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Construction of noncanonical representations of a Brownian motion

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ABSTRACT. Give a Brownian motion $B = \{B(t); t \in [0,1]\}$. For any linearly independent system $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$ in $L^2[0,1]$, we construct a Brownian motion $\bar{B}_{\mathbf{g}} = \{\bar{B}_{\mathbf{g}}(t); t \in [0,1]\}$ which is noncanonical with respect to B. In detail, the orthogonal complement of $H_t(\bar{B}_{\mathbf{g}})$ in $H_t(B)$ is the linear span of $\{\int_0^t g_1(u)dB(u), \int_0^t g_2(u)dB(u), \dots, \int_0^t g_N(u)dB(u)\}$. As a special case, Lévy's examples of noncanonical representations of a Brownian motion are included. For the construction of $\bar{B}_{\mathbf{g}}$, we use the theory of a partial isometry. A generalized Hardy inequality is derived and applied as an important lemma.

0. Introduction.

The theory of canonical representation for a Gaussian process has been presented for the first time by Lévy [9] and later developed by Hida [5] and Cramér [2]. Especially Hida has given a systematic method for the theory of multiplicity of the canonical representation. The main results on the canonical representation after their initial articles are referred to the book of Hida and Hitsuda [6]. On the other hand, Lévy [10] has given some nontrivial examples of the noncanonical representations of a Brownian motion with respect to a given Brownian motion which is used as a standard in order to emphasize the importance of the canonical representation.

The aim of the present article is to give a general method so as to obtain a noncanonical representation of a Brownian motion, which has its own interest in connection with a generalized Hardy inequality in $L^2[0,1]$ or in $L^2[0,\infty)$.

We give here a review of Lévy [10] in connection with the present problem. Let $B = \{B(t); t \in [0,1]\}$ be a Brownian motion. It is proved that for each $q > -1/2$ and $q \neq 0$ a Gaussian process $\bar{B}_q = \{\bar{B}_q(t); t \in [0,1]\}$ defined by

$$\bar{B}_q(t) = \int_0^t \left(\frac{2q+1}{q} \frac{u^q}{t^q} - \frac{q+1}{q} \right) dB(u) \tag{1}$$

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is again a Brownian motion. In fact, the law of the Gaussian process $\bar{B}_q$ is equal to the one of the Brownian motion. But the representation of the right-hand side of (1) is not canonical with respect to B, because, for each t, the random variable $\int_0^t u^q\,dB(u)$ is orthogonal to the linear span $H_t(\bar{B}_q)$ by $\{\bar{B}_q(u); u \in [0,t]\}$. Even in case of $q = 0$, we can give the noncanonical representation of a Brownian motion $\bar{B}_0$ by

$$\bar{B}_0(t) = \int_0^t \left(1 + \log\frac{u}{t}\right) dB(u). \tag{2}$$

It may be interesting that $B(t)$ itself is orthogonal to the linear span $H_t(\bar{B}_0)$ for each $t \in [0,1]$. We note that for any $q > -1/2$ these examples are expressed in the unified form by the use of an integral operator K_q defined by a kernel k_q:

$$\bar{B}_q(t) = B(t) - \int_0^t \int_0^s k_q(s,u)\,dB(u)\,ds, \tag{3}$$

where

$$k_q = k_q(s,u) \equiv \begin{cases} (2q+1)u^q/s^{q+1}, & s \geq u, \\ 0, & s < u. \end{cases}$$

In terms of white noise, (3) is informally written as in the form

$$\dot{\bar{B}}_q = (I - K_q)\dot{B}, \tag{4}$$

with an initial condition $\bar{B}_q(0) = 0$. Though the exact meaning of the operator $K_q : L^2[0,1] \to L^2[0,1]$ and the adjoint one K_q^* will be defined in Section 1 as a much more general form, it should be noted that

$$\bar{B}_q(t) = \int_0^1 (I - K_q^*)1_{[0,t]}(u)\,dB(u), \qquad t \in [0,1],$$

where 1_A denotes the indicator function of an interval A.

The idea above of Lévy's construction will be generalized as in the following method. For any natural number N and for any linearly independent system $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$ of $L^2[0,1]$, we will construct in Section 1 a partial isometry $I - K_\mathbf{g}$ whose initial subspace is the orthogonal complement of the linear span of $\{g_1, g_2, \dots, g_N\}$ and whose final subspace is $L^2[0,1]$. Then the adjoint $I - K_\mathbf{g}^*$ is an isometry on $L^2[0,1]$. In the course of defining the operator $K_\mathbf{g}$, we naturally need a generalized form of the Hardy inequality in $L^2[0,1]$. It may be worth pointing out that the operator K_q of the above example of Lévy is equal to $K_\mathbf{g}$ for $N = 1$ and $g_1(t) = t^q$.

In Section 2, we will be able to construct a Brownian motion $\bar{B}_{\mathbf{g}} = \{\bar{B}_{\mathbf{g}}(t); t \in [0,1]\}$ for the linearly independent system $\mathbf{g}$ such that the orthogonal complement of $H_t(\bar{B}_{\mathbf{g}})$ in $H_t(B)$ coincides with the linear span of $\{\int_0^t g_1(u)dB(u), \int_0^t g_2(u)dB(u), \dots, \int_0^t g_N(u)dB(u)\}$. The Brownian motion $\bar{B}_{\mathbf{g}}$ is explicitly represented as in the form

$$\bar{B}_{\mathbf{g}}(t) = \int_0^1 (I - K_{\mathbf{g}}^*) 1_{[0,t]}(u) dB(u), \qquad t \in [0,1]$$

in terms of the isometry $I - K_{\mathbf{g}}^*$.

1. The operator $K_{\mathbf{g}}$ and a generalized Hardy inequality.

In the beginning, let us define an operator $K_{\mathbf{g}}$ depending on a fixed finite system $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$, $g_i \in L^2[0,1]$, $i = 1, 2, \dots, N$. For the sake of convenience, we use notations

$$\mathbf{g}(t) = {}^{\tau}(g_1(t), g_2(t), \dots, g_N(t)), \tag{5}$$

$$G(t) = \int_0^t \mathbf{g}(u)\,{}^{\tau}\mathbf{g}(u) du = \left(\int_0^t g_i(u) g_j(u) du \right)_{i,j=1,2,\dots N}, \tag{6}$$

$$\int_0^t \mathbf{g}(s)\alpha(s) ds = {}^{\tau}\left(\int_0^t g_1(s)\alpha(s) ds, \dots, \int_0^t g_N(s)\alpha(s) ds \right), \tag{7}$$

where ${}^{\tau}\mathbf{g}$ means the transposed vector of a vertical vector $\mathbf{g}$. Note that the rank $R(t)$ of the matrix $G(t)$ is an integer-valued and nondecreasing function which is left-continuous in $t > 0$. In the first stage, we assume the following.

ASSUMPTION A. The rank function $R(t)$ is constant N:

$$R(t) = N, \qquad t > 0.$$

REMARK 1. The assumption above is not essential. It will be easily removed at the final theorem in the present section.

Theorem 1.1 below guarantees that it is permitted to define a bounded operator $K_{\mathbf{g}}$ on $L^2[0,1]$ by

$$K_{\mathbf{g}}\alpha(s) = {}^{\tau}\mathbf{g}(s) G(s)^{-1} \int_0^s \mathbf{g}(u)\alpha(u) du, \qquad \alpha \in L^2[0,1], \tag{8}$$

though the right-hand side of (8) has a singularity at $s = 0$. Namely, the operator $K_{\mathbf{g}}$ is regarded as an integral operator with kernel $k_{\mathbf{g}}(t,s) = {}^{\tau}\mathbf{g}(t) G(t)^{-1} \mathbf{g}(s)$.

Denote Q_ε by

$$Q_\varepsilon = \{\alpha \in L^2[0,1]; \alpha(t) = 0 \text{ for } t \le \varepsilon\},$$

for each $\varepsilon > 0$.

LEMMA 1.1 *Under Assumption A, we have*

$$\int_0^1 \left({}^\tau\mathbf{g}(s) G(s)^{-1} \int_0^s \mathbf{g}(u)\alpha(u)du \right)^2 ds \le 4\|\alpha\|^2 \tag{9}$$

for $\alpha \in Q_\varepsilon, \varepsilon > 0$.

PROOF. Let us note that $G(t) - G(s) = \left(\int_s^t g_i(u)g_j(u)du\right)$ is nonnegative definite, so the minimum eigenvalue $\lambda(t)$ of $G(t)$ is a nondecreasing function in t. As a result, the maximum eigenvalue $1/\lambda(t)$ of the inverse matrix $G(t)^{-1}$ of $G(t)$ is nonincreasing. The left-hand side of (9) is equal to the integral $\int_\varepsilon^1 ({}^\tau\mathbf{g}(s)G(s)^{-1} \int_\varepsilon^s \mathbf{g}(u)\alpha(u)du)^2 ds$ and this value is finite, because α is in Q_ε and $1/\lambda(\varepsilon)$ is finite. Thus the left-hand side of (9) can be rewritten as

$$\int_\varepsilon^1 \left(\int_\varepsilon^s {}^\tau\mathbf{g}(u)\alpha(u)du\, G(s)^{-1}\mathbf{g}(s){}^\tau\mathbf{g}G(s)^{-1} \int_\varepsilon^s \mathbf{g}(u)\alpha(u)du \right) ds$$

$$= -\int_\varepsilon^1 \left\{ \int_\varepsilon^s {}^\tau\mathbf{g}(u)\alpha(u)du (G(s)^{-1})' \int_\varepsilon^s \mathbf{g}(u)\alpha(u)du \right\} ds$$

$$= \left[-\int_\varepsilon^s {}^\tau\mathbf{g}(u)\alpha(u)du\, G(s)^{-1} \int_\varepsilon^s \mathbf{g}(u)\alpha(u)du \right]_{s=\varepsilon}^{s=1}$$

$$+ 2\int_\varepsilon^1 {}^\tau\mathbf{g}(s)G(s)^{-1} \int_\varepsilon^s \mathbf{g}(u)\alpha(u)du\alpha(s)ds \text{ (integration by parts).}$$

Since $\int_\varepsilon^1 {}^\tau\mathbf{g}(u)\alpha(u)du\, G(1)^{-1} \int_\varepsilon^1 \mathbf{g}(u)\alpha(u)du \ge 0$, the last expression is less than

$$2\|\alpha\| \left\{ \int_0^1 \left({}^\tau\mathbf{g}(s)G(s)^{-1} \int_0^s \mathbf{g}(u)\alpha(u)du \right)^2 ds \right\}^{1/2}$$

by the use of Schwarz's inequality. Thus we get the result (9). □

THEOREM 1.1. *For any* $\alpha \in L^2[0,1]$, *the function* $K_\mathbf{g}\alpha$ *defined by* (8) *belongs to* $L^2[0,1]$ *and is evaluated by*

$$\|K_\mathbf{g}\alpha\| \le 2\|\alpha\|. \tag{10}$$

PROOF. For $\alpha \in Q_\varepsilon, \varepsilon > 0$, the evaluation has been completed by the preceding lemma. For $\alpha \in L^2[0,1]$, define α_n by

$$\alpha_n(t) = \begin{cases} 0, & t \le 1/n, \\ \alpha(t), & t > 1/n. \end{cases}$$

Then α_n belongs to $Q_{1/n}$ and $K_{\mathbf{g}}\alpha_n$ belongs to $L^2[0,1]$, satisfying $\|K_{\mathbf{g}}\alpha_n\| \leq 2\|\alpha_n\|$. Thus the sequence $\{\alpha_n\}$ converges to α in $L^2[0,1]$ and $\{K_{\mathbf{g}}\alpha_n\}$ is a Cauchy sequence in $L^2[0,1]$. On the other hand, $K_{\mathbf{g}}\alpha_n(t) = {}^{\tau}\mathbf{g}(t)G(t)^{-1}\int_{1/n}^{t}\mathbf{g}(s)\alpha(s)ds$ converges to ${}^{\tau}\mathbf{g}(t)G(t)^{-1}\int_0^t \mathbf{g}(s)\alpha(s)ds$ for each $t \in [0,1]$. As a result, we know that $K_{\mathbf{g}}\alpha$ is well-defined and that $K_{\mathbf{g}}$ is expressed as the integral operator (8). □

REMARK 2. The inequality (10) is a generalization of the Hardy inequality in $L^2[0,1]$. When $N = 1$ and $g_1(t) \equiv 1$, (10) becomes Hardy's one [3]; see also Yor [11] discussing an innovation problem for a pinned Brownian motion (a Brownian bridge). In Jeulin and Yor [8] as well as in Yor [11], some results were obtained in connection with a stochastic linear differential equation.

In the next step, we will prove some essential properties of the integral operator $K_{\mathbf{g}}$ and the adjoint operator $K_{\mathbf{g}}^*$.

LEMMA 1.2. *The operator $K_{\mathbf{g}}$ is invariant under the selection of basis $\{g_1, g_2, \ldots, g_N\}$. In other words, if the linear span of* $\mathbf{g}$ coincides with the one *of $\tilde{\mathbf{g}} = (\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, \ldots, \tilde{\mathbf{g}}_N)$, then $K_{\mathbf{g}} = K_{\tilde{\mathbf{g}}}$.*

PROOF. If the linear spans of $\mathbf{g}$ and of $\tilde{\mathbf{g}}$ are the same, then there exists a regular $N \times N$-matrix A so that $\mathbf{g} = A\tilde{\mathbf{g}}$. Thus the result is clear. □

LEMMA 1.3. *The adjoint operator $K_{\mathbf{g}}^*$ for $K_{\mathbf{g}}$ is expressed as*

$$K_{\mathbf{g}}^*\alpha(u) = {}^{\tau}\mathbf{g}(u)\int_u^1 G(s)^{-1}\mathbf{g}(s)\alpha(s)ds, \qquad \alpha \in L^2[0,1]. \tag{11}$$

PROOF. The calculation as in the proof of Lemma 1.1 tells us that the formal adjoint of $K_{\mathbf{g}}$ is indeed the adjoint operator. □

REMARK 3. The kernel of $K_{\mathbf{g}}^*$ above is a Goursat kernel of order N.

THEOREM 1.2. *Under Assumption A,*
(I) For any $\alpha \in L^2[0,1]$,

$$\|(I - K_{\mathbf{g}})\alpha\|^2 = \|\alpha\|^2 - {}^{\tau}(\alpha, \mathbf{g})G(1)^{-1}(\alpha, \mathbf{g})(\leq \|\alpha\|^2), \tag{12}$$

here the notation $(\alpha, \mathbf{g})$ means $((\alpha, g_1), (\alpha, g_2), \ldots, (\alpha, g_N))$ for convenience' sake, and

$$Ker(I - K_{\mathbf{g}}) = LS\{g_1, g_2, \ldots, g_N\}. \tag{13}$$

(II) For any $\alpha \in L^2[0,1]$,

$$\|(I - K_{\mathbf{g}}^*)\alpha\| = \|\alpha\|. \tag{14}$$

Proof. (I) For any $\alpha \in L^2[0,1]$, we get

$$\begin{aligned}\|K_{\mathbf{g}}\alpha\|^2 &= \left[-\int_0^s {}^\tau\mathbf{g}(u)\alpha(u)du\, G(s)^{-1}\int_0^s \mathbf{g}(u)\alpha(u)du\right]_0^1 \\ &\quad + 2\int_0^1 {}^\tau\mathbf{g}(s)\alpha(s)G(s)^{-1}\int_0^s \mathbf{g}(u)\alpha(u)du\,ds \quad \text{(integration by parts)} \\ &= -{}^\tau(\alpha,\mathbf{g})G(1)^{-1}(\alpha,\mathbf{g}) + 2(\alpha, K_{\mathbf{g}}\alpha).\end{aligned}$$

Since $\|(I-K_{\mathbf{g}})\alpha\|^2 = \|\alpha\|^2 - 2(\alpha, K_{\mathbf{g}}\alpha) + \|K_{\mathbf{g}}\alpha\|^2$, (12) is obtained.

Suppose $\alpha = c_1 g_1 + c_2 g_2 + \cdots + c_N g_N$. Then

$${}^\tau(\alpha,\mathbf{g})G(1)^{-1}(\alpha,\mathbf{g}) = {}^\tau\mathbf{c}G(1)G(1)^{-1}G(1)\mathbf{c} = {}^\tau\mathbf{c}G(1)\mathbf{c} = \|\alpha\|^2,$$

where ${}^\tau\mathbf{c} = (c_1, c_2, \dots, c_N)$. So in this case, the right-hand side of (12) is vanished. On the other hand, if α is orthogonal to the linear span of $\{g_1, g_2, \dots, g_N\}$, then $(\alpha, \mathbf{g}) = 0$. The result follows.

(II) It is sufficient to prove that $\|K_{\mathbf{g}}^*\alpha\|^2 = 2(\alpha, K_{\mathbf{g}}^*\alpha)$. This is done by the use of integration by parts as in the proof of (I):

$$\begin{aligned}\|K_{\mathbf{g}}^*\alpha\|^2 &= \int_0^1\left(\int_u^1 \alpha(s){}^\tau\mathbf{g}(s)G(s)^{-1}ds\,\mathbf{g}(u){}^\tau\mathbf{g}(u)\int_u^1 G(s)^{-1}\mathbf{g}(s)\alpha(s)ds\right)du \\ &= \left[\int_u^1 \alpha(s){}^\tau\mathbf{g}(s)G(s)^{-1}ds\,G(u)\int_u^1 G(s)^{-1}\mathbf{g}(s)\alpha(s)ds\right]_{u=0}^{u=1} \\ &\quad + 2\int_0^1\left(\alpha(u){}^\tau\mathbf{g}(u)\int_u^1 G(s)^{-1}\mathbf{g}(s)\alpha(s)ds\right)du \\ &= 2(\alpha, K_{\mathbf{g}}^*\alpha).\end{aligned}$$

In the arguments above, we used the facts that $\mathbf{g}(u){}^\tau\mathbf{g}(u) = (G(u))'$ and $[\cdots]_0^1 = 0$. $\square$

Remark 4. The operator $I - K_{\mathbf{g}}$ is a partial isometry and the initial subspace is $L^2[0,1] \ominus LS\{g_1, g_2, \dots, g_N\}$ and the final subspace is $L^2[0,1]$ (see [1]).

In the remainder of this section, we state the same results as Theorem 1.2 without Assumption A. The notice has been given in the remark just after the assumption. Here only the main outline is presented, since the essence is already included in the arguments above and the method is very similar to the preceding one.

The first task is to define an integral operator $K_{\mathbf{g}}$ for a given linearly independent system $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$ in $L^2[0,1]$. Let the integer-valued function $R(t) = \operatorname{rank}\left(\int_0^t g_i(s)g_j(s)ds\right)$ be

$$R(t) = N_k, \qquad t_{k-1} < t \le t_k,\ k = 1, 2, \dots, L,$$

here $1 \le N_{k-1} < N_k$, $N_L = N$, $t_0 = 0$ and $t_L = 1$. Without loss of generality, we can assume that, for any $t \in (t_{k-1}, t_k]$, the system $\{g_1, g_2, \dots, g_{N_k}\}$ is linearly independent in $L^2[0, t]$. Thus the Gramian matrix

$$G_k(t) = \left(\int_0^t g_i(u)g_j(u)du\right)_{i,j=1,2,\dots,N_k}$$

is regular for $t \in (t_{k-1}, t_k]$, so it has the inverse matrix $G_k(t)^{-1}$. The operator $K_{\mathbf{g}} : L^2[0, 1] \to L^2[0, 1]$ is defined by

$$K_{\mathbf{g}}\alpha(s) = {}^\tau\mathbf{g}_k(s)G_k(s)^{-1}\int_0^s \mathbf{g}_k(u)\alpha(u)du, \qquad s \in (t_{k-1}, t_k], \tag{15}$$

where ${}^\tau\mathbf{g}_k(s) = (g_1(s), g_2(s), \dots, g_{N_k}(s))$. Then the operator is well-defined and has the properties of Lemma 1.1 and Theorem 1.2 (*I*). For the proofs of these facts, we need to apply the method of integration by parts for each subinterval $(t_{k-1}, t_k]$, $k = 1, 2, \dots, L$. As for the property of Lemma 1.2, we can justify the analogous result after a slight modification of the statements. As a preparatory result, we can obtain the next lemma.

LEMMA 1.4. *The adjoint operator $K_{\mathbf{g}}^*$ for $K_{\mathbf{g}}$ defined by* (15) *is given by*

$$K_{\mathbf{g}}^*\alpha(u) = \sum_{i=1}^{L} {}^\tau\mathbf{g}_i(u)\int_{t_{i-1}}^{t_i} G_i(s)^{-1}\mathbf{g}_i(s)1_{(u,1]}(s)\alpha(s)ds, \qquad \alpha \in L^2[0, 1]. \tag{16}$$

According to the expression of $K_{\mathbf{g}}^*$ in (16), we can get the same result as Theorem 1.2 (II) without Assumption A.

THEOREM 1.3. *For any linearly independent system $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$ in $L^2[0, 1]$, the operator $I - K_{\mathbf{g}}^*$ is isometric*:

$$\|(I - K_{\mathbf{g}}^*)\alpha\| = \|\alpha\|, \qquad \alpha \in L^2[0, 1]. \tag{17}$$

2. Noncanonical representation of a Brownian motion.

The theorem below gives a general scheme of noncanonical representations of a Brownian motion of Lévy's type with respect to a given Brownian motion $B = \{B(t); t \in [0, 1]\}$.

THEOREM 2.1. *For any linearly independent system* $\mathbf{g} = \{g_1, g_2, \dots, g_N\}$ *in* $L^2[0,1]$, *the Gaussian process* $\bar{B}_{\mathbf{g}}$ *defined by*

$$\bar{B}_{\mathbf{g}}(t) = \int_0^1 (I - K_{\mathbf{g}}^*) 1_{[0,t]}(u) dB(u), \qquad t \in [0,1], \tag{18}$$

is a Brownian motion having a property

$$H_t(\bar{B}_{\mathbf{g}}) = H_t(B) \ominus LS\left\{\int_0^t g_i(u) dB(u); i = 1, 2, \dots, N\right\}, \tag{19}$$

where $LS\{\cdots\}$ *means the linear span of* $\{\cdots\}$.

PROOF. We get

$$E[\bar{B}_{\mathbf{g}}(t)\bar{B}_{\mathbf{g}}(s)] = ((I - K_{\mathbf{g}}^*)1_{[0,t]}, (I - K_{\mathbf{g}}^*)1_{[0,s]}) = (1_{[0,t]}, 1_{[0,s]}) = t \wedge s,$$

by the use of (17). Thus $\bar{B}_{\mathbf{g}}$ is a Brownian motion. The orthogonal property (19), where we note that $LS\{\int_0^t g_i(u)dB(u); i = 1, 2, \dots, N\}$ has dimension $R(t) = N_k$ if $t_{k-1} < t \le t_k$, is clear from the analogy of Theorem 1.2 (*I*). □

The idea of the theorem above clearly includes a more general result on the noncanonical representation as follows.

PROPOSITION 2.1. *Let a Gaussian process* $X = \{X(t); t \in [0,1]\}$ *have the canonical representation*

$$X(t) = \int_0^t F_t(u) dB(u) \tag{20}$$

in the sense of Lévy [9]. *Then*

$$\bar{X}_{\mathbf{g}}(t) = \int_0^t (I - K_{\mathbf{g}}^*) F_t(u) dB(u) \tag{21}$$

gives a noncanonical representation of X with respect to the Brownian motion B, satisfying

$$H_t(\bar{X}_{\mathbf{g}}) = H_t(B) \ominus LS\left\{\int_0^t g_i(u) dB(u); i = 1, 2, \dots, N\right\}. \tag{22}$$

REMARK 5. (*I*) In the theorem above, the noncanonical representation is essentially unique. If $\bar{B}$ is a Brownian motion whose linear span $H_t(\bar{B})$ is given by the right-hand side of (19), then $\bar{B}$ is represented as

$$\bar{B}(t) = \int_0^1 (I - K_{\mathbf{g}}^*) 1_{[0,t]}(u) d\tilde{B}(u), \qquad t \in [0,1], \tag{23}$$

where $\tilde{B}$ is a Brownian motion satisfying $H_t(\tilde{B}) = H_t(B)$.

(II) The representation (18) is rewritten into the form

$$\bar{B}_{\mathbf{g}}(t) = B(t) - \sum_{k=1}^{L} \int_{t_{k-1}}^{t_k} 1_{[0,t]}(s)^{\tau} \mathbf{g}_k(s) G_k(s)^{-1} \int_0^s \mathbf{g}_k(u) dB(u) ds. \tag{24}$$

It has an informal meaning similar to (4):

$$\dot{\bar{B}}_{\mathbf{g}} = (I - K_{\mathbf{g}}) \dot{B}, \tag{25}$$

with the initial condition $\bar{B}_{\mathbf{g}}(0) = 0$.

(III) Let us consider the transformation

$$\dot{X} = (I - K)\dot{B}, \qquad X(0) = 0,$$

associated with a Volterra operator K. The transformation has its own interest in connection with the canonical representation of a Gaussian process. If the kernel $k(s,u)$ of K satisfies the condition $\int_0^1 \int_0^s k(s,u)^2 du\, ds < \infty$, then $I - K$ is invertible. Such a operator is useful to characterize the canonical representation of a Gaussian process $X = \{X(t); t \in [0,1]\}$ which is equivalent to a Brownian motion (see Hitsuda [7] or Hida and Hitsuda [6]). On the contrary, the kernel $k_{\mathbf{g}}(s,u)$ in (25) is not square integrable.

3. Concluding commentaries.

1. In Section 1 and Section 2, we have considered the noncanonical representation of the Brownian motion $\bar{B}_{\mathbf{g}}$ in time interval [0,1]. It is easy to construct a noncanonical representation of a Brownian motion $\bar{B}_{\mathbf{g}}$ in time interval $[0,\infty)$. In the case of $[0,\infty)$, the function g_i in the given independent system $\mathbf{g} = \{g_1, g_2, \ldots, g_N\}$ can be chosen from $L^2_{\text{loc}}[0,\infty) = \{g; \int_0^t g(u)^2 du < \infty, \text{for any } t \in [0,\infty)\}$. It is not so difficult to testify to validity of the results in Section 1 and Section 2 after a slight modification of the arguments.

2. In this paper, we picked up only such a noncanonical representation that the codimension of $H_t(\bar{B}_{\mathbf{g}})$ in $H_t(B)$ is finite for each t. In their another paper [4], the authors will give an example of the noncanonical representation of a Brownian motion $\bar{B}$ with respect to B such that the codimension $H_t(B) \ominus H_t(\bar{B})$ is infinite.

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