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# On a characterization of $L^p$-norm and a converse of Minkowski's inequality

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ABSTRACT. Let $\mathbf{C}$ be a cone in a linear space. Under some weak regularity conditions we show that every subadditive function $\mathbf{p}: \mathbf{C} \to \boldsymbol{R}$ such that $\mathbf{p}(rx) \leqslant r\mathbf{p}(x)$ for some $r \in (0, 1)$ and all $x \in \mathbf{C}$ must be positively homogenous. As an application we obtain a new characterization of $L^p$-norm. This permits us to prove among other things the following converse of Minkowski's inequality.

Let $(\Omega, \Sigma, \mu)$ be a measure space such that there exist disjoint sets $A, B \in \Sigma$ satisfying the condition $\mu(B) = 1/\mu(A)$, $\mu(A) \neq 1$. If $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is an arbitrary bijection such that

$$\varphi^{-1}\left(\int_\Omega \varphi \circ (x + y) d\mu\right) \leqslant \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu\right) + \varphi^{-1}\left(\int_\Omega \varphi \circ y d\mu\right)$$

for all the $\mu$-integrable step functions $x, y: \Omega \to \boldsymbol{R}_+$ then $\varphi$ is a power function.

## Introduction

Let $\boldsymbol{R}$, $\boldsymbol{R}_+$ and $\boldsymbol{N}$ denote respectively the set of reals, nonnegative reals and positive integers.

For a measure space $(\Omega, \Sigma, \mu)$ let $\mathbf{S} = \mathbf{S}(\Omega, \Sigma, \mu)$ stand for the linear space of all the $\mu$-integrable step functions $x: \Omega \to \boldsymbol{R}$ and let $\mathbf{S}_+ := \{x \in S: x \geqslant 0\}$.

It can be easily verified that for every bijection $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that $\varphi(0) = 0$ the functional $\boldsymbol{P}_\varphi: \mathbf{S} \to \boldsymbol{R}_+$ given by the formula

$$(1) \qquad \boldsymbol{P}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| d\mu\right), \qquad x \in \mathbf{S},$$

is well defined. In [4] we have proved the following *converse of Minkowski's inequality.*

*Let $(\Omega, \Sigma, \mu)$ be a measure space with two sets $A, B \in \Sigma$ such that*

$$(2) \qquad 0 < \mu(A) < 1 < \mu(B) < \infty$$

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*and* $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *a bijection such that* $\varphi(0) = 0$. *If* $\varphi^{-1}$ *is continuous at* 0 *and*

$$\boldsymbol{P}_\varphi(x + y) \leqslant \boldsymbol{P}_\varphi(x) + \boldsymbol{P}_\varphi(y), \qquad x,\ y \in \mathbf{S}_+, \tag{3}$$

*then* $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

It has also been shown that condition (2) is essential. In this paper we show that modifying the definition of $\boldsymbol{P}_\varphi$ one can eliminate the assumption $\varphi(0) = 0$. The remaining assumption of the continuity of $\varphi^{-1}$ at 0 plays a key but technical role. We conjecture that the above result is valid without this assumption. However it seems to be a difficult problem to get rid of it completely.

In a recent paper [7] we have attempted to replace the continuity of $\varphi^{-1}$ at 0 by the following assumption: there exist disjoint sets $C$, $D \in \Sigma$ of positive measures such that $\mu(C) + \mu(D) = 1$. This approach leads to some open problems in the theory of convex functions. Nevertheless we were able to prove that in the case when $\mu(C) = \mu(D)$ the continuity of $\varphi^{-1}$ at 0 is superfluous.

In section 3 of the present paper we show that the continuity of $\varphi^{-1}$ at 0 together with assumption (2) can be replaced by one of the following conditions:

(i) *there exist* $n \in \boldsymbol{N}$, $n > 1$, *and* $A$, $B \in \Sigma$ *such that*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{1}{n}; \qquad \mu(B) = n,$$

or

(ii) *there exist* $n$, $m \in \boldsymbol{N}$, $n \neq m$, $n > 1$, *and* $A$, $B$, $C \in \Sigma$ *such that*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{m}{n}; \qquad \mu(B) = \frac{n}{m}; \qquad \mu(C) = n.$$

The proof of this theorem is based on the following characterization of $L^p$-norm which is the main result of section 2.

*If* $(\Omega, \Sigma, \mu)$ *is a measure space with two disjoint sets* $A$, $B \in \Sigma$ *such that* $\mu(A) = \mu(B) = 1$; *a function* $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is bijective, inequality* (3) *holds and there exists an* $r \in (0, 1)$ *such that* $\boldsymbol{P}_\varphi(rx) \leqslant r\boldsymbol{P}_\varphi(x)$ *for all* $x \in S_+$ *then* $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

This is a partial generalization of a theorem in [5] where $\boldsymbol{P}_\varphi$ is supposed to be positively homogeneous. A keystone of the proof is a recently obtained theorem which roughly speaking states that (under some weak regularity conditions) every real subadditive function $\mathbf{p}$ defined on a cone $\mathbf{C}$ in a linear space satisfying condition that there exists an $r \in (0, 1)$ such that $\mathbf{p}(rx) \leqslant r\mathbf{p}(x)$ for every $x \in \mathbf{C}$ must be positively homogeneous (cf. [8] and [9]). In the preparatory section 1 we give a sketch of the proof of this result.

## 1. Auxiliary results

Let $\mathbf{X}$ be a real linear space. A set $\mathbf{C} \subset \mathbf{X}$ is said to be a cone in $\mathbf{X}$ if $\mathbf{C} + \mathbf{C} \subset \mathbf{C}$ and $t\mathbf{C} \subset \mathbf{C}$ for every $t > 0$.

LEMMA 1. *Let $\mathbf{X}$ be a real linear space and $\mathbf{C}$ a cone in $\mathbf{X}$. If $\mathbf{p}: \mathbf{C} \to \boldsymbol{R}$ satisfies the following conditions:*

1°. *$\mathbf{p}$ is subadditive i.e. $\mathbf{p}(x+y) \leqslant \mathbf{p}(x) + \mathbf{p}(y)$ for all $x, y \in \mathbf{C}$;*

2°. *for every $x \in \mathbf{C}$ the function $f_x: (0, \infty) \to \boldsymbol{R}$ given by the formula*

$$f_x(t) := \mathbf{p}(tx), \qquad t > 0,$$

*is bounded above in a neighbourhood of a point;*

3°. *there exists an $r \in (0, 1)$ such that*

$$\mathbf{p}(rx) \leqslant r\mathbf{p}(x), \qquad x \in \mathbf{C},$$

*then $\mathbf{p}$ is positively homogeneous i.e. $\mathbf{p}(tx) = t\mathbf{p}(x)$ for all $t > 0$, $x \in \mathbf{C}$.*

PROOF. (Sketch) Take an arbitrary $x \in \mathbf{C}$. By 1° the function $f := f_x$ is subadditive in $(0, \infty)$. This together with 2° implies that $f$ is locally bounded above, (i.e. bounded above on every compact subset of $(0, \infty)$), and, consequently, locally bounded. Therefore (cf. [2], Theorem 7.6.1, p. 244 and the remark coming after its proof; also [3], p. 407)

$$\lim_{t \to \infty} \frac{f(t)}{t} = \inf_{t > 0} \frac{f(t)}{t}. \tag{4}$$

By induction from 3° we have

$$\frac{f(t)}{t} \leqslant \frac{f(r^{-n}t)}{r^{-n}t}, \qquad t > 0; \quad n \in \boldsymbol{N}.$$

Letting $n \to \infty$ and making use of (4) we hence obtain for all $t > 0$

$$\frac{f(t)}{t} \leqslant \inf_{t > 0} \frac{f(t)}{t},$$

which means that $f(t) = f(1)t$ for all $t > 0$. Now by the definition of $f$ we have

$$\mathbf{p}(tx) = f_x(t) = f(t) = f(1)t = f_x(1)t = \mathbf{p}(x)t$$

which was to be shown.

REMARK 1. The same argument permits us to get more general result. Namely, instead of 1° we can assume that for every $x \in \mathbf{C}$ the function $f_x$ is subadditive in $(0, \infty)$ and instead of 2° that for every $x \in \mathbf{C}$ there is $r_x \in (0, 1)$

such that every $t > 0$ we have $f_x(r_x tx) \leqslant r_x f_x(tx)$, (cf. [8] where a detailed proof is given).

We quote the following result due to T. Świątkowski and the present author (cf. [6]).

LEMMA 2. *Let $f: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a subadditive bijection. If $f$ is continuous at* 0 *then it is a homeomorphism of $\boldsymbol{R}_+$.*

REMARK 2. Let $x \in \mathrm{S}$. Then there exist disjoint $A_1, \ldots, A_k \in \Sigma$ and $x_1, \ldots, x_k \in \boldsymbol{R}$ such that

$$x = \sum_{i=1}^{k} x_i \chi_{A_i}; \qquad \mu(A_i) < \infty, \qquad (i = 1, \ldots, k).$$

($\chi_E$ denotes the characteristic function of a set $E$). For an arbitrary bijection $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ we have

$$\varphi \circ |x| = \sum_{i=1}^{k} \varphi(|x_i|)\chi_{A_i} + \varphi(0)\chi_{\Omega - A_i}.$$

If $\varphi(0) = 0$ then $x \in \mathrm{S} \Rightarrow \varphi \circ |x| \in \mathrm{S}_+$ and, consequently, the functional $\boldsymbol{P}_\varphi$ is well defined for every measure space $(\Omega, \Sigma, \mu)$.

It is easily seen that in the case when $\mu(\Omega) < \infty$ the functional $\boldsymbol{P}_\varphi$ is well defined by the formula (1) even when the condition $\varphi(0) = 0$ fails to hold. One can also avoid this assumption in the case $\mu(\Omega) = \infty$ modifying the formula (1) as follows

$$\boldsymbol{P}_\varphi(x) := \varphi^{-1}\left(\int_{\Omega_x} \varphi \circ |x| \, d\mu\right), \qquad x \in \mathrm{S},$$

where $\Omega_x := \{\omega \in \Omega: x(\omega) \neq 0\}$. Thus the assumption $\varphi(0) = 0$ in [4] was made to simplify the notations. From the next lemma it follows that it could be done without any loss of generality.

LEMMA 3. *Let $(\Omega, \Sigma, \mu)$ be a measure space with at least one set $A \in \Sigma$ of positive finite measure such that $\mu(A) \neq 1$ and $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ an arbitrary bijection satisfying inequality* (3). *Then $\varphi(0) = 0$.*

PROOF. Let $a := \mu(A)$. Putting in (3) $x = y := t\chi_A$, $t \geqslant 0$, we obtain

$$\varphi^{-1}(a\varphi(2t)) \leqslant 2\varphi^{-1}(a\varphi(t)), \qquad t \geqslant 0,$$

which means that the function $f := \varphi^{-1} \circ (a\varphi)$ satisfies the inequality

$$f(2t) \leqslant 2f(t), \qquad t \geqslant 0.$$

Since $f$ is a bijection of $\boldsymbol{R}_+$ there is a $t_0 \in \boldsymbol{R}_+$ such that $f(t_0) = 0$. From the above inequality we infer that $f(2t_0) = 0$ and, consequently, $f(2t_0) = f(t_0)$. Now the bijectivity of $f$ implies that $t_0 = 0$. Hence we get $\varphi^{-1}(a\varphi(0)) = 0$ and, since $a \neq 1$, $\varphi(0) = 0$. This completes the proof.

## 2. A characterization of $L^p$-norm

In this section we prove the following

THEOREM 1. *Let* $(\Omega, \Sigma, \mu)$ *be a measure space with at least two sets* $A$, $B \in \Sigma$ *such that*

$$A \cap B = \varnothing, \qquad \mu(A) = \mu(B) = 1, \tag{5}$$

*and suppose that* $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is bijective. If*

$$\boldsymbol{P}_\varphi(x + y) \leqslant \boldsymbol{P}_\varphi(x) + \boldsymbol{P}_\varphi(y), \qquad x,\ y \in \mathbf{S}_+, \tag{6}$$

*and there exists an* $r \in (0, 1)$ *such that for every* $x \in \mathbf{S}_+$

$$\boldsymbol{P}_\varphi(rx) \leqslant r\boldsymbol{P}_\varphi(x), \tag{7}$$

*then* $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

PROOF. To apply Lemma 1 put $\mathbf{X} := \boldsymbol{R}^2$, $\mathbf{C} := \boldsymbol{R}_+^2$ and define $\mathbf{p}\colon \mathbf{C} \to \boldsymbol{R}$ by

$$\mathbf{p}(x) := \boldsymbol{P}_\varphi(x_1\chi_A + x_2\chi_B), \qquad x = (x_1, x_2) \in \boldsymbol{R}_+^2.$$

From (6) and (7) the assumptions 1° and 3° of Lemma 1 are satisfied. To verify that condition 2° of this lemma is also fulfilled, we note that by the definitions of $\mathbf{p}$ and $\boldsymbol{P}_\varphi$ and (5) we get

$$\mathbf{p}(x) = \varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x = (x_1, x_2) \in \boldsymbol{R}_+^2. \tag{8}$$

As $\mathbf{p}$ is subadditive in $\mathbf{C}$ we have

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all nonnegative $x_1$, $x_2$, $y_1$, $y_2$. Since $\mu(A \cup B) = 2$ it follows from Lemma 3 that $\varphi(0) = 0$. Therefore substituting $y_1 = x_2 := 0$ we obtain $\varphi^{-1}(\varphi(x_1) + \varphi(y_2)) \leqslant x_1 + y_2$ or, equivalently,

$$\mathbf{p}(x) = \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) \leqslant x_1 + x_2, \qquad x_1,\ x_2 \geqslant 0. \tag{9}$$

Hence $f_x(t) := \mathbf{p}(tx) = \varphi^{-1}(\varphi(tx_1) + \varphi(tx_2)) \leqslant t(x_1 + x_2)$ which shows that condition 2° of Lemma 1 is fulfilled. According to this lemma we have $\mathbf{p}(tx) = t\mathbf{p}(x)$ for all $x \in \mathbf{C}$ and $t > 0$ which, in view of (8), can be written as

$$\varphi^{-1}(\varphi(tx_1) + \varphi(tx_2)) = t\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x_1,\ x_2 \geqslant 0; \quad t > 0.$$

Replacing here $x_1$ by $\varphi^{-1}(x_1)$ and $x_2$ by $\varphi^{-1}(x_2)$ and making use of the bijectivity of $\varphi$ we obtain

$$\varphi(t\varphi^{-1}(x_1 + x_2)) = \varphi(t\varphi^{-1}(x_1)) + \varphi(t\varphi^{-1}(x_2)), \qquad x_1, x_2 \geqslant 0, \quad t > 0,$$

which means that for every $t > 0$ the function $\varphi \circ (t\varphi^{-1})$ is additive. Since $\varphi \circ (t\varphi^{-1})$ is nonnegative, it must be a linear function (cf. J. Aczél [1], p. 34). Consequently, for every $t > 0$, there exists an $m(t) > 0$ such that

(10)
$$\varphi(t\varphi^{-1}(x)) = m(t)x, \qquad x \geqslant 0.$$

Note that this relation remains valid if we additionally define $m(0) := 0$. Take arbitrary $s, t \geqslant 0$. Composing the functions $\varphi \circ (s\varphi^{-1})$ and $\varphi \circ (t\varphi^{-1})$ and making use of relation (10) we get

$$\varphi(st\varphi^{-1}(x)) = m(s)m(t)x, \qquad x \geqslant 0.$$

On the other hand the same relation says that

$$\varphi(st\varphi^{-1}(x)) = m(st)x, \qquad x \geqslant 0.$$

Hence we infer that

$$m(st) = m(s)m(t), \qquad s, t \geq 0,$$

i.e. $m$ is multiplicative, and, in view of (10), $m$ is bijective and

$$\varphi^{-1}(t) = \varphi^{-1}(1)m^{-1}(t), \qquad t \geqslant 0.$$

Now from (8) and from the multiplicativity of $m$ and $m^{-1}$ we have

$$\mathbf{p}(x) = m^{-1}(m(x_1) + m(x_2)), \qquad x = (x_1, x_2) \in \boldsymbol{R}_+^2,$$

and, as $\mathbf{p}$ is subadditive,

(11)
$$m^{-1}(m(x_1 + y_1) + m(x_2 + y_2)) \leqslant m^{-1}(m(x_1) + m(x_2)) + m^{-1}(m(y_1) + m(y_2))$$

for all $x_1, x_2, y_1, y_2 \geqslant 0$. Setting here $x_1 = y_1 := s$ and $x_2 = y_2 := t$, we get

$$m^{-1}(2m(s + t)) \leqslant 2m^{-1}(m(s) + m(t)), \qquad s, t \geqslant 0.$$

From the multiplicativity of $m^{-1}$ we obtain

$$m^{-1}(2)(s + t) \leqslant 2m^{-1}(m(s) + m(t)), \qquad s, t \geqslant 0.$$

This implies that for $s, t \geqslant 0$ and $c := m^{-1}(2)/2$ we have

$$cm^{-1}(t) \leqslant m^{-1}(s + t), \qquad c > 0,$$

and, consequently,

$$c \cdot \limsup_{t \to 0} m^{-1}(t) \leqslant \inf \{m^{-1}(s): s > 0\}.$$

Since $m$ is bijective it follows that

$$\lim_{t \to 0} m^{-1}(t) = 0 = m^{-1}(0)$$

i.e. the function $m^{-1}$ is continuous at 0. Setting in (11): $x_1 := m^{-1}(s)$, $y_2 := m^{-1}(t)$, $x_2 = y_1 := 0$ we get

$$m^{-1}(s+t) \leqslant m^{-1}(s) + m^{-1}(t), \qquad s,\ t \geqslant 0,$$

i.e. $m^{-1}$ is subadditive in $\boldsymbol{R}_+$. By Lemma 2, $m^{-1}$ is a homeomorphism of $\boldsymbol{R}_+$. Consequently (cf. J. Aczél [1], p. 41), there is a $p > 0$ such that $m(t) = t^p$ for all $t \geqslant 0$. Hence $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, which completes the proof.

REMARK 3. It is quite obvious that condition (7) of Theorem 1 is fulfilled if there exists an $r > 1$ such that for every $x \in \mathbf{S}_+$:

$$\boldsymbol{P}_\varphi(rx) \geqslant r\boldsymbol{P}_\varphi(x).$$

Moreover, according to Remark 1, both these conditions can be replaced by more general ones.

Taking in Theorem 1 the measure space $(\Omega, \Sigma, \mu)$ such that $\Omega := \{1, 2\}$; $\Sigma := 2^\Omega$; $\mu(\{1\}) = \mu(\{2\}) := 1$ and making use of Remark 3 we obtain the following

COROLLARY 1. *Let* $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *be a bijection such that*

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

*for all nonnegative* $x_1, y_1, x_2, y_2$. *If there exists an* $r \in (0, 1)$, *(resp.* $r > 1$*), such that*

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) \leqslant r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x_1, x_2 \geqslant 0,$$

*(resp. the reversed inequality holds), then* $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, *for some* $p \geqslant 1$.

REMARK 4. If a bijection $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfies the functional equation

$$\varphi(rt) = \rho\varphi(t), \qquad t > 0,$$

for some positive $r$ and $\rho$, $r \neq \rho$, then

$$\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) = r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)), \qquad x_1, x_2 \geqslant 0.$$

Indeed, we have $\varphi^{-1}(\rho t) = r\varphi^{-1}(t)$, $(t > 0)$, and, therefore

$$\begin{aligned}\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) &= \varphi^{-1}(\rho[\varphi(x_1) + \varphi(x_2)]) \\ &= r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)).\end{aligned}$$

## 3. A converse of Minkowski's inequality

In the previous section we have proved that if the functional $\boldsymbol{P}_\varphi$ satisfies the triangle inequality and a kind of substitute of the homogeneity condition, (cf. e.g. (7)), then $\varphi$ must be a power function. Now we assume that $\boldsymbol{P}_\varphi$ satisfies only the triangle inequality.

The main result of this section reads as follows.

THEOREM 2. *Let $(\Omega, \Sigma, \mu)$ be a measure space such that there exist $A$, $B$, $C \in \Sigma$ and $m$, $n \in \boldsymbol{N}$, $m \neq n$, satisfying the following conditions:*

$$A \cap B = \varnothing; \qquad \mu(A) = \frac{m}{n}; \qquad \mu(B) = \frac{n}{m}; \qquad \mu(C) = n.$$

*If $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a bijection such that*

$$\boldsymbol{P}_\varphi(x + y) \leqslant \boldsymbol{P}_\varphi(x) + \boldsymbol{P}_\varphi(y), \qquad x,\ y \in \mathbf{S}_+,$$

*then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.*

PROOF. By Lemma 3 we have $\varphi(0) = 0$. Hence, substituting in the assumed triangle inequality

$$x := \varphi^{-1}\left(\frac{s}{\mu(A)}\right)\chi_A; \qquad y := \varphi^{-1}\left(\frac{t}{\mu(B)}\right)\chi_B,$$

we get

$$\varphi^{-1}(s + t) \leqslant \varphi^{-1}(s) + \varphi^{-1}(t), \qquad s,\ t \geqslant 0,$$

i.e. $\varphi^{-1}$ is subadditive. By induction we have for every $k \in \boldsymbol{N}$

$$\varphi^{-1}(t_1 + \cdots + t_k) \leqslant \varphi^{-1}(t_1) + \cdots + \varphi^{-1}(t_k), \qquad t_1, \ldots, t_k \geqslant 0.$$

Setting here $t_1 = \cdots = t_k := t$ we get $\varphi^{-1}(kt) \leqslant k\varphi^{-1}(t)$ and, consequently,

$$\varphi^{-1}(k\varphi(t)) \leqslant kt, \qquad k \in \boldsymbol{N}, \quad t \geqslant 0.$$

This implies that for every $k \in \boldsymbol{N}$ the function $\varphi^{-1} \circ (k\varphi)$ is continuous at 0. Substituting in the triangle inequality in turn

$$x := s\chi_A, \quad y := t\chi_A; \qquad x := s\chi_B, \quad y := t\chi_B; \qquad x := s\chi_C, \quad y := t\chi_C,$$

we infer that the functions $\varphi^{-1} \circ \left(\frac{m}{n}\varphi\right)$, $\varphi^{-1} \circ \left(\frac{n}{m}\varphi\right)$ and $\varphi^{-1} \circ (n\varphi)$ are subadditive in $\boldsymbol{R}_+$. From Lemma 2 it follows that $\varphi^{-1} \circ (n\varphi)$ is a homeomorphism of $\boldsymbol{R}_+$. Since the composition of an increasing subadditive function and subadditive one is subadditive, the relation

$$\varphi^{-1} \circ (m\varphi) = (\varphi^{-1} \circ (n\varphi)) \circ \left(\varphi^{-1} \circ \left(\frac{m}{n}\varphi\right)\right)$$

implies that $\varphi^{-1} \circ (m\varphi)$ is subadditive and, by Lemma 2, a homeomorphism of $\boldsymbol{R}_+$. The function $\varphi^{-1} \circ \left(\frac{1}{n}\varphi\right)$ being the inverse of $\varphi^{-1} \circ (n\varphi)$ is a homeomorphism of $\boldsymbol{R}_+$. Now the relation

$$\varphi^{-1} \circ \left(\frac{m}{n}\varphi\right) = (\varphi^{-1} \circ (m\varphi)) \circ \left(\varphi^{-1} \circ \left(\frac{1}{n}\varphi\right)\right)$$

implies that $\varphi^{-1} \circ \left(\frac{m}{n}\varphi\right)$ and its inverse $\varphi^{-1} \circ \left(\frac{n}{m}\varphi\right)$ are homeomorphisms. Because these functions are inverses of one another and subadditive, they must be superadditive and, consequently, additive. Therefore (cf. J. Aczél [1], p. 34) there exists an $r > 0$ such that

$$\varphi^{-1}\left(\frac{m}{n}\varphi(t)\right) = rt, \qquad t \geqslant 0.$$

Denoting $a := \mu(A) = \frac{m}{n}$, we hence get

$$a\varphi(t) = \varphi(rt), \qquad \varphi^{-1}(at) = r\varphi^{-1}(t), \qquad t \geqslant 0. \tag{12}$$

Setting in the triangle inequality

$$x := x_1\chi_A + x_2\chi_B, \qquad y := y_1\chi_A + y_2\chi_B; \qquad x_1,\ x_2,\ y_1,\ y_2 \geqslant 0,$$

and taking into account that $A \cap B = \varnothing$ and $\mu(B) = \frac{1}{a}$ we obtain

$$\varphi^{-1}\left(a\varphi(x_1 + y_1) + \frac{1}{a}\varphi(x_2 + y_2)\right) \leqslant \varphi^{-1}\left(a\varphi(x_1) + \frac{1}{a}\varphi(x_2)\right) + \varphi^{-1}\left(a\varphi(y_1) + \frac{1}{a}\varphi(y_2)\right).$$

Applying (12) we can write this inequality as follows

$$\varphi^{-1}\left(\varphi(rx_1 + ry_1) + \varphi\left(\frac{1}{r}x_2 + \frac{1}{r}y_2\right)\right) \leqslant \varphi^{-1}\left(\varphi(rx_1) + \varphi\left(\frac{1}{r}x_2\right)\right) + \varphi^{-1}\left(\varphi(ry_1) + \varphi\left(\frac{1}{r}y_2\right)\right).$$

Replacing here $rx_1$, $r^{-1}x_2$, $ry_1$, $r^{-1}y_2$ resp. by $x_1$, $x_2$, $y_1$, $y_2$ we get

$$\varphi^{-1}(\varphi(x_1 + y_1) + \varphi(x_2 + y_2)) \leqslant \varphi^{-1}(\varphi(x_1) + \varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + \varphi(y_2))$$

for all nonnegative $x_1$, $x_2$, $y_1$, $y_2$. Applying once more (12) we obtain

$$\begin{aligned}\varphi^{-1}(\varphi(rx_1) + \varphi(rx_2)) &= \varphi^{-1}(a\varphi(x_1) + a\varphi(x_2)) \\ &= \varphi^{-1}(a[\varphi(x_1) + \varphi(x_2)]) \\ &= r\varphi^{-1}(\varphi(x_1) + \varphi(x_2)).\end{aligned}$$

Now our theorem results from Corollary 1 because, clearly, $r \neq 1$.

If in the above theorem $n = 1$ we can take $C = B$. Therefore we have the following

COROLLARY 2. *Let $(\Omega, \Sigma, \mu)$ be a measure space such that there exist $A$, $B \in \Sigma$ and $m \in N$, $m \neq 1$, satisfying the following conditions*:

$$A \cap B = \varnothing; \qquad \mu(A) = m; \qquad \mu(B) = \frac{1}{m}.$$

*If $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a bijection such that*

$$\boldsymbol{P}_\varphi(x + y) \leqslant \boldsymbol{P}_\varphi(x) + \boldsymbol{P}_\varphi(y), \qquad x,\ y \in S_+,$$

*then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.*

Finally let us note that using Lemma 3 we can write the converse of Minkowski's inequality quoted in the introduction in a little more general form (cf. [4]).

THEOREM 3. *Let $(\Omega, \Sigma, \mu)$ be a measure space with at least two sets $A$, $B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$. If $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a bijection such that $\varphi^{-1}$ is continuous at 0 and*

$$\boldsymbol{P}_\varphi(x + y) \leqslant \boldsymbol{P}_\varphi(x) + \boldsymbol{P}_\varphi(y), \qquad x,\ y \in S_+,$$

*then $\varphi(t) = \varphi(1)t^p$, $(t \geqslant 0)$, for some $p \geqslant 1$.*

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