cf. Harish–Chandra [9], Warner [16]), we obtain the following:

$$A_0(\lambda) = 1, \tag{5.3}$$

$$\begin{aligned} \frac{1}{4}k(2\lambda - k)A_k(\lambda) &= (n-1)\sum_{l\geq 1}(\lambda - n - k + 2l)A_{k-2l}(\lambda) \\ &+ \sum_{l\geq 1}(\lambda - n - k + 4l)A_{k-4l}(\lambda) - 2(2r+1)\sum_{l\geq 1} lA_{k-2l}(\lambda) \\ &- m(1-m)\sum_{l\geq 1}(2l-1)A_{k+2-4l}(\lambda) \\ &- (2m^2 + 2m - 1)\sum_{l\geq 1} lA_{k-4l}(\lambda). \end{aligned} \tag{5.4}$$

From the above expression, it is clear that $A_{2p+1}(\lambda) = 0$ $(p = 0, 1, 2, \dots)$ and the poles of the function $\lambda \to A_{2p}(\lambda)$ are contained in $\{1, 2, \dots, p\}$. From (4.7), we have

$$a_-(\lambda) = \frac{(n-1)!\,2^{\lambda+n-1}\Gamma(-\lambda)(-1)(\lambda + n - m - 2r)}{\Gamma\left(\frac{-\lambda + n - m + 2}{2}\right)\Gamma\left(\frac{-\lambda + n + m}{2}\right)}. \tag{5.5}$$

Let $\lambda \in S_+ \backslash \{-n + m + 2r\}$. Then since $a_-(\lambda) \neq 0$, there exists $\mu \geq 2\lambda$ such that the leading characters of $E_\tau(T_\sigma, \lambda, x)$ are $e^{(-\lambda-\rho)(\log a)}$ and $e^{(\lambda-\rho-\mu)(\log a)}$. Therefore, from theorem 5.1, we see that the norm of $E_\tau(T_\sigma, \lambda, x)$ is square-integrable.

On the other hand, from the functional equation for the Eisenstein integral (cf. Harish–Chandra [7], [8]), we obtain the following:

$$-a_+(-\lambda)^{-1}a_-(\lambda)E_\tau(T_\sigma, -\lambda, x) = E_\tau(T_\sigma, \lambda, x), \qquad (\lambda \in \mathfrak{a}_C^*). \tag{5.6}$$

Noting that $a_+(-\lambda)/a_-(\lambda) = (\lambda - n + m + 2r)/(\lambda + n - m - 2r)$ and the poles of $a_+(-\lambda)/a_-(\lambda)$ correspond to the zeros of the function $\lambda \to E_\tau(T_\sigma, \lambda, a)$, we see that $E_\tau(T_\sigma, \lambda, x)$ vanishes at $\lambda = -n + m + 2r$.

We next consider the case $\tau_1 = \tau_{m,r}$. We write σ_1 for $\sigma_{m,r}$ and σ_2 for $\sigma_{m+1,r-1}$. Then $\hat{M}(\tau_1, \tau_2) = \{\sigma_1, \sigma_2\}$ (i.e. $\dim E^M = 2$) and there exist rational functions $\lambda \to A_k^1(\lambda)$ and $\lambda \to A_k^2(\lambda)$ such that $\Gamma_k(\lambda - \rho)(T_{\sigma_1}) = A_k^1(\lambda) T_{\sigma_1} + A_k^2(\lambda) T_{\sigma_2}$. From the definition of Γ_k, we obtain the following:

$$A_0^1(\lambda) = 1\,, \tag{5.7}$$

$$\begin{aligned} \frac{1}{4}k(2\lambda - k)A_k^1(\lambda) &= (n-1)\sum_{l\geq 1}(\lambda - n - k + 2l)A_{k-2l}^1(\lambda) \\ &\quad + \sum_{l\geq 1}(\lambda - n - k + 4l)A_{k-4l}^1(\lambda) \\ &\quad + 2(n-r)\left\{\sum_{l\geq 1}(2l-1)A_{k+1-2l}^2(\lambda) - \sum_{l\geq 1} 2lA_{k-2l}^1(\lambda)\right\} \\ &\quad - (1-m^2)\sum_{l\geq 1}(-1)^l l A_{k-2l}^1(\lambda)\,, \end{aligned} \tag{5.8}$$

$$A_0^2(\lambda) = 0\,, \tag{5.9}$$

$$\begin{aligned} &\frac{1}{4}\{k(2\lambda - k) - 2n + 2m + 4r - 1\}A_k^2(\lambda) \\ &\quad = (n-1)\sum_{l\geq 1}(\lambda - n - k + 2l)A_{k-2l}^2(\lambda) \\ &\qquad + \sum_{l\geq 1}(\lambda - n - k + 4l)A_{k-4l}^2(\lambda) \\ &\qquad + 2r\left\{\sum_{l\geq 1}(2l-1)A_{k+1-2l}^1(\lambda) - \sum_{l\geq 1} 2lA_{k-2l}^2(\lambda)\right\} \\ &\qquad - m^2\sum_{l\geq 1}(-1)^l l A_{k-2l}^1(\lambda)\,. \end{aligned} \tag{5.10}$$

Then from the above expression, it is clear that $A_{2p+1}^1(\lambda) = A_{2p}^2(\lambda) = 0$ ($p = 0, 1, 2, \ldots$) and the poles of the function $\lambda \to A_{2p}^1(\lambda)$ and $\lambda \to A_{2p+1}^2(\lambda)$ are contained in

$$\{1, 2, \ldots, p, n - m - 2r + 1, \ldots, (n - m - 2r + 1 - 2p(p+1))/(2p+1)\}\,.$$

From (4.8) we have

$$a_-(\lambda) = \frac{(n-1)!\,2^{\lambda+n-1}\Gamma(-\lambda)(-1)(\lambda - n + m + 2r)}{\Gamma\left(\dfrac{-\lambda + n - m + 2}{2}\right)\Gamma\left(\dfrac{-\lambda + n + m}{2}\right)}\,. \tag{5.11}$$

Let $\lambda \in S_+$. Then since $(2p+1)\lambda - n + m + 2r - 1 - 2p(p+1) \neq 0$ and $a_-(\lambda) \neq 0$, there exists $\mu \geq 2\lambda$ such that the leading characters of $E_\tau(T_\sigma, \lambda, x)$

are $e^{(-\lambda-\rho)(\log a)}$ and $e^{(\lambda-\rho-\mu)(\log a)}$. Therefore, from theorem 5.1, we see that the norm of $E_\tau(T_\sigma, \lambda, x)$ is square-integrable. This completes the proof of the theorem.

REMARK. (1) The above theorem gives information about discrete series, which is already known to be true in a more general situation (cf. Enright [3]). Our result implies that the information of discrete series can also be obtained from the zeros of Harish-Chandra's C-function.

(2) Via the correspondence between the C-function and the intertwining operators we see that, at the points where the C-function vanishes, the intertwining operator has nontrivial kernel, and hence the induced representation is reducible.

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