HIROSHIMA MATH. J.
27 (1997), 221–228

Symmetricity of the Whitehead element

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

Mitsunori IMAOKA* and Yusuke KAWAMOTO

(Received February 14, 1996)

ABSTRACT. We study the symmetricity of the Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$ for an odd prime p. It is shown that w_n considered as a map $S^{2np-3} \to S^{2n-1}$ factors through the p-fold covering map $\sigma: S^{2np-3} \to L^{2np-3}$ only when n is a power of p, and that w_{p^i} actually factors through σ if $0 \leq i \leq 4$. This is some of an odd prime version of the results of Randall and Lin for the projectivity of the Whitehead product $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$.

1. Introduction

Let p be a prime, and $\sigma: S^{2n+1} \to L^{2n+1}$ denote the p-fold covering, where $L^{2n+1} = S^{2n+1}/\boldsymbol{Z}_p$ is the standard lens space. For any space X, an element $\alpha \in \pi_{2n+1}(X)$ is defined to be *symmetric*, if α considered as a map $S^{2n+1} \to X$ factors through $\sigma: S^{2n+1} \to L^{2n+1}$, that is, there exists a map $g: L^{2n+1} \to X$ with $\alpha = [g\sigma]$. Mimura-Mukai-Nishida [8] have shown that all elements in the positive dimensional stable homotopy groups of spheres are symmetric.

In this paper, we study the symmetricity of the Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$ for an odd prime p. Hence, all spaces are assumed to be localized at an odd prime p. We recall the definition of w_n (cf. [3], [4]). Let $\varepsilon: C(n) \to S^{2n-1}$ be the homotopy fiber of the double suspension map $\Sigma^2: S^{2n-1} \to \Omega^2 S^{2n+1}$. It is known that $C(n)$ is $(2np-4)$-connected and $\pi_{2np-3}(C(n)) \cong \boldsymbol{Z}_p$. For a generator $z \in \pi_{2np-3}(C(n))$, w_n is given by $w_n = \varepsilon_*(z) \in \pi_{2np-3}(S^{2n-1})$. Then, our results are stated as follows:

THEOREM A. *If the Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$ is symmetric, then $n = p^i$ for some $i \geq 0$.*

THEOREM B. *The Whitehead element $w_{p^i} \in \pi_{2p^{i+1}-3}(S^{2p^i-1})$ is symmetric for $0 \leq i \leq 4$.*

1991 *Mathematics Subject Classification.* Primary 55Q15, secondary 55S10
Key words and phrases. Whitehead element, symmetric, lens space
*Partially supported by Grant-Aid for Scientific Research, No. 07404002

Theorem A corresponds to the result of Randall [9], who shows that the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ is symmetric, for the prime 2, only when n or $n+1$ is a power of 2. In this case, the symmetric is refered to as the projective. Milgram-Zvengrowski [7] have shown that $[\iota_{2^i}, \iota_{2^i}]$ is projective iff $i = 0, 1, 2$, and Lin [6] has concluded that $[\iota_{2^i-1}, \iota_{2^i-1}]$ is actually projective for any $i > 0$. Theorem B corresponds to such solutions, but the whole analogy with the methods in [6] does not hold in the case of odd primes. We shall show that the cases as in Theorem B are obtainable applying the results of Cohen [1].

We prove Theorem A in §2, Theorem B in §3, and §4 is devoted to establish a key lemma for the proof of Theorem B. Throughout the paper, $\boldsymbol{Z}_p$ denotes the cyclic group of order p and also the additive group of the mod p integers.

The authors wish to express their thanks to Professor Takao Matumoto for his valuable suggestions.

2. Proof of Theorem A

We shall apply the following proposition in the case that X is a stunted lens space, and the proposition is crucial also in the proof of Theorem B.

PROPOSITION 1 [4; Prop. C]. *Suppose that a CW-complex X is $(2n-1)$-connected and* $\dim X \leq 2np-3$. *Then, for any map $\eta: S^{2np-3} \to X$ with $\eta_* = 0: H_{2np-3}(S^{2np-3}; \boldsymbol{Z}_p) \to H_{2np-3}(X; \boldsymbol{Z}_p)$, the following conditions* (1) *and* (2) *are equivalent*:

(1) *There exists a map $\kappa: X \to S^{2n-1}$ with $w_n = [\kappa\eta]$;*

(2) *There exists a map $\omega: \Sigma^2 C_\eta \to S^{2n+1}$ with $\mathscr{P}^n \neq 0$ on $H^{2n+1}(C_\omega; \boldsymbol{Z}_p)$, where C_α is the cofiber of $\alpha = \eta$ or ω and $\mathscr{P}^n \in \mathscr{A}$ is the Steenrod operation over $\boldsymbol{Z}_p$.*

Let $L = S^\infty/\boldsymbol{Z}_p$ be the infinite dimensional lens space, and L^a for $a \geq 0$ denote the a-skeleton of L. Then, $L_l^k = K^k/L^{l-1}$ for $0 < l \leq k$ is the stunted lens space, and the composition of the double covering map $\sigma: S^{2k-1} \to L^{2k-1}$ with the collapsing map $L^{2k-1} \to L_l^{2k-1}$ is the attaching map $\sigma: S^{2k-1} \to L_l^{2k-1}$ of the top cell in L_l^{2k}. Recall that $H^*(L; \boldsymbol{Z}_p) = \Lambda_{\boldsymbol{Z}_p}(x) \otimes \boldsymbol{Z}_p[y]$ with $\beta x = y$, where the degrees of x and y are 1 and 2 respectively and β is the Bockstein operation. Then, we remark

LEMMA 2. *w_n is symmetric if and only if there exists a map $\kappa: L_{2n}^{2np-3} \to S^{2n-1}$ with $w_n = [\kappa\sigma]$ for the attaching map $\sigma: S^{2np-3} \to L_{2n}^{2np-3}$.*

PROOF. The if part is clear, so we assume that w_n is symmetric. Then, by the dimensional reason, there exists a map $g: L_{2n-1}^{2np-3} \to S^{2n-1}$ with $w_n =$

$[g\sigma]$. For the inclusion $i: S^{2n-1} \to L^{2np-3}_{2n-1}$, we have $p[gi] = 0 \in \pi_{2n-1}(S^{2n-1})$, because L^{2n}_{2n-1} is the cofiber of a map $S^{2n-1} \to S^{2n-1}$ of degree p. Hence, $[gi] = 0$ and we have a required map κ with $w_n = [\kappa\sigma]$. $\square$

Now, put $n = p^t + u$ for $0 < u < p^t(p-1)$, and assume that the Whitehead element $w_n \in \pi_{2np-3}(S^{2n-1})$ is symmetric. We shall verify Theorem A by inducing a contradiction from this assumption.

By applying Proposition 1 in the case of $X = L^{2np-3}_{2n}$ and using Lemma 2, we have a map $\omega: \Sigma^2 L^{2np-2}_{2n} \to S^{2n+1}$ with $\mathscr{P}^n \neq 0: H^{2n+1}(C_\omega; Z_p) \to H^{2np+1}(C_\omega; Z_p)$. Then, by the cofiber sequence $S^{2n+1} \to C_\omega \to \Sigma^3 L^{2np-2}_{2n}$, we have isomorphisms $H^{2n+1}(C_\omega; Z_p) \cong Z_p$ and $H^i(C_\omega; Z_p) \cong H^{i-3}(L^{2np-2}_{2n}; Z_p)$ for $i \geq 2n+3$. We denote the generator of $H^{2n+1}(C_\omega; Z_p) \cong Z_p$ by a, and identify the generator of $H^{2k+3}(C_\omega; Z_p)$ for $n \leq k \leq np-1$ with $y^k \in H^{2k}(L^{2np-2}_{2n}; Z_p) \cong Z_p$. Then, $\mathscr{P}^n(a) \equiv y^{np-1}$ up to unit.

Let $u = u_1 p^{t_1} + \cdots + u_l p^{t_l}$ be the p-adic expansion of u. Thus, $0 < u_i \leq p-1$, $t \geq t_1 > \cdots > t_l \geq 0$, and $0 < u_1 \leq p-2$ if $t_1 = t$. The Adem relation gives

$$\mathscr{P}^u \mathscr{P}^{p^t}(a) = \sum_{i=0}^{[u/p]} (-1)^{u+i} c_i \mathscr{P}^{n-i} \mathscr{P}^i(a) \qquad \text{for } c_i = \binom{(p-1)(p^t-i)-1}{u-pi}. \tag{2.1}$$

Then,

$$\begin{aligned} c_0 &= \binom{(p-1)p^t - 1}{u} \\ &= \binom{(p-2)p^t + (p-1)p^{t-1} + \cdots + (p-1)p + (p-1)}{u_1 p^{t_1} + \cdots + u_l p^{t_l}} \not\equiv 0 \mod p, \end{aligned}$$

and thus

$$c_0 \mathscr{P}^n(a) \neq 0. \tag{2.2}$$

On the other hand, $\mathscr{P}^{p^t}(a) = a_{p^t} y^{p^{t+1}+u-1}$ for some $a_{p^t} \in Z_p$, and $\mathscr{P}^u(y^{p^{t+1}+u-1}) = \binom{p^{t+1}+u-1}{u} y^{np-1} = 0$. Hence,

$$\mathscr{P}^u \mathscr{P}^{p^t}(a) = 0. \tag{2.3}$$

For $1 \leq i \leq [u/p]$ and some $a_i \in Z_p$, we have $\mathscr{P}^i(a) = a_i y^{n-1+i(p-1)}$ and $\mathscr{P}^{n-i}(y^{n-1+i(p-1)}) = b_i y^{np-1}$ for $b_i = \binom{n-i+ip-1}{n-i}$. Then, $b_i \not\equiv 0 \bmod p$ if and only if $\alpha_p(n-i) + \alpha_p(ip-1) = \alpha_p(n-i+ip-1)$, where $\alpha_p(k) = \sum_{j=0}^h k_j$ for the p-adic expansion of an integer $k = \sum_{j=0}^h k_j p^j$. If we put $i = i_1 p^{j_1} + \cdots + i_m p^{j_m}$ for $j_1 > \cdots > j_m$ as the p-adic expansion of i, then we have the following:

$$ip - 1 = i_1 p^{j_1+1} + \cdots + i_{m-1} p^{j_{m-1}+1} + (i_m - 1)p^{j_m+1} + (p-1)p^{j_m} + \cdots + (p-1);$$

$$n - i = (p^t + u_1 p^{t_1} + \cdots + u_l p^{t_l}) - (i_1 p^{j_1} + \cdots + i_m p^{j_m}).$$

Hence, if $\alpha_p(n-i) + \alpha_p(ip-1) = \alpha_p(n-i+ip-1)$, then $t_l = j_m$ and $u_l = i_m$, and we can set $u = vp^{b+1} + dp^b$ and $i = jp^{b+1} + dp^b$ in this case for some v, $j > 0$ and $0 < d \leq p-1$, where $b = t_l = j_m$. Then, we have

$$c_i \equiv \binom{ep^{b+1} + (d-1)p^b + (p-1)p^{b-1} + \cdots + (p-1)}{fp^{b+1} + dp^b} \equiv 0 \mod p$$

for some e, $f > 0$. Thus, for $1 \leq i \leq [u/p]$, we have

$$c_i \mathscr{P}^{n-i} \mathscr{P}^i(a) = 0. \tag{2.4}$$

(2.2)–(2.4) contradict (2.1), and we have completed the proof of Theorem A.

3. Proof of Theorem B

First, we remark that $w_1 = 0$ and that, by [10; Th. 7.1], $w_p \in \pi_{2p^2-3}(S^{2p-1})$ is divisible by p. If $w_p = pw$, then $w_p = w[q\sigma]$ for the collapsing map $q: L^{2p^2-3} \to S^{2p^2-3}$, and thus Theorem B trivially holds for w_1 and w_p.

We shall show that w_i for $2 \leq i \leq 4$ is symmetric, by applying a method due to Lin [6] and some results of Cohen [1]. For $m \geq 1$, let $B(p^m)$ be a spectrum whose cohomology is given by

$$H^*(B(p^m); \boldsymbol{Z}_p) \cong \mathscr{A}/\mathscr{A}\{\chi(\beta^\varepsilon \mathscr{P}^j) | \varepsilon + j > p^{m-1}\}$$

as $\mathscr{A}$-modules, where χ is the canonical anti-automorphism of $\mathscr{A}$. We may call $B(p^m)$ the Brown-Gitler spectrum, although it is slightly different from the original one. The existence of the spectrum $B(p^m)$ is established in [1], and also the following is shown in [1; Ch. 4, Th. 2.1]:

PROPOSITION 3. *For $m \geq 2$, there exists a stable map $\zeta_m : \Sigma^{2p^{m-1}(p^2-p-1)} B(p^{m-1}) \to S^0$ with $\mathscr{P}^{p^m} \neq 0 : H^0(C_{\zeta_m}; \boldsymbol{Z}_p) \to H^{2p^m(p-1)}(C_{\zeta_m}; \boldsymbol{Z}_p)$.*

Henceforce, we assume that, for a given integer $i > 0$, the integers t and s always denote

$$t = 2p^{i+1} - 2 \quad \text{and} \quad s = 2p^{i+1} - 2p^{i-1} - 1. \tag{3.1}$$

By Proposition 1, if we show that there exists a map $\xi: \Sigma^2 L_s^t \to S^{2p^i+1}$ for $2 \leq i \leq 4$ with $\mathscr{P}^{p^i} \neq 0 : H^{2p^i+1}(C_\xi; \boldsymbol{Z}_p) \to H^{2p^{i+1}+1}(C_\xi; \boldsymbol{Z}_p)$, then we get a map $\kappa: L_s^{2p^{i+1}-3} \to S^{2p^i-1}$ with $w_{p^i} = [\kappa\sigma]$, which establishes Theorem B. Here, we remark that it is enough to find the map ξ as is a stable map

$$\xi: L_s^t \to S^{2p^i-1} \quad \text{with} \quad \mathscr{P}^{p^i} \neq 0 : H^{2p^i-1}(C_\xi; \boldsymbol{Z}_p) \to H^{2p^{i+1}-1}(C_\xi; \boldsymbol{Z}_p). \tag{3.2}$$

In fact, the suspension homomorphism $[\Sigma^2 L_s^t, S^{2p^i+1}] \to [\Sigma^{2N} L_s^t, S^{2N+2p^i-1}]$ is bijective for any $N \geq 1$, because $C(p^i + m)$ is $(2(p^i + m)p - 4)$-connected for any $m \geq 1$.

Thus, Theorem B follows from the following proposition, in which ζ_i is the stable map of Proposition 3.

PROPOSITION 4. *For $2 \leq i \leq 4$, there exists a stable map $\psi : L_s^t \to \Sigma^s B(p^{i-1})$ such that a stable map ξ of (3.2) is taken as the composition $(\Sigma^{2p^i-1}\zeta_i)\psi$.*

We prepare some lemmas concerning the stunted lens spaces before the proof of Proposition 4. When $a < 0$ and $a \leq b$, the stunted lens space L_a^b means a spectrum $\Sigma^{-2p^N} L_{2p^N+a}^{2p^N+b}$ for sufficiently large $N > 0$ using the James periodicity. Indeed, since the J-order of the canonical complex line bundle over L^{b-a} is $p^{[(b-a)/(p-1)]}$ by [5], we have only to take N satisfying $N \geq [(b-a)/(p-1)]$ and $2p^N + a > 0$.

For a given $i > 0$ and $0 < a < b \leq 2p^{i+1}$, we define $\bar{L}_a^b$ to be the spectrum $\Sigma^{2p^{i+1}} L_{-2p^{i+1}+a}^{-2p^{i+1}+b}$. Then, by taking $M = p^{2(p^{i+1}-1)/(p-1)-(i+1)} - 1$, it is also represented $\bar{L}_a^b = \Sigma^{-2Mp^{i+1}} L_{2Mp^{i+1}+a}^{2Mp^{i+1}+b}$. We put $\bar{y}^j = y^{Mp^{i+1}+j} \in H^{2j}(\bar{L}_a^b; Z_p)$ for $a \leq 2j \leq b$. Define a map $\Phi : H^*(L_a^b; Z_p) \to H^*(\bar{L}_a^b; Z_p)$ by $\Phi(x^\varepsilon y^j) = x^\varepsilon \bar{y}^j$ for $a \leq \varepsilon + 2j \leq b$ and $\varepsilon = 0$ or 1. Then, it is easy to show the following lemma, by which $H^*(\bar{L}_a^b; Z_p)$ is an unstable $\mathscr{A}$-module:

LEMMA 5. *For any $i > 0$ and $0 < a < b \leq 2p^{i+1}$, $\Phi : H^*(L_a^b; Z_p) \to H^*(\bar{L}_a^b; Z_p)$ is an isomorphism of $\mathscr{A}$-modules.*

The following is the key lemma for the proof of Proposition 4, and Lemma 5 is used in the proof of the lemma.

LEMMA 6. *For $2 \leq i \leq 4$, there exists a stable map $\varphi : S^{2p^{i-1}} \to B(p^{i-1}) \wedge \bar{L}_1^{2p^{i-1}}$ such that $\varphi^*(1 \otimes \bar{y}^{p^{i-1}}) \neq 0$.*

We postpone the proof of Lemma 6 until the next section, and complete the proof of Proposition 4 by assuming Lemma 6.

PROOF OF PROPOSITION 4. Since there is a Spainer-Whitehead duality $D : S^0 \to \bar{L}_1^{2p^{i-1}} \wedge \Sigma^{-2p^{i+1}+1} L_s^t$, we have an isomorphism $\{L_s^t, \Sigma^s B(p^{i-1})\} \cong \pi_{2p^{i-1}}^S(B(p^{i-1}) \wedge \bar{L}_1^{2p^{i-1}})$, where t and s are the integers of (3.1). Hence, corresponding to φ of Lemma 6, there exists a stable map $\psi : L_s^t \to \Sigma^s B(p^{i-1})$ which satisfies

$$\psi^* \neq 0 : H^s(\Sigma^s B(p^{i-1}); Z_p) \to H^s(L_s^t, Z_p).$$

Thus, $\psi^*(1) \equiv xy^{p^{i+1}-p^{i-1}-1}$ up to unit. Then, it also holds that

$$\psi^* \neq 0 : H^t(\Sigma^s B(p^{i-1}); Z_p) \to H^t(L_s^t; Z_p). \tag{3.3}$$

In fact, by Davis [2], the equality $\chi(\mathscr{P}^{p^j}\cdots\mathscr{P}^p\mathscr{P}^1) = \mathscr{P}^{p^j+\cdots+p+1}$ holds for any $j \geq 0$. Then, $\psi^*(\chi(\mathscr{P}^{p^{i-2}}\cdots\mathscr{P}^p\mathscr{P}^1\beta)) = \beta\mathscr{P}^{p^{i-2}+\cdots+p+1}\psi^*(1) \equiv y^{(t/2)}$ up to unit, and thus (3.3) follows. Now, we can show that ψ is the required map.

Let $\xi : L_s^t \to S^{2p^i-1}$ be the composition of $\psi : L_s^t \to \Sigma^s B(p^{i-1})$ and $\Sigma^{2p^i-1}\zeta_i$: $\Sigma^s B(p^{i-1}) \to S^{2p^i-1}$, where ζ_i is the stable map of Proposition 3. Then, we have the following commutative diagram:

$$\begin{array}{ccccc}
H^{2p^i-1}(\Sigma^{2p^i-1}C_{\zeta_i}) & \xrightarrow{\mathscr{P}^{p^i}} & H^{2p^{i+1}-1}(\Sigma^{2p^i-1}C_{\zeta_i}) & \xleftarrow[\cong]{} & H^t(\Sigma^s B(p^{i-1})) \\
\cong\downarrow & & \downarrow & & \downarrow\psi^* \\
H^{2p^i-1}(C_\xi) & \xrightarrow{\mathscr{P}^{p^i}} & H^{2p^{i+1}-1}(C_\xi) & \xleftarrow[\cong]{} & H^t(L_s^t),
\end{array}$$

where all cohomology groups are taken with Z_p-coefficients. Since $H^t(\Sigma^s B(p^{i-1}); Z_p) \cong Z_p$ is generated by $\chi(\mathscr{P}^{p^{i-2}}\cdots\mathscr{P}^p\mathscr{P}^1\beta)$, Proposition 3 and (3.3) yield $\mathscr{P}^{p^i} \neq 0 : H^{2p^i-1}(C_\xi; Z_p) \to H^{2p^{i+1}-1}(C_\xi; Z_p)$, and we have completed the proof. □

4. An Adams spectral sequence

In this section, we stablish Lemma 6. Let $\{E_r^{q,u}(p^k, X)\} \Rightarrow \pi_*^S(B(p^k) \wedge X)$, for a spectrum X, be an Adams spectral sequence given as in [1]. In [1] the spectral sequence is used in the case of $X = L$ the infinite dimensional lens space, but we shall apply the spectral sequence for the stunted lens spaces.

More precisely, the E_1-term of it is given by

$$E_1^{q,u}(p^k, X) = \sum_{j\geq 0} \Lambda_{u-q-j}^q(p^k) \otimes H_j(X; Z_p).$$

Here, $\Lambda_a^b(p^k)$ is an algebra given as follows: Let Λ be the Λ-algebra, that is, Λ is an associative graded algebra over Z_p with generators λ_m of degree $2m(p-1)-1$ for $m \geq 1$; μ_n of degree $2n(p-1)$ for $n \geq 0$; subject to the so-called Adem relations (see [1; Ch. 1, §1]), where we have changed the notations and the gradings from those in [1] (λ_m and μ_n are denoted in [1] by λ_{m-1} and μ_{n-1} of degrees $-2m(p-1)+1$ and $-2n(p-1)$ respectively). Let $I(k)$ be the left ideal generated by $\{\lambda_m, \mu_n | m \leq p^{k-1}, n \leq p^{k-1}-1\}$. Then, $(\Lambda/I(k))^b$ denotes the submodule of $\Lambda/I(k)$ generated by the monomials of λ_m or μ_n with length b, and $\Lambda_a^b(p^k)$ is the component of degree a in $(\Lambda/I(k))^b$.

As a Z_p-vector space, $\Lambda_a^b(p^k)$ has a basis formed by some admissible monomials. Let $v_m = \lambda_m$ or μ_m. Then, the monomial $v_{m_1}\cdots v_{m_b}$ of $(\Lambda/I(k))^b$ is admissible if, for each j with $1 \leq j \leq b-1$, $pm_j \geq m_{j+1}+1$ or $pm_j \geq m_{j+1}$ holds according as $v_{m_j} = \lambda_{m_j}$ or $v_{m_j} = \mu_{m_j}$ ([1; Ch. I, §1]). Then, a basis of $\Lambda_a^b(p^k)$ consists of the admissible monomials $v_{m_1}\cdots v_{m_b}$ of degree a with $m_b \geq p^{k-1}+1$ or p^{k-1} according as $v_{m_b} = \lambda_{m_b}$ or μ_{m_b} by [1; Ch. III, Lemma 3.1]. As a result, the element which has the lowest degree in $(A/I(k))^b$ is $\mu_{p^{k-b}}\mu_{p^{k-b+1}}\cdots\mu_{p^{k-2}}\mu_{p^{k-1}}$. Thus, we have the following:

LEMMA 7. *$A_a^b(p^k) = 0$ if $a < 2(p^k - p^{k-b})$.*

Now, for a fixed $l \geq 0$, we put $L(l, k) = \Sigma^{-2Mp^{l+1}} L_{2Mp^{l+1}+1}^{2Mp^{l+1}+2p^k}$ for $0 \leq k \leq l$, where $M = p^{2(p^{l+1}-1)/(p-1)-(l+1)} - 1$, and consider the spectral sequence

$$E_r^{q,u}(n, k) = E_r^{q,u}(p^n, L(l, k)) \Rightarrow \pi_*^S(B(p^n) \wedge L(l, k)).$$

Let $(y^{p^m})^* \in H_{2p^m}(L(l, k); Z_p)$ be the element dual to $\bar{y}^{p^m}$ for $0 \leq m \leq k$. Then, by [1; Ch. III, Lemma 3.5], we see that

$$(3.4) \qquad d_1(1 \otimes (y^{p^m})^*) = 0 \qquad \text{in} \quad E_1^{1,2p^m}(m, m).$$

By [1; Ch. III, Th. 4.1], there exists a stable map $f_k: B(p^k) \to \Sigma^{2p^{k-1}(p-1)} B(p^{k-1})$ for $k \geq 2$ such that $(f_k)^*: H^*(B(p^{k-1}); Z_p) \to H^{*+2p^{k-1}(p-1)}(B(p^k); Z_p)$ is multiplication on the right by $\chi(\mathscr{P}^{p^{k-1}})$. Put $h_k = f_k \wedge 1: B(p^k) \wedge L(l, k) \to \Sigma^{2p^{k-1}(p-1)} B(p^{k-1}) \wedge L(l, k)$. Then, by [1; Ch. III, Lemma 3.8] and using Lemma 5, we have

$$(3.5) \qquad (h_k)_*(1 \otimes (y^{p^k})^*) = (1 \otimes (y^{p^{k-1}})^*).$$

Also, by [1; Ch. III, Cor. 3.7], if $q \geq 1$ and $u < q + 2p^k$, then

$$(3.6) \qquad (h_k)_* = 0: E_1^{q,u}(k, k) \to E_1^{q,u-2p^{k-1}(p-1)}(k-1, k).$$

We remark that the inclusion $i: L(k-1, k-1) \to L(k-1, k)$ induces a cohomology isomorphism up to dimension $2p^{k-1}$, and thus $i_*: E_r^{q,u-2p^{k-1}(p-1)}(k-1, k-1) \to E_r^{q,u-2p^{k-1}(p-1)}(k-1, k)$ is an isomorphism if $u < q + 2p^k$ and $q \geq 1$ or if $(q, u) = (0, 2p^k)$. Hence, by the identification through i_* for these q and u, $(h_k)_*$ can be regarded as $(h_k)_*: E_r^{q,u}(k, k) \to E_r^{q,u-2p^{k-1}(p-1)}(k-1, k-1)$. Then, applying (3.4)–(3.6), we have

LEMMA 8. *$1 \otimes (y^{p^k})^* \in E_{l-k+2}^{0,2p^k}(k, k)$ for $1 \leq k \leq l$.*

PROOF. Let k be fixed. By (3.4), $1 \otimes (y^{p^m})^* \in E_2^{0,2p^m}(m, m)$ for any m with $k \leq m \leq l$. Inductively, assume that, for some r with $2 \leq r \leq l - k$, $1 \otimes (y^{p^m})^* \in E_r^{0,2p^m}(m, m)$ holds for any m with $k \leq m \leq l + 2 - r$. Then, for any n with $k \leq n \leq l + 2 - (r + 1)$, $d_r(1 \otimes (y^{p^n})^*) = (h_{n+1})_*(d_r(1 \otimes (y^{p^{n+1}})^*)) = 0$ by (3.5) and (3.6), and hence $1 \otimes (y^{p^n})^* \in E_{r+1}^{0,2p^n}(n, n)$. Therefore, as for $1 \otimes (y^{p^k})^*$, we have $d_r(1 \otimes (y^{p^k})^*) = 0$ for $1 \leq r \leq l - k + 1$, which establishes the required result. □

Now, we can complete the proof of Lemma 6. Let $2 \leq i \leq 4$, and $(y^{p^{i-1}})^*$ denote the dual of $\bar{y}^{p^{i-1}} \in H^{2p^{i-1}}(\bar{L}_1^{2p^{i-1}}; Z_p)$. Then, applying Lemma 8 in the case of $l = i + 1$ and $k = i - 1$, we obtain that $1 \otimes (y^{p^{i-1}})^* \in E_4^{0,2p^{i-1}}(p^{i-1}, \bar{L}_1^{2p^{i-1}})$.

However, for $2 \leq i \leq 4$ and any $r \geq 4$, $E_1^{r,2p^{i-1}+r-1}(p^{i-1}, \bar{L}_1^{2p^{i-1}}) = 0$ by Lemma 7, and hence $d_r(1 \otimes (y^{p^{i-1}})^*) \in E_r^{r,2p^{i-1}+r-1}(p^{i-1}, \bar{L}_1^{2p^{i-1}}) = 0$. Therefore, $1 \otimes (y^{p^{i-1}})^*$ for $2 \leq i \leq 4$ is a permanent cycle, and represents an element $[\varphi] \in \pi_{2p^{i-1}}^S(B(p^{i-1}) \wedge \bar{L}_1^{2p^{i-1}})$. Then, we have $\varphi^*(1 \otimes \bar{y}^{p^{i-1}}) \neq 0$. Thus we have completed the proof.

REMARK. In our proof of Theorem B, the condition $i \leq 4$ is necessary only to show that $d_r(1 \otimes (y^{p^{i-1}})^*) = 0$ for any $r \geq 4$. However, it seems not so easy to deduce whether such differentials still vanish for $i \geq 5$ or not. Also, some formulas like those in [6; Prop. 2.4, 2.5] which are useful in the case of $p = 2$ do not have straightforward analogy for odd primes.

References

[1] R. L. Cohen, Odd primary infinite families in stable homotopy theory, Memoirs. A. M. S. **242** (1981).

[2] D. M. Davis, The antiautomorphism of the Steenrod algebra, Proc. Amer. Math. Soc. **44** (1974), 135–228.

[3] B. Gray, Unstable families related to the image of J, Cambridge Philos. Soc. Proc. **96** (1984), 95–113.

[4] Y. Hemmi and Y. Kawamoto, Decomposability of the mod p-Whitehead element, Hiroshima. Math. J. **26** (1996), 623–633.

[5] T. Kambe, H. Matsunaga and H. Toda, A note on stunted lens space, J. Math. Kyoto Univ. **5** (1966), 143–149.

[6] W. H. Lin, The Whitehead squares $[\iota_{2^i-1}, \iota_{2^i-1}] \in \pi_{2^{i+1}-3}(S^{2^i-1})$ are projective, Topology **34** (1995), 411–422.

[7] R. J. Milgram and P. Zvengrowski, Even Whitehead squares are not projective, Can J. Math. **29** (1977), 957–962.

[8] M. Mimura, J. Mukai and G. Nishida, Representing elements of stable homotopy groups by symmetric maps, Osaka. J. Math. **11** (1974), 105–111.

[9] D. Randall, Projectivity of the Whitehead squares, Proc. Amer. Math. Soc. **40** (1973), 606–611.

[10] H. Toda, On iterated suspensions I, J. Math. Kyoto Univ., **5** (1965), 87–142.

Department of Mathematics Education
Faculty of Education
Hiroshima University,
Higashi-Hiroshima 739, *Japan*
e-mail: imaoka@ipc.hiroshima-u.ac.jp

Department of Mathematics
Faculty of Science
Hiroshima University,
Higashi-Hiroshima 739, *Japan*
e-mail: r7m05@math.sci.hiroshima-u.ac.jp