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The chromatic E_1-term $H^1M_1^1$ at the prime 3

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ABSTRACT. In this paper, we determine the E_1-term $H^1M_1^1$ of the chromatic spectral sequence converging to the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(M)$ of the mod 3 Moore spectrum M. At the prime $p > 3$, the E_1-term $H^1M_1^1$ plays a central role determining the homotopy groups $\pi_*(L_2M)$ of the $v_2^{-1}BP$-localized mod p Moore spectrum.

1. Introduction

Let M denote the mod p Moore spectrum and L_n the Bousfield localization functor with respect to $v_n^{-1}BP$. Here BP is the Brown-Peterson ring spectrum at a prime number p and v_n $(n = 1, 2, \ldots)$ denotes the generator of $\pi_*(BP)$ with $|v_n| = 2p^n - 2$. Consider the spectrum N^1 obtained as a cofiber of the localization map $M \to L_1M$. In [12] and [9] H. Tamura and the second author determined the homotopy groups $\pi_*(L_2N^1)$ by using the Adams-Novikov spectral sequence at the prime $p > 3$. For $p > 3$ the Adams-Novikov filtration is at most 4 and the homotopy groups of L_2N^1 is determined by E_2-term [9]. At the prime $p = 3$, on the other hand, it is known that for any large integer $s_0 > 0$ there exists an integer $s > s_0$ such that the E_2-term $E_2^{s,*} \neq 0$ by the Morava structure theorem [8, Th. 6.2.10 (c)].

In this paper we will determine the first line of the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_2N^1)$ at the prime 3. The E_2-term is an Ext group $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M_1^1)$ for a $BP_*(BP)$-comodule $BP_*(L_2N^1) = M_1^1$ which will be denoted by $H^*M_1^1$ following the paper on chromatic spectral sequences due to Miller, Ravenel and Wilson [6].

In order to state the result, we define integers $a(n)$, $a'(n)$ and a_n for $n \geq 0$ by:

$$a(0) = 2 \quad \text{and} \quad a(n) = 6 \cdot 3^{n-1} + 1 \quad (n > 0);$$

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$$a'(0) = 10 \quad \text{and} \quad a'(n) = 28 \cdot 3^{n-1} \quad (n > 0); \quad \text{and}$$

$$a_0 = 1 \quad \text{and} \quad a_n = 4 \cdot 3^{n-1} - 1 \quad (n > 0).$$

Furthermore we use the notation:

$$k(n)_* = F_3[v_n] \quad \text{and} \quad K(n)_* = v_n^{-1}k(n)_* = F_3[v_n, v_n^{-1}],$$

where F_3 denotes the prime field of characteristic 3.

THEOREM 1.1. *$H^1M_1^1$ is isomorphic to the direct sum of the $k(1)_*$-modules*

$$K(1)_*/k(1)_* \bigoplus K(1)_*/k(1)_*$$

and

$$\bigoplus_{k\geq 0,t} (k(1)_*/(v_1^{a(k)}) \bigoplus k(1)_*/(v^{a'}{}_1^{(k)})) \bigoplus \bigoplus_t k(1)_*/(v_1^2) \bigoplus \bigoplus_{k\geq 0,u} k(1)_*/(v_1^{a_k}),$$

where t, $u \in \mathbf{Z}$ with $3 \nmid u$.

The generators of each cyclic $k(1)_*$-module will be given in Theorem 6.1 which is a finer restatement of Theorem 1.1.

As in [12], we can apply this theorem to the nontriviality problem of the products of β-elements in the homotopy groups $\pi_*(M)$, as we will discuss in a forthcoming paper. We hope that this will be the first milestone to determine the homotopy groups $\pi_*(L_2S^0)$ of L_2-localized spheres L_2S^0 at the prime 3 as in the case for the prime >3 (*cf.* [6], [12], [9], [14]).

In **2** we restate a key lemma given in [6] to fit our situation so that it suffices to find some elements in order to discribe Ext-group. After giving some preparatory computations in **3**, we define new elements for the case $p = 3$ in **4** (*cf.* [12]). Then we get the desired elements in **5** which satisfy the condition given in **2**.

2. Key lemma

Let (A, Γ) denote the Hopf algebroid such that Γ is A-flat as an A-module. Then the category of Γ-comodules has enough injectives (*cf.* [6, Lemma A.1.2.2]) and Ext group $\mathrm{Ext}_\Gamma^i(M, N)$ is defined to be the i-th derived functor of Hom-functor $\mathrm{Hom}_\Gamma(M, N)$ for comodules M and N. Let C^*M denote an injective resolution of a comodule M. Then the Ext group $\mathrm{Ext}_\Gamma^*(A, M)$ is a cohomology of the resolution, that is, the homology of the complex $\mathrm{Hom}_\Gamma(A, C^*M)$. Here we use a cobar resolution, and the resulting cobar complex Ω_Γ^*M is given by:

$$\Omega_\Gamma^n M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (n \text{ copies of } \Gamma)$$

with the differential $d_r\colon \Omega_\Gamma^n M \to \Omega^{n+1}M$ defined by

$$\begin{aligned} d_r(m\otimes x_1\otimes\cdots\otimes x_n) &= \psi(m)\otimes x_1\otimes\cdots\otimes x_n \\ &\quad+\sum_{i=1}^{n}(-1)^i m\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes\Delta(x_i)\otimes x_{i+1}\otimes\cdots\otimes x_n \\ &\quad-(-1)^n m\otimes x_1\otimes\cdots\otimes x_n\otimes 1. \end{aligned}$$

Here $\psi\colon M\to M\otimes_A\Gamma$ and $\Delta\colon \Gamma\to\Gamma\otimes_A\Gamma$ denote the structure map of M and the diagonal map of the Hopf algebroid Γ, respectively.

An example of such a Hopf algebroid (A,Γ) is $(BP_*, BP_*(BP))$ associated to the Brown-Peterson spectrum BP at the prime 3. In this case we abbreviate $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M)$ by H^*M for a $BP_*(BP)$-comodule M.

We recall [6] the comodules

$$M_2^0 = v_2^{-1}BP_*/(3, v_1) \quad\text{and}$$
$$M_1^1 = v_2^{-1}BP_*/(3, v_1^\infty) = \{x/v_1^j : j>0,\ x\in M_2^0\}.$$

Then we have the short exact sequence

$$0\to M_2^0 \xrightarrow{\varphi} M_1^1 \xrightarrow{v_1} M_1^1 \to 0$$

of comodules, where $\varphi(x)=x/v_1$. This gives rise to the long exact sequence

$$(2.1)\quad 0\to H^0M_2^0 \xrightarrow{\varphi_*} H^0M_1^1 \xrightarrow{v_1} H^0M_1^1 \xrightarrow{\delta_0} H^1M_2^0 \xrightarrow{\varphi_*} H^1M_1^1 \xrightarrow{v_1} H^1M_1^1 \xrightarrow{\delta_1} H^2M_2^0 \to\cdots.$$

Following the computation of $H^1M_1^1$ in [12] at $p>3$, we will work in the category of $E(2)_*(E(2))$-comodules. Here $E(2)_* = Z_{(3)}[v_1, v_2, v_2^{-1}]$ and the action of BP_* is induced by sending v_i to v_i for $i\le 2$ and to zero for $i>2$. $E(2)$ is a ring spectrum representing the homology theory $E(2)_*(X) = E(2)_*\otimes_{BP_*}BP_*(X)$. Then

$$E(2)_*(E(2)) = E(2)_*\otimes_{BP_*}BP_*(BP)\otimes_{BP_*}E(2)_*$$

and the Hopf algebroid structure of $(E(2)_*, E(2)_*(E(2)))$ is induced by the one of $(BP_*, BP_*(BP))$. Since $E(2)_*(E(2))$ is flat over $E(2)_*$, we can use homological algebra in the category and define Ext groups as derived functors of Hom. Then we have a change of rings theorem ([5]):

$$H^iM = \mathrm{Ext}^i_{E(2)_*(E(2))}(E(2)_*, M\otimes_{BP_*}E(2)_*)$$

for any v_2-local comodule M. Remark that M_k^j is v_2-local when $j+k=2$. By virtue of this theorem, we will hereafter abbreviate $M_j^i\otimes_{BP_*}E(2)_*$ to M_j^i:

$$M_2^0 = K(2)_* = E(2)_*/(3, v_1) \quad\text{and}\quad M_1^1 = E(2)_*/(3, v_1^\infty),$$

and $H^iM_k^j$ will denote

$$\mathrm{Ext}^i_{E(2)_*(E(2))}(E(2)_*, M_k^j).$$

There will be no serious confusion, since these H^*M's coincide as long as $M = M_k^j$ with $j + k = 2$.

In [7] (*cf.* [8]), Ravenel claimed to have determined the structure of $H^*M_2^0$ at the prime 3, which turned out to be wrong as pointed out by Henn. There are independent corrections, see [2], [3], [11] and [16]. In particular they show that Ravenel's result is correct up to dimension 2 and we will use only this part:

THEOREM 2.2. a) *$H^1M_2^0$ is the $K(2)_*$-vector space generated by*

$$h_{10},\ h_{11} \quad \text{and} \quad \zeta_2.$$

b) *$H^2M_2^0$ is the $K(2)_*$-vector space generated by*

$$\xi,\ b_0,\ b_1,\ h_{10}\zeta_2 \quad \text{and} \quad h_{11}\zeta_2.$$

These generators have degrees

$$|h_{10}| = 4,\ |h_{11}| = |b_0| = 12,\ |b_1| = 36,\ |\zeta_2| = 0 \quad \text{and} \quad |\xi| = 8,$$

and are represented by cocycles as follows

$$\begin{aligned} h_{10} &= [t_1], \\ h_{11} &= [v_2^{-1}t_1^3], \\ \zeta_2 &= [v_2^{-1}t_2 + v_2^{-3}t_2^3 - v_2^{-1}t_1^4], \\ b_i &= [-t_1^{3^i} \otimes t_1^{2\cdot 3^i} - t_1^{2\cdot 3^i} \otimes t_1^{3^i}] \quad \text{and} \\ \xi &= [v_2^{-3}t_1 \otimes t_3 + v_2^{-10}t_3^3 \otimes t_1^3 + \cdots]. \end{aligned} \tag{2.3}$$

Here the generator ξ is represented by any cocycle whose leading term is $v_2^{-3}t_1 \otimes t_3 + v_2^{-10}t_3^3 \otimes t_1^3$. These representatives are in the cobar complex $\Omega^*_{E(2)_*(E(2))}K(2)_*$. In this paper the same symbol will be used to denote a cohomology class and its representative as is done in [12].

To compute $H^1M_1^1$ by the exact sequence (2.1) we need the following lemma that can be proved by an easy diagram chasing:

LEMMA 2.4. (*cf.* [6, Remark 3.11]) *Consider a $k(1)_*$-submodule B of $H^1M_1^1$ that fits into the following commutative diagram*:

$$\begin{array}{ccccccc}
H^1M_2^0 & \xrightarrow{\varphi} & B & \xrightarrow{v_1} & B & \xrightarrow{\delta} & H^2M_2^0 \\
\| & & \cap & & \cap & & \| \\
H^1M_2^0 & \xrightarrow{\varphi} & H^1M_1^1 & \xrightarrow{v_1} & H^1M_1^1 & \xrightarrow{\delta} & H^2M_2^0.
\end{array}$$

If the upper sequence is exact, then

$$H^1M_1^1 = B.$$

To construct the desired $k(1)_*$-module B, we first note that $B \supset \operatorname{Im} \varphi$. Since $\operatorname{Im} \varphi$ is isomorphic to Coker $(\delta\colon H^0M_1^1 \to H^1M_2^0)$ and Ker $(v_1\colon H^1M_1^1 \to H^1M_1^1)$, we start with

LEMMA 2.5. *Ker v_1 in $H^1M_1^1$ is the F_3-vector space with basis consisting of the cocycles represented by*:

$$v_2^{3^k s}t_1/v_1,\ v_2^{3t-1}t_1^3/v_1,\ v_2^t\zeta_2/v_1 \quad \textit{and} \quad t_1/v_1$$

for $k \geq 0$, $t \in \boldsymbol{Z}$ and $s \in \boldsymbol{Z}$ such that either $s \equiv 1(3)$ or $s \equiv -1(9)$.

This lemma is shown by using the elements $x_i \in v_2^{-1}BP_*$ defined in [6] such that

$$x_i \equiv v_2^{3^i} \bmod(3, v_1) \quad \text{and}$$

(2.6)
$$d_0(x_i) \equiv \begin{cases} v_1t_1^3 \bmod(3, v_1^3) & i = 0, \\ v_1^3v_2^2t_1 - v_1^4v_2(\tau + v_2\zeta_2) \bmod(3, v_1^5) & i = 1, \\ v_1^{a_i}v_2^{2\cdot 3^{i-1}}\sigma \bmod(3, v_1^{2+a_i}) & i > 1, \end{cases}$$

where a_i is an integer such that $a_0 = 1$ and $a_i = 4\cdot 3^{i-1} - 1$ for $i > 0$. In fact, the lemma follows from Theorem 2.2 a) and the fact that δ is computed [6, (5.9)] to be

$$\delta(x_0^s/v_1) = sv_2^{s-1}t_1^3,$$
$$\delta(x_1^s/v_1^3) = sv_2^{3s-1}t_1 \quad \text{and}$$
$$\delta(x_i^s/v_1^{a_i}) = -sv_2^{3^{i-1}(3s-1)}t_1, \quad i > 1,$$

and $H^0M_1^1$ is generated by $x_i^s/v_1^{a_i}$ and $1/v_1^j$, $j \geq 1$ [6, Th. 5.3]. These elements may also be considered in $E(2)_*$ and satisfy the same formula there. Hence we do not distinguish them either.

Consider the pairs $(w(i), e(i)) = (x(i), a(i)), (y(i), b(i))$ and $(z(i), c(i))$ of elements $x(i)$, $y(i)$ and $z(i)$ in $\Omega^1_{E(2)_*(E(2))}E(2)_*/(3) = E(2)_*/(3) \otimes_{E(2)_*} E(2)_*(E(2))$ and positive integers (including ∞) $a(i)$, $b(i)$ and $c(i)$ such that

$$x(i) \equiv v_2^i t_1 \bmod(3, v_1),$$
$$y(i) \equiv v_2^i t_1^3 \bmod(3, v_1),$$
$$z(i) \equiv v_2^i \zeta_2 \bmod(3, v_1) \quad \text{and}$$

$e(i) = \infty$ if $\delta(w(i)/v_1^j) = 0$ for any $j > 0$, otherwise $e(i) = \min(e'(i))$ such that $\delta(w(i)/v_1^{e'(i)}) \neq 0$.

We also consider the subsets of $\boldsymbol{Z}$:

$$\Lambda = \{i: i = 3^k s \text{ with } s \equiv 1(3) \text{ or } s \equiv -1(9)\} \quad \text{and} \quad \Lambda' = \{i: i \equiv -1(3)\}.$$

Now Lemma 2.4 implies the following

COROLLARY 2.7. *With the above notation, let B be the $k(1)_*$-module generated by $w(i)/v_1^{e(i)}$'s for i such that*

$$i \in \Lambda \text{ if } w = x, \quad i \in \Lambda' \text{ if } w = y \quad \text{and} \quad i \in \boldsymbol{Z} \text{ if } w = z.$$

If the set

$$\{\delta(w(i)/v_1^{e(i)})\} = \{\delta(x(i)/v_1^{a(i)}), \delta(y(j)/v_1^{b(j)}), \delta(z(k)/v_1^{c(k)}): i \in \Lambda, j \in \Lambda', k \in \boldsymbol{Z}\} \subset H^2 M_2^0$$

is linearly independent over F_3, then $H^1 M_1^1 = B$.

3. Preparatory computations

Before proceeding we need some computations. First we give some formulae on the right unit $\eta_R: BP_* \to BP_*(BP)$ by tensoring the rational numbers $\boldsymbol{Q}$. Note that

$$BP_* \otimes \boldsymbol{Q} = \boldsymbol{Q}[m_1, m_2, \dots] \quad \text{and} \quad BP_*(BP) \otimes \boldsymbol{Q} = (BP_* \otimes \boldsymbol{Q})[t_1, t_2, \dots],$$

where the generators have the internal degrees $|m_i| = 2(3^i - 1) = |t_i|$. Recall [1] that

$$\eta_R(m_1) = m_1 + t_1, \ \eta_R(m_2) = m_2 + m_1 t_1^3 + t_2 \quad \text{and}$$
$$\eta_R(m_3) = m_3 + m_2 t_1^9 + m_1 t_2^3 + t_3;$$

and [4] that

$$v_1 = 3m_1, \ v_2 = 3m_2 - m_1 v_1^3 \quad \text{and} \quad v_3 = 3m_3 - m_2 v_1^9 - m_1 v_2^3.$$

Since $9m_2 \equiv v_1^4 \bmod(3)BP_*$, we have

$$\eta_R(v_1) = v_1 + 3t_1,$$
$$(3.1) \quad \eta_R(v_2) = v_2 + v_1 t_1^3 + 3t_2 - v_1(3v_1^2 t_1 + 9v_1 t_1^2 + 9t_1^3) - t_1 \eta_R(v_1^3) \quad \text{and}$$
$$\eta_R(v_3) \equiv v_3 + v_2 t_1^9 + v_1 t_2^3 - v_1^9 t_2 + v_1^2 V - t_1 \eta_R(v_2^3) \bmod(3).$$

Here V is an element of $BP_*(BP)$ which satisfies

$$3v_1V \equiv v_2^3 + v_1^3t_1^9 - v_1^9t_1^3 - \eta_R(v_2^3) \bmod(9). \tag{3.2}$$

Next we will calculate Ext group $\mathrm{Ext}^{*,*}_{E(2)_*(E(2))}(E(2)_*, M_1^1)$. Following [6], we will write

$$x \equiv y \bmod(3, v_1^j)$$

for x, $y \in \Omega^{*,*}_{E(2)_*(E(2))}E(2)_*$, if $pr(x) = pr(y)$ in $\Omega^{*,*}_{E(2)_*(E(2))}E(2)_*/(3, v_1^j)$ where $pr\colon \Omega^{*,*}_{E(2)_*(E(2))}E(2)_* \to \Omega^{*,*}_{E(2)_*(E(2))}E(2)_*/(3, v_1^j)$ denotes the natural projection.

The definition of cobar complex shows that the differentials d_i: $\Omega^i_{E(2)_*(E(2))}E(2)_*/(3, v_1^j) \to \Omega^{i+1}_{E(2)_*(E(2))}E(2)_*/(3, v_1^j)$ for $i = 0, 1$ are given by

$$\begin{aligned} d_0(m) &= \eta_R(m) - m \quad \text{and} \\ d_1(x) &= 1 \otimes x - \Delta(x) + x \otimes 1 \end{aligned} \tag{3.3}$$

for $m \in E(2)_*$ and $x \in E(2)_*(E(2))$, where $\eta_R\colon E(2)_*/(3, v_1^j) \to E(2)_*/(3, v_1^j) \otimes_{E(2)_*} E(2)_*(E(2))$ is induced by the right unit η_R of the Hopf algebroid $BP_*(BP)$ and $\Delta\colon E(2)_*(E(2)) \to E(2)_*(E(2)) \otimes_{E(2)_*} E(2)_*(E(2))$ is the diagonal of the Hopf algebroid. Note that

$$\begin{aligned} x \otimes vy &= x\eta_R(v) \otimes y \quad \text{and} \\ \Delta(xy) &= \Delta(x)\Delta(y). \end{aligned} \tag{3.4}$$

We also note that (3.3) shows the derivative formula:

$$d_1(mx) = d_0(m) \otimes x + md_1(x)$$

for $m \in E(2)_*$ and $x \in E(2)_*(E(2))$.

Furthermore, (*cf.* [12, (2.3.2), (2.3.5)])

$$\begin{aligned} d_1(x\eta_R(m)) &= d_1x \otimes m - x \otimes d_0(m), \\ d_1(t_2) &\equiv -t_1 \otimes t_1^3 - v_1b_0 \bmod(3) \quad \text{and} \\ d_1(t_3) &\equiv -t_1 \otimes t_2^3 - t_2 \otimes t_1^9 - v_2b_1 \bmod(3, v_1). \end{aligned} \tag{3.5}$$

Here $b_i \in E(2)_*(E(2))^{\otimes 2}$ is defined by

$$b_i = -t_1^{3^i} \otimes t_1^{2\cdot 3^i} - t_1^{2\cdot 3^i} \otimes t_1^{3^i}. \tag{3.6}$$

For example, (3.2) is written as $3v_1V = v_1^3t_1^9 - v_1^9t_1^3 - d_0(v_2^3)$, and so we obtain by (3.1)

$$d_1(V) \equiv v_1^2b_1 - v_1^8b_0 \bmod(3). \tag{3.7}$$

The last formula of (3.1) yields the relation in $E(2)_*(E(2))$:

$$v_2 t_1^9 + v_1 t_2^3 - v_1^9 t_2 + v_1^2 V - t_1 \eta_R(v_2^3) = 0. \tag{3.8}$$

In fact, $\eta_R(v_3) = 0$ in $E(2)_*(E(2)) = E(2)_*[t_1, t_2, \dots]/(\eta_R(v_k): k > 2)$. More generally, the relation $\eta_R(v_k) = 0$ for $k > 2$ in $E(2)_*(E(2))$ implies (*cf.* [12, (3.2.2)]):

$$t_n^9 = v_2^{3^n-1} t_n - v_1 v_2^{-1} t_{n+1}^3 \in E(2)_*(E(2))/(3, v_1^2). \tag{3.9}$$

Now we see that

LEMMA 3.10. *By definition, we have*

$$b_0^3 \equiv b_1 \bmod(3).$$

Furthermore, we have

$$b_1^3 \equiv v_2^6 b_0 \bmod(3, v_1^2)$$

up to homology. That is, there exists an elements ω_1 *such that*

$$d_1(\omega_1) \equiv -b_1^3 + v_2^6 b_0 \bmod(3, v_1^2).$$

REMARK 3.11. *Moreover, we can show*

$$b_1^3 \equiv v_2^6 b_0 + v_1^2 v_2^4 b_1 \bmod(3, v_1^3)$$

up to homology.

PROOF OF LEMMA 3.10. Rewriting b_1^3 in $E(2)_*(E(2)) \otimes_{E(2)_*} E(2)_*(E(2))/(3, v_1^2)$ by (3.4) and (3.9), we get

$$\begin{aligned} -b_1^3 &= t_1^9 \otimes t_1^{18} + t_1^{18} \otimes t_1^9 \\ &\equiv (v_2^2 t_1 - v_1 v_2^{-1} t_2^3) \otimes (v_2^4 t_1^2 + v_1 v_2 t_1 t_2^3) \\ &\quad + (v_2^4 t_1^2 + v_1 v_2 t_1 t_2^3) \otimes (v_2^2 t_1 - v_1 v_2^{-1} t_2^3) \bmod(3, v_1^2) \\ &\equiv v_2^6 t_1 \otimes t_1^2 + v_1 v_2^5 t_1^4 \otimes t_1^2 + v_1 v_2^3 t_1 \otimes t_1 t_2^3 - v_1 v_2^3 t_2^3 \otimes t_1^2 \\ &\quad + v_2^6 t_1^2 \otimes t_1 - v_1 v_2^5 t_1^5 \otimes t_1 - v_1 v_2^3 t_1^2 \otimes t_2^3 + v_1 v_2^3 t_1 t_2^3 \otimes t_1 \bmod(3, v_1^2). \end{aligned}$$

On the other hand we have by (3.4) and (3.9),

$$\Delta(t_2^3) = t_2^3 \otimes 1 + v_2^2 t_1^3 \otimes t_1 + 1 \otimes t_2^3 \bmod(3, v_1). \tag{3.12}$$

Therefore, using the formula $d_1(xy) = d_1(x)(y \otimes 1 + 1 \otimes y) + \Delta(x) d_1(y) - x \otimes y - y \otimes x$, we compute

$$\begin{aligned}-d_1(v_1v_2^3t_1^2t_2^3) &\equiv -v_1v_2^3d_1(t_1^2t_2^3) \bmod(3, v_1^2)\\
&\equiv -v_1v_2^3(t_1\otimes t_1)(t_2^3\otimes 1 + 1\otimes t_2^3)\\
&\quad + v_1v_2^3(t_1^2\otimes 1 - t_1\otimes t_1 + 1\otimes t_1^2)(v_2^2t_1^3\otimes t_1)\\
&\quad + v_1v_2^3t_1^2\otimes t_2^3 + v_1v_2^3t_2^3\otimes t_1^2 \bmod(3, v_1^2)\\
&\equiv -v_1v_2^3(t_1t_2^3\otimes t_1 + t_1\otimes t_1t_2^3)\\
&\quad + v_1v_2^3(v_2^2t_1^5\otimes t_1 - v_2^2t_1^4\otimes t_1^2 + v_2^2t_1^3\otimes t_1^3)\\
&\quad + v_1v_2^3t_1^2\otimes t_2^3 + v_1v_2^3t_2^3\otimes t_1^2 \bmod(3, v_1^2).\end{aligned}$$

Moreover we have

$$-d_1(v_1v_2^5t_1^6) \equiv -v_1v_2^5t_1^3\otimes t_1^3 \bmod(3, v_1^2).$$

Collecting these terms we get the desired homologous relation:

$$d_1(\omega_1) \equiv -b_1^3 + v_2^6b_0 \bmod(3, v_1^2),$$

by defining $\omega_1 = v_1v_2^3t_1^2t_2^3 + v_1v_2^5t_1^6$. q.e.d.

4. The elements X and Y

In this section, we define the elements X and Y, which will yield the generators of $H^1M_1^1$. Note that there are no corresponding elements for $p > 3$.

By (3.2) we compute

(4.1) $$\begin{aligned}\eta_R(v_2^6) &\equiv (v_2^3 + v_1^3t_1^9 - v_1^9t_1^3 - 3v_1V)^2 \bmod(9)\\
&\equiv v_2^6 - 6v_1v_2^3V + 2v_1^3v_2^3t_1^9 - 6v_1^4t_1^9V\\
&\quad + v_1^6t_1^{18} - 2v_1^9v_2^3t_1^3 + 6v_1^{10}t_1^3V - 2v_1^{12}t_1^{12} + v_1^{18}t_1^6 \bmod(9).\end{aligned}$$

LEMMA 4.2. *For the element $x_2 = v_2^9 - v_1^8v_2^7$ of $E(2)_*$, we obtain*

$$\begin{aligned}d_0(x_2) &\equiv -v_1^{11}v_2^6t_1 - v_1^{12}v_2^6(v_2^{-3}t_2^3 - v_2^{-3}t_1^{12} + v_2^{-9}t_2^9 + v_1v_2^{-3}V)\\
&\quad - v_1^{14}t_1^{18}\eta_R(v_2) + v_1^{15}v_2^3t_1^9 - v_1^{17}v_2^4t_1^3 \bmod(3, v_1^{18})\end{aligned}$$

in $\Omega^1_{E(2)_*(E(2))}E(2)_* = E(2)_*(E(2))$.

PROOF. Using (3.3), (3.1) and (3.8), we compute $\bmod(3, v_1^{18})$:

$$\begin{aligned}d_0(v_2^9) &\equiv v_1^9t_1^{27}\\
&\equiv v_1^9(v_2^6t_1^3 - v_1^3v_2^{-3}t_2^9 - v_1^6v_2^{-3}V^3)\end{aligned}$$

$$\begin{aligned}-d_0(v_1^8v_2^7) &\equiv -v_1^8(v_2^3+v_1^3t_1^9-v_1^9t_1^3)^2(v_2+v_1t_1^3-v_1^3t_1)+v_1^8v_2^7\\ &\equiv -v_1^8(v_2^6-v_1^3v_2^3t_1^9+v_1^6t_1^{18}+v_1^9v_2^3t_1^3)(v_2+v_1t_1^3-v_1^3t_1)+v_1^8v_2^7\\ &\equiv -v_1^9v_2^6t_1^3+v_1^{11}v_2^6t_1+v_1^{11}v_2^4t_1^9+v_1^{12}v_2^3t_1^{12}\\ &\quad -v_1^{14}v_2^3t_1^{10}-v_1^{14}t_1^{18}\eta_R(v_2)-v_1^{17}v_2^4t_1^3\\ &\equiv -v_1^9v_2^6t_1^3-v_1^{11}v_2^6t_1+v_1^{11}v_2^4(-v_1v_2^{-1}t_2^3-v_1^2v_2^{-1}V+v_1^3v_2^{-1}t_1^{10})\\ &\quad +v_1^{12}v_2^3t_1^{12}-v_1^{14}v_2^3t_1^{10}-v_1^{14}t_1^{18}\eta_R(v_2)-v_1^{17}v_2^4t_1^3.\end{aligned}$$

Now by summing up we get the desired congruence, since $V\equiv -v_2^6t_1^9 \bmod(3, v_1^3)$.
q.e.d.

Next we define an element X of $E(2)_*(E(2))$ such that

(4.3) $$3v_1^3X \equiv -v_1^9x - 3v_1^9y - d_0(x_2) \bmod(9, v_1^{15}).$$

Here x and y are defined by

$$x = v_1^2v_2^6t_1 + v_1^3v_2^6\zeta' + v_1^4v_2^3V + v_1^5v_2t_1^{18} \quad\text{and}\quad y = v_1v_2^6t_1^2 + v_1^3v_2^3t_1V.$$

By [10, Lemma 2.6] there exists an element ζ' such that

$$\zeta' \equiv \zeta_2^3 \bmod(3, v_1^3) \quad\text{and}\quad d_1(\zeta') \equiv 0 \bmod(9, v_1^3).$$

For the following computations note that $(9, v_1^{15})$ is an invariant ideal.

THEOREM 4.4. *The element* $X \in E(2)_*(E(2))$ *satisfies*:

$$X \equiv -v_2^8t_1 \bmod(3, v_1),$$

and

$$d_1(X) \equiv v_1^{10}v_2^5b_0 + v_1^{10}v_2^5t_1^3 \otimes \zeta_2^3 \bmod(3, v_1^{11}).$$

PROOF. Note that $(9, v_1^6)$ is an invariant ideal, and we have

$$3v_1^3X \equiv -d_0(v_2^9) \bmod(9, v_1^6).$$

Now the first formula follows from

$$d_0(v_2^9) \equiv 3v_1^3v_2^6t_1^9 \equiv 3v_1^3v_2^8t_1 \bmod(9, v_1^4)$$

implied by (3.3), (3.1) and (3.8).

For the second formula we see from $d_1d_0 = 0$ and (4.3) that

$$3v_1^3d_1(X) \equiv -v_1^9d_1(x+3y) \bmod(9, v_1^{15}).$$

Thus it suffices to compute $d_1(x)$ and $d_1(3y) \bmod(9, v_1^5)$. First we compute

$d_1(x)$. Note that the definitions (3.1), (3.2) and (3.3) show

$$d_1(V) \equiv 0 \bmod(9, v_1).$$

Using (3.1), (3.5) and (4.1), we compute $\bmod(9, v_1^5)$:

$$\begin{aligned} d_1(v_1^2v_2^6t_1) &\equiv 6v_1t_1 \otimes v_2^6t_1 + v_1^2(-6v_1v_2^3V + 2v_1^3v_2^3t_1^9 - 6v_1^4t_1^9V) \otimes t_1 \\ &\equiv 6v_1v_2^6t_1 \otimes t_1 + 3v_1^4v_2^3t_1^{10} \otimes t_1 - 6v_1^3v_2^3V \otimes t_1, \\ d_1(v_1^3v_2^6\zeta') &\equiv -6v_1^4v_2^3V \otimes \zeta', \\ d_1(v_1^4v_2^3V) &\equiv 3v_1^3v_2^3t_1 \otimes V \quad \text{and} \\ d_1(v_1^5v_2t_1^{18}) &\equiv 6v_1^4v_2t_1 \otimes t_1^{18}. \end{aligned}$$

Summing up, we have

$$\begin{aligned} d_1(x) \equiv\ & 6v_1v_2^6t_1 \otimes t_1 - 6v_1^3v_2^3V \otimes t_1 + 3v_1^4v_2^3t_1^{10} \otimes t_1 - 6v_1^4v_2^3V \otimes \zeta' \\ & + 3v_1^3v_2^3t_1 \otimes V + 6v_1^4v_2t_1 \otimes t_1^{18} \bmod(9, v_1^5). \end{aligned}$$

For $d_1(3y)$, we will use $d_1(V) \equiv 0 \bmod(3, v_1^2)$ as seen in (3.7) and compute similarly $\bmod(3, v_1^5)$,

$$\begin{aligned} d_1(v_1v_2^6t_1^2) &\equiv 2v_1^4v_2^3t_1^9 \otimes t_1^2 - 2v_1v_2^6t_1 \otimes t_1 \quad \text{and} \\ d_1(v_1^3v_2^3t_1V) &\equiv -v_1^3v_2^3(t_1 \otimes V + V \otimes t_1). \end{aligned}$$

Now note that $t_1^9 \equiv v_2^2t_1 \bmod(3, v_1)$ by (3.8) and $V \equiv -v_2^2t_1^3 - v_1v_2t_1^6 + v_1^2v_2^2t_1 \bmod(3, v_1^3)$ by (3.1) and (3.2). Then, we compute $\bmod(9, v_1^5)$,

$$\begin{aligned} 3v_1^4v_2^3t_1^{10} \otimes t_1 + 6v_1^4v_2t_1 \otimes t_1^{18} + 6v_1^4v_2^3t_1^9 \otimes t_1^2 &\equiv -3v_1^4v_2^5b_0 \quad \text{and} \\ -6v_1^4v_2^3V \otimes \zeta' &\equiv 6v_1^4v_2^5t_1^3 \otimes \zeta_2^3. \end{aligned}$$

This shows

$$d_1(x + 3y) \equiv -3v_1^4v_2^5b_0 + 6v_1^4v_2^5t_1^3 \otimes \zeta_2^3 \bmod(9, v_1^5).$$

Substitute this to $3v_1^3d_1(X) \equiv -v_1^9d_1(x + 3y) \bmod(9, v_1^{14})$, and we obtain the result. q.e.d.

Next we define the element Y. First we need a lemma.

LEMMA 4.5. *For $i = 1, 2$ there exist elements κ_i of $E(2)_*(E(2))$ such that*

$$d_1(\kappa_1) \equiv v_1^3v_2^{-3}(t_2^3 \otimes t_1^9 - t_1^3 \otimes t_1^{18}) - v_1^6(v_2^{-30}t_3^9 \otimes t_1^9 - v_2^{-12}t_2^9 \otimes t_1^{18}) \bmod(3, v_1^7)$$

and

$$\begin{aligned} d_1(\kappa_2) \equiv\ & v_1v_2^{-2}(t_1^3 \otimes t_2^3 - t_1^6 \otimes t_1^9 + v_2^3b_0) \\ & + v_1^2v_2^{-9}t_1^3 \otimes (c(t_3^3) + v_2^6t_1^3t_2^3) - v_1^2\zeta_2^3 \otimes t_1^6 + v_1^2t_1^3\zeta_2^3 \otimes t_1^3 \bmod(3, v_1^3). \end{aligned}$$

PROOF. Define

$$\kappa_1 = v_1^3v_2^{-27}t_1^9t_2^{27} - v_1^3v_2^{-27}t_3^9 - v_1V + \zeta_2^9\eta_R(v_2^3) + v_1^3v_2^{-18}\omega_1^3.$$

We continue our computation using the formulae (3.1), (3.3) and (3.5). We recall the relation in $E(2)_*(E(2))$: $v_2^{3^n}t_n \equiv v_2t_n^9 + v_1t_{n+1}^3 \bmod(3, v_1^2)$ given in (3.9). Then the right hand side of the first congruence in the lemma is:

(4.6) $$v_1^3v_2^{-3}(t_2^3 \otimes t_1^9 - t_1^3 \otimes t_1^{18}) - v_1^6(v_2^{-30}t_3^9 \otimes t_1^9 - v_2^{-12}t_2^9 \otimes t_1^{18})$$
$$\equiv v_1^3v_2^{-27}t_2^{27} \otimes t_1^9 - v_1^3v_2^{-9}t_1^{27} \otimes t_1^{18} \bmod(3, v_1^7).$$

Now mod $(3, v_1^7)$,

$$d_1(v_1^3v_2^{-27}t_1^9t_2^{27}) \equiv -v_1^3v_2^{-27}(t_1^{36} \otimes t_1^{81} + t_1^{27} \otimes t_1^{90} + t_1^9 \otimes t_2^{27} + t_2^{27} \otimes t_1^9),$$
$$d_1(-v_1^3v_2^{-27}t_3^9) \equiv v_1^3v_2^{-27}(t_1^9 \otimes t_2^{27} + t_2^9 \otimes t_1^{81} + v_2^9b_1^9),$$
$$d_1(-v_1V) \equiv -v_1(v_1^2b_1) \quad \text{and}$$
$$d_1(\zeta_2^9\eta_R(v_2^3)) \equiv -\zeta_2^9 \otimes (v_1^3t_1^9).$$

Noticing that $\zeta_2^9 \equiv v_2^{-9}(t_2^9 - t_1^{36}) + v_2^{-27}t_2^{27} \bmod(3)$, $90 = 81 + 9$ and $t_1^{81} \equiv v_2^{18}t_1^9$ for our modulo, the sum of these terms equals the right hand side of (4.6). Thus the first part follows from the congruence $d_1(v_1^3v_2^{-18}\omega_1^3) \equiv -v_1^3v_2^{-18}b_1^9 + v_1^3b_1 \bmod(3, v_1^9)$ obtained by Lemma 3.10.

To prove the second assertion, put

$$\kappa_2 = v_1v_2^{-2}t_1^3t_2^3 + v_1v_2^{-8}c(t_3^3) + \eta_R(v_2^2)\zeta_2^3.$$

Where $c: E(2)_*(E(2)) \to E(2)_*(E(2))$ denotes the Hopf conjugation. Note that c satisfies $\Delta c = (c \otimes c)T\Delta$ for the switching map T (*cf.* [1], [8], [15]). Furthermore, $c(t_1) = -t_1$ and $c(t_2) = \tau = t_1^4 - t_2$ (*cf.* [8]). Thus mod$(3, v_1^3)$:

$$d_1(v_1v_2^{-2}t_1^3t_2^3) \equiv v_1^2v_2^{-3}t_1^3 \otimes t_1^3t_2^3 - v_1v_2^{-2}(t_1^6 \otimes t_1^9 + t_1^3 \otimes t_1^{12} + t_1^3 \otimes t_2^3 + t_2^3 \otimes t_1^3),$$
$$d_1(v_1v_2^{-8}c(t_3^3)) \equiv v_1^2v_2^{-9}t_1^3 \otimes c(t_3^3)$$
$$- v_1v_2^{-8}(t_1^{27} \otimes t_2^3 - t_1^{27} \otimes t_1^{12} - \tau^9 \otimes t_1^3 - v_2^3b_1^3) \quad \text{and}$$
$$d_1(\eta_R(v_2^2)\zeta_2^3) \equiv -\zeta_2^3 \otimes (-v_1v_2t_1^3 + v_1^2t_1^6).$$

Here we used $\zeta_2 \equiv v_2^{-1}t_2 - v_2^{-3}\tau^3 \bmod(3, v_1)$. Thus the second congruence follows. q.e.d.

Now define an element Y' of $E(2)_*(E(2))$ by

(4.7) $$Y' = z + w + v_1\kappa_1 + v_1^5\kappa_2$$

for

$$z = v_2t_1^9 + v_1t_2^3 + v_1^2V \quad \text{and} \quad w = v_1^3v_2^{-2}t_1^{18} + v_1^5v_2^{-1}t_2^3.$$

Theorem 4.8. *There is an element* $Y \in E(2)_*(E(2))$ *which satisfies*:

$$Y \equiv v_2^3 t_1 \bmod(3, v_1),$$

and

$$d_1(Y) \equiv v_1^7 v_2 \xi \bmod(3,\ v_1^8).$$

Proof. First we compute for z:

$$d_1(z) \equiv (v_1 t_1^3 - v_1^3 t_1) \otimes t_1^9 - v_1(t_1^3 \otimes t_1^9 + v_1^3 b_1) + v_1^2(v_1^2 b_1)$$

mod$(3, v_1^8)$ by (3.5) and (3.7), which equals $-v_1^3 t_1 \otimes t_1^9$. By (3.8), we also have $v_2^3 t_1 \equiv v_2 t_1^9 + v_1 t_2^3 + v_1^2 V - v_1^3 t_1^{10} \bmod(3, v_1^9)$. Therefore,

$$d_1(z) \equiv -v_1^3(v_2^{-2} t_1^9 + v_1 v_2^{-3} t_2^3 + v_1^2 v_2^{-3} V - v_1^3 v_2^{-3} t_1^{10}) \otimes t_1^9 \bmod(3, v_1^8).$$

Now we consider the other elements:

$$d_1(w) \equiv v_1^3(v_1 v_2^{-3} t_1^3 - v_1^3 v_2^{-3} t_1 - v_1^3 v_2^{-5} t_1^9 - v_1^4 v_2^{-6} t_1^{12}) \otimes t_1^{18} + v_1^3 v_2^{-2} t_1^9 \otimes t_1^9$$
$$- v_1^6 v_2^{-2} t_1^3 \otimes t_2^3 + v_1^7 v_2^{-3} t_1^6 \otimes t_2^3 - v_1^5 v_2^{-1} t_1^3 \otimes t_1^9 \bmod(3, v_1^8).$$

Lemma 4.5 then shows

$$d_1(v_1 k_1) \equiv v_1^4 v_2^{-3}(t_2^3 \otimes t_1^9 - t_1^3 \otimes t_1^{18})$$
$$- v_1^7(v_2^{-30} t_3^9 \otimes t_1^9 - v_2^{-12} t_2^9 \otimes t_1^{18}) \bmod(3, v_1^8) \quad \text{and}$$
$$d_1(v_1^5 \kappa_2) \equiv v_1^6 v_2^{-2}(t_1^3 \otimes t_2^3 - t_1^6 \otimes t_1^9 + v_2^3 b_0) + v_1^7 t_1^3 \zeta_2^3 \otimes t_1^3$$
$$+ v_1^7 v_2^{-9} t_1^3 \otimes (c(t_3^3) + v_2^6 t_1^3 t_2^3) - v_1^7 \zeta_2^3 \otimes t_1^6 \bmod(3, v_1^8).$$

Notice that

$$v_1^6 v_2^{-3} t_1^{10} \otimes t_1^9 - v_1^6 v_2^{-3} t_1 \otimes t_1^{18} - v_1^6 v_2^{-5} t_1^9 \otimes t_1^{18} \equiv -v_1^6 v_2 b_0 + v_1^7 \rho \bmod(3, v_1^8).$$

Since $t_1^9 \equiv v_2^2 t_1 - v_1 v_2^{-1} t_2^3 \bmod(3, v_1^2)$ by (3.8), ρ does not involve t_3. Therefore we obtain

$$d_1(Y') \equiv v_1^7 \Xi \bmod(3, v_1^8),$$

where Ξ involves $-v_2^{-30} t_3^9 \otimes t_1^9 - v_2^{-9} t_1^3 \otimes t_3^3 \equiv -v_2^{-2} t_3 \otimes t_1 - v_2^{-9} t_1^3 \otimes t_3^3 \bmod(3, v_1)$ which is the characterization of the homology class $[v_2 \xi]$. So there is a cochain ω_2 such that $d_1(\omega_2) = -\Xi + v_2 \xi$. Now we define the element Y by $Y = Y' + v_1^7 \omega_2$. q.e.d.

5. Construction of $w(i)$

First we consider ξ. The generator ξ is represented by a cocycle whose leading term is $t_1 \otimes t_3 + t_3^3 \otimes t_1^3$ and there is no other generator of the same degree. So any cocycles whose leading terms are $t_1 \otimes t_3 + t_3^3 \otimes t_1^3$ are homologous. Therefore, (3.9) shows that we have a cochain ω_3 such that

$$d_1(\omega_3) \equiv \xi^3 + v_2\xi \bmod(3, v_1). \tag{5.1}$$

Now we define $x(m) \in E(2)_*(E(2))$ for $m = 3^k s$ with $k \geq 0$ and for $s \in \boldsymbol{Z}$ with $s \equiv 1$ or $s \equiv -1(9)$ by

$$x(1) = v_2 t_1 + v_1 \tau,$$

$$x(3) = Y,$$

$$v_1^3 x(3^{n+1}) = v_1 x(3^n)^3 + (-1)^k v_1^{3a(n)+1} v_2^{3i(n)} \omega_3 - d_0(v_2^{3^{n+1}+1}) \quad n > 0, \quad \text{and}$$

$$x(3^n(3t+1)) = x_{n+1}^t x(3^n) \quad n \geq 0,\ t \in \boldsymbol{Z}; \quad \text{and}$$

$$x(9t-1) = -x_2^{t-1} X,$$

$$v_1^3 x(3(9t-1)) = v_1 x(9t-1)^3 + v_1^{29} v_2^{27t-12} V - d_0(v_2^{3(9t-1)+1}) \quad \text{and}$$

$$v_1^3 x(3^{n+1}(9t-1)) = v_1 x(3^n(9t-1))^3 + v_1^{3a'(n)} v_2^{3i'(t;n)+1} \zeta_2^{3^{n+2}} - d_0(v_2^{3^{n+1}(9t-1)+1}) \quad n > 0$$

for the integers $a(k)$ and $a'(k)$ for $k \geq 0$ defined in the introduction, and $i(k)$ and $i'(t; k)$ for $k \geq 0$ and $t \in \boldsymbol{Z}$ defined by

$$i(0) = 0, \quad i(k+1) = 3i(k) + 1,$$

$$i'(t; 0) = 9t - 4, \quad i'(t; k+1) = 3^k(9(3t-1) - 1).$$

PROPOSITION 5.2. *Let s denote an integer such that $s \equiv 1(3)$ and $s \equiv -1(9)$. Then there exist elements $x(m)$ of $E(2)_*(E(2))/(3)$ for m satisfying $m = 3^k s$ such that*

$$x(m) \equiv v_2^m t_1 \bmod(3, v_1),$$

and for $n \geq 0$ and $t \in \boldsymbol{Z}$,

$$d_1(x(3t+1)) \equiv v_1^2 v_2^{3t} b_0 \bmod(3, v_1^3),$$

$$d_1(x(3^n(3t+1))) \equiv -(-1)^n v_1^{a(n)} v_2^{3^{n+1}t+i(n)} \xi \bmod(3, v_1^{a(n)+1})(n > 0); \quad \textit{and}$$

$$d_1(x(9t-1)) \equiv -v_1^{10} v_2^{9t-4} b_0 - v_1^{10} v_2^{9t-4} t_1^3 \otimes \zeta_2^3 \bmod(3, v_1^{11}),$$

$$d_1(x(3^n(9t-1))) \equiv -v_1^{a'(n)} v_2^{i'(t;n)} t_1 \otimes \zeta_2^{3^{n+1}} \bmod(3, v_1^{a'(n)+1})(n > 0).$$

PROOF. The first part is proved by induction as follows: First $x(1) \equiv v_2 t_1 \bmod(3, v_1)$ follows from the definition. Theorem 4.8 shows the assertion for $x(3)$. Since $d_0(v_2^{3^{n+1}+1}) \equiv v_2^{3^{n+1}}(v_1 t_1^3 - v_1^3 t_1) \bmod(3, v_1^4)$, we have $v_1^3 x(3^{n+1}) \equiv$

$v_1^3 v_2^{3^{n+1}} t_1 \bmod(3, v_1^4)$ for $n > 1$ if we assume that $x(3^n) \equiv v_2^{3^n} t_1 \bmod(3, v_1)$. Noticing that $x_n \equiv v_2^{3^n} \bmod(3, v_1)$, we see

$$x(3^n(3t+1)) \equiv v_2^{3^n(3t+1)} t_1 \bmod(3, v_1).$$

Second, $x(9t-1) \equiv v_2^{9t-1} t_1 \bmod(3, v_1)$ follows from Theorem 4.4. The other part is shown inductively as above.

For the second part, first we get the congruence $d_1(x(1)) \equiv v_1^2 b_0 \bmod(3, v_1^3)$ from the fact $d_1(\tau) = -t_1^3 \otimes t_1 + v_1 b_0$.

Next we will prove

$$d_1(x(3^n)) \equiv -(-1)^n v_1^{a(n)} v_2^{i(n)} \xi \bmod(3, v_1^{a(n)+1}),$$

inductively. For $n = 1$, this follows immediately from Theorem 4.8.

Now suppose that $d_1(x(3^k)) \equiv -(-1)^k v_1^{a(k)} v_2^{i(k)} \xi \bmod(3, v_1^{a(k)+1})$ with $i(k) \equiv 1(3)$. Then $d_1(x(3^k)^3) \equiv -(-1)^k v_1^{3a(k)} v_2^{3i(k)} \xi^3 \bmod(3, v_1^{3a(k)+3})$. On the other hand,

$$d_1(v_1^{3a(k)+1} v_2^{3i(k)} \omega_3) \equiv v_1^{3a(k)+1} v_2^{3i(k)} (\xi^3 + v_2 \xi) \bmod(3, v_1^{3a(k)+2})$$

by (5.1), which completes the induction for $x(3^n)$.

Note that $a_{k+1} > a(k)$, where $a_k = 3^k + 3^{k-1} - 1$. Then we have the congruence for $d_1(x(3^k(3t+1)))$ since $d_0(x_{k+1}^t) \equiv 0 \bmod(3, v_1^{a(k)+1})$ for $t \geq 1$ and $k \geq 0$.

Now we turn to the last claim. By Theorem 4.4, we see that

$$d_1(x(9t-1)) \equiv -v_1^{10} v_2^{9t-4} b_0 - v_1^{10} v_2^{9t-4} t_1^3 \otimes \zeta_2^3 \bmod(3, v_1^{11}).$$

Cubing this we obtain

$$d_1(x(9t-1)^3) \equiv -v_1^{30} v_2^{3(9t-4)} b_1 - v_1^{30} v_2^{9(3t-1)-1} t_1 \otimes \zeta_2^{3^2} \bmod(3, v_1^{33}).$$

On the other hand,

$$d_1(v_1^{29} v_2^{3(9t-4)} V) \equiv v_1^{31} v_2^{3(9t-4)} b_1 \bmod(3, v_1^{32}).$$

This gives rise to the formula for $d_1(x(3(9t-1)))$.

For $k > 0$, we may suppose inductively that

$$d_1(x(3^k(9t-1))) \equiv -v_1^{a'(k)} v_2^{i'(t;k)} t_1 \otimes \zeta_2^{3^{k+1}} \bmod(3, v_1^{a'(k)+1}).$$

Then, cubing this again, we obtain

$$d_1(x(3^k(9t-1))^3) \equiv -v_1^{3a'(k)} v_2^{3i'(t;k)} t_1^3 \otimes \zeta_2^{3^{k+2}} \bmod(3, v_1^{3a'(k)+3}).$$

Finally, we compute

$$d_1(v_1^{3a'(k)} v_2^{3i'(t;k)+1} \zeta_2^{3^{k+2}}) \equiv v_1^{3a'(k)} v_2^{3i'(t;k)} (v_1 t_1^3 - v_1^3 t_1) \otimes \zeta_2^{3^{k+2}} \bmod(3, v_1^{3a'(k)+3}),$$

and get the assertion for $k+1$, which completes the induction. q.e.d.

The other terms $y(i)$ and $z(i)$ are defined similarly to the corresponding ones in [12]:

$$y(3t-1) = v_2^{3t-3}V \quad \text{and} \quad z(3^n u) = x_n^u \zeta_2^{3^{n+1}} \quad \text{for} \quad 3 \nmid u \in \mathbf{Z},$$

where x_n is the element of (2.6). Next we verify the following

PROPOSITION 5.3. *For $t, u \in \mathbf{Z}$ with $3 \nmid u$,*

$$d_1(y(3t-1)) \equiv v_1^2 v_2^{3t-3} b_1 \bmod (3, v_1^3),$$

$$d_1(z(3^n u)) \equiv \begin{cases} u v_1 v_2^{u-1} t_1^3 \otimes \zeta_2^3 & n = 0, \\ u v_1^3 v_2^{3u-1} t_1 \otimes \zeta_2^9 & n = 1, \\ -u v_1^{a_n} v_2^{3^{n-1}(3u-1)} t_1 \otimes \zeta_2^{3^{n+1}} & n > 1. \end{cases}$$

6. $H^1 M_1^1$

Recall the notation:

$$k(1)_* = F_3[v_1] \quad \text{and} \quad K(1)_* = F_3[v_1, v_1^{-1}].$$

From Corollary 2.7 we deduce our main theorem:

THEOREM 6.1. *$H^1 M_1^1$ is the direct sum of $k(1)_*$-modules A and B. Here A is isomorphic to $K(1)_*/k(1)_* \oplus K(1)_*/k(1)_*$, in which each factor is generated by t_1 and ζ_2. B is the direct sum of cyclic $k(1)_*$-modules generated by*

$$x(3^k(3t+1))/v_1^{a(k)}, \ x(3^k(9t-1))/v_1^{a'(k)}, \ y(3t-1)/v_1^2 \quad \textit{and} \quad z(3^k u)/v_1^{a_k}$$

for $k \geq 0$ and $t, u \in \mathbf{Z}$ with $3 \nmid u$.

Note that a cyclic $k(1)_*$-module generated by x/v_1^a is isomorphic to the truncated polynomial algebra $k(1)_*/(v_1^a)$.

PROOF. By Corollary 2.7 it suffices to show that δ-images of these generators are linearly independent. Notice that

$$d_1(x) \equiv v_1^a y \bmod (3, v_1^{a+1}) \quad \text{implies} \quad \delta(x/v_1^a) = y,$$

for a representative y of a generator of $H^2 M_2^0$ (see Theorem 2.2). Propositions 5.2 and 5.3 yield the classification of the set of δ-images of the generators as follows:

(I)	$v_2^{3t} b_0, \ v_2^{9t-4} b_0 + v_2^{9t-3} h_{11} \zeta_2$	$(t \in \mathbf{Z})$,
(II)	$v_2^{3^{n+1}t + i(n)} \xi$	$(n > 0, t \in \mathbf{Z})$,
(III)	$v_2^{i'(t;n)} h_{10} \zeta_2, \ v_2^{3^{n-1}(3u-1)} h_{10} \zeta_2$	$(n > 0, t \in \mathbf{Z}, u \in \mathbf{Z} - 3\mathbf{Z})$,
(IV)	$v_2^{3t-3} b_1$	$(t \in \mathbf{Z})$,
(V)	$v_2^{u-1} h_{11} \zeta_2$	$(u \in \mathbf{Z} - 3\mathbf{Z})$.

Here we remark that the cocycles $\zeta_2^{3^n}$, $n \geq 1$, represent the same cohomology class ζ_2 in $H^1M_2^0$. Every element in the same class has an distinct power of v_2. It is clear that the elements classified in different classes are linearly independent and so are δ-images of the generators. q.e.d.

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