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Eisenstein series on orthogonal groups $O(1, m+1)$ and $O(2, m+2)$

Yoshikazu Hirai

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Abstract. In this paper we will study two kinds of Eisenstein series: One for the orthogonal groups of signature $(1, m+1)$, and one for the orthogonal groups of signature $(2, m+2)$. We give an explicit Fourier expansion by means of Shimura's method.

0. Introduction

Let S be an even integral negative definite symmetric matrix of rank m and assume that S is maximal. We put

$$S_1 = \begin{pmatrix} & & 1 \\ & S & \\ 1 & & \end{pmatrix}, \qquad S_2 = \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix},$$

$$G = O(S), \qquad G_1 = O(S_1) \qquad \text{and} \qquad G_2 = O(S_2).$$

Put $K_{1,p} = G_{1,p} \cap GL_{m+2}(\mathbf{Z}_p)$ and $K_{2,p} = G_{2,p} \cap GL_{m+4}(\mathbf{Z}_p)$. $G^0_{1,\infty}$, the identity component of the real point of G_1, acts on $\mathfrak{X} := \mathbf{R}^m \times \mathbf{R}^\times_+$ ($\mathbf{R}^\times_+$ is the set of positive real numbers) transitively by

$$g_1 \cdot \mathbf{X}^\sim = (g_1\langle\mathbf{X}\rangle)^\sim \cdot j(g_1, \mathbf{X}),$$

$$\mathbf{X}^\sim := \begin{pmatrix} r - S[X]/2 \\ X \\ 1 \end{pmatrix} \in \mathbf{R}^{m+2} \qquad (g_1 \in G^0_{1,\infty}, \mathbf{X} = (X, r) \in \mathfrak{X}).$$

$G^0_{2,\infty}$, the identity component of the real point of G_2, acts on

$$\mathfrak{D} := \left\{ Z \in \mathbf{C}^{m+2} \,\middle|\, S_1[\mathrm{Im}(Z)] > 0, S_1(Y_0, \mathrm{Im}(Z)) > 0, \, Y_0 = \begin{pmatrix} 1 \\ 0_m \\ 1 \end{pmatrix} \right\}$$

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transitively by

$$g \cdot Z^{\sim} = (g\langle Z\rangle)^{\sim} \cdot J(g, Z), \quad Z^{\sim} := \begin{pmatrix} -S_1[Z]/2 \\ Z \\ 1 \end{pmatrix} \in \mathbf{C}^{m+4} \quad (g \in G^0_{2,\infty}, Z \in \mathfrak{D}).$$

We fix a point $\mathbf{X}_0 = (0_m, 1) \in \mathfrak{X}$ and denote by $K_{1,\infty}$ the stabilizer subgroup of $\mathbf{X}_0$ in $G^0_{1,\infty}$. Let P_1 [resp. P_2] be a maximal parabolic subgroup of G_1 [resp. G_2] defined by (1.1) [resp. (1.2)]. By the Iwasawa decomposition for $G_{1,A}$, each $g_1 \in G_{1,A}$ is written in the form

$$g_1 = \begin{pmatrix} t_1(g_1) & * & * \\ & h_1(g_1) & * \\ & & t_1(g_1)^{-1} \end{pmatrix} k_1(g_1), \qquad t_1(g_1) \in \mathbf{Q}_A^{\times},$$

$$h_1(g_1) \in G_A, \qquad k_1(g_1) \in \prod_{v \leq \infty} K_{1,v}.$$

Then the Eisenstein series on $G_{1,A}$ is defined by

$$\mathscr{E}(g_1; s) := \sum_{\gamma_1 \in P_{1,\mathbf{Q}} \backslash G_{1,\mathbf{Q}}} |t_1(\gamma_1 g_1)|^{s+m/2}, \tag{0.1}$$

which converges absolutely in a right half plane $\{s \in \mathbf{C} | \mathrm{Re}\, s > m/2\}$.

Let l be a non-negative even integer. We denote by $M_l(\Gamma)$ the space of holomorphic automorphic forms on $\mathfrak{D}$ of weight l with respect to $\Gamma := G_{2,\mathbf{Q}} \cap G^0_{2,\infty} \prod_{p<\infty} K_{2,p}$. The real analytic Eisenstein series on $\mathfrak{D}$ of weight l with respect to Γ is defined by

$$E_l(Z, s) = \left(\frac{S_1[\mathrm{Im}\, Z]}{2}\right)^{(2s-2l+m+2)/4} \sum_{\gamma \in (P_{2,\mathbf{Q}} \cap \Gamma) \backslash \Gamma} |J(\gamma, Z)|^{-s+l-m/2-1} J(\gamma, Z)^{-l}. \tag{0.2}$$

which converges absolutely in a right half plane $\{s \in \mathbf{C} | \mathrm{Re}\, s > m/2 + 1\}$. In particular if $l > m + 2$, we can define the holomorphic Eisenstein series $E_l(Z) := E_l(Z, l - m/2 - 1) \in M_l(\Gamma)$.

In §1 we introduce two kinds of Eisenstein series (0.1) and (0.2). Applying Shimura's method, we write the Fourier expansion in terms of adelic language (Proposition 1.3 and Proposition 1.4).

§2–§4 give the local theory to write Fourier expansions in Proposition 1.3 and Proposition 1.4 more explicitly. In §2 we calculate the contribution of non-archimedean part which commonly appears in two types of Eisenstein series. The local Hecke algebra which is studied by Sugano [12] plays important roles to prove the main theorem (Theorem 2.1) in §2. In §3 we introduce confluent hypergeometric functions and calculate the contribution in archimedean part. In various aspects of our argument, we use the properties of confluent hypergeometric functions studied by Shimura [10]. In §4 we calculate the contribution which only appears in the type of Eisenstein series defined by (0.2).

In §5 and §6 we study Eisenstein series on $O(1,m+1)$ and $O(2,m+2)$ defined by (0.1) and (0.2), respectively. Combining the results in §1–§4, we write the Fourier expansion of the Eisenstein series explicitly (Theorem 5.2 and Theorem 6.2). We prove the continuation and the functional equation of the Eisenstein series without using Langlands' theory [2].

THEOREM 0.1 (THEOREM 5.4). *Let s be a complex number with $\mathrm{Re}\, s > m/2$. We normalize the Eisenstein series $\mathscr{E}^*(g_1,s)$ by*

$$\mathscr{E}^*(g_1,s) := \xi(S;s+1)\mathscr{E}(g_1,s)\cdot\begin{cases}1 & \text{if } m \text{ is even}\\ \xi(2s+1) & \text{if } m \text{ is odd}\end{cases},$$

where $\xi(S;s)$ is the (global) standard L-function attached to the constant function defined by (5.2) and $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\mathscr{E}^*(g_1,s)$ has a meromorphic continuation in s to the whole s-plane and is invariant under $s\mapsto -s$.

THEOREM 0.2 (THEOREM 6.4). *Let s be a complex number with $\mathrm{Re}\, s > m/2+1$. We normalize the Eisenstein series $E_l^*(Z,s)$ by*

$$E_l^*(Z,s) := P_l(s)\xi(S_1;s+1)E_l(Z,s)\cdot\begin{cases}1 & \text{if } m \text{ is even}\\ \xi(2s+1) & \text{if } m \text{ is odd}\end{cases},$$

where $P_l(s)$ is a polynomial in s defined in (3.10). Then $E_l^*(Z,s)$ has a meromorphic continuation in s to the whole s-plane and is invariant under $s\mapsto -s$.

Although the above assertions have been proved by Langland's theory, our proof seems to be new and elementary.

The absolute convergence of (0.2) at $s=l-m/2-1$ is not guaranteed if $l\le m+2$. However, as in Shimura [11], we obtain the holomorphic Eisenstein series of smaller weights. Since the Eisenstein series $E_l(Z,s)$ is regular at $s=l-m/2-1$ $(l>(m+4)/2)$, we can define $E_l(Z):=E_l(Z,s)|_{s=l-m/2-1}$.

THEOREM 0.3 (THEOREM 6.5).

$$E_l(Z)\in M_l(\Gamma)\qquad\text{for}\qquad l>(m+4)/2.$$

Moreover we give an explicit formula for the Fourier coefficients of holomorphic Eisenstein series $E_l(Z)$ (Theorem 6.6). By this formula, we verify that Fourier coefficients of $E_l(Z)$ are rational numbers whose denominators are bounded (Corollary 6.7).

In §7 we consider the Eisenstein series on $O(2, m+2)$ in the case of $\mathbf{Q}$-rank 1 to complete our results in this paper.

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Notation. We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$, respectively, the ring of integers, the rational number field, the real number field, and the complex number field. For an associative ring R with an identity element, $R^{\times}$ denotes the group of all invertible elements of R and $M_m(R)$ the ring of all matrices of size m with coefficients in R. We put $GL_m(R) = M_m(R)^{\times}$. If $X \in M_m(R)$, ${}^tX$ and $\mathrm{Tr}(X)$ stand for its transpose and trace. If R is commutative, $\det(X)$ stands for its determinant, and we denote by $SL_m(R)$ the special linear group G of degree m. For each place v of $\mathbf{Q}$, we denote by $\mathbf{Q}_v$ the v-completion of $\mathbf{Q}$, and by $|x|_v$ the module of x for an $x \in \mathbf{Q}_v^{\times}$. $\mathbf{Q}_A$ [resp. $\mathbf{Q}_A^{\times}$] means the adele ring [resp. the idele group] of $\mathbf{Q}$ and for $x = (x_v) \in \mathbf{Q}_A^{\times}$ put $|x|_A = \prod_v |x_v|_v$. For an algebraic group G defined over $\mathbf{Q}$, we denote by $G_{\mathbf{Q}}$ the group of $\mathbf{Q}$-rational points of G. We abbreviate $G_{\mathbf{Q}_v}$ to G_v. Let ∞ and f denote the sets of archimedean primes and non-archimedean primes of $\mathbf{Q}$, respectively. We denote by G_A, G_f and G_{∞}^0 the adelized group of G, the finite part of G_A, and the identity component of G_{∞}, respectively. Similar notations are used for an algebra or a vector space. When Q is a symmetric matrix of degree m, for X and Y in $M_{m,n}$, we put $Q(X, Y) = {}^tXQY$ and $Q[X] = {}^tXQX$. We set $e[x] = e^{2\pi ix}$ for $x \in \mathbf{C}$. The cardinality of a finite set S is denoted by $\# S$. The disjoint union of sets $Z_1, \dots, Z_s$ is denoted by $\coprod_{i=1}^{s} Z_i$. We denote by $\delta((*)) = 1$ or 0 according as the condition $(*)$ is satisfied or not. For $a \in \mathbf{R}$, the symbol $[a]$ denotes the integer not greater than a.

1. Definition of Eisenstein series

Let $S \in M_m(\mathbf{Q})$ be an even integral negative definite symmetric matrix and assume that S is maximal, namely, $S[g^{-1}]$ is not even integral for any $g \in GL_m(\mathbf{Q}) \cap M_m(\mathbf{Z})$ with $\det g \neq \pm 1$. We denote by G the orthogonal group of S and by G_1 [resp. G_2] the orthogonal group of

$$S_1 = \begin{pmatrix} & & 1 \\ & S & \\ 1 & & \end{pmatrix} \quad \left[\text{resp. } S_2 = \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix}\right].$$

Put $L = \mathbb{Z}^m$, $L^* = S^{-1}L$, $L_1 = \mathbb{Z}^{m+2}$, and $L_1^* = S_1^{-1}L_1$. We define maximal compact subgroups $K_p := G_p \cap GL_m(\mathbb{Z}_p)$, $K_{1,p} := G_{1,p} \cap GL_{m+2}(\mathbb{Z}_p)$, and $K_{2,p} := G_{2,p} \cap GL_{m+4}(\mathbb{Z}_p)$. Let ∞ be the archimedean place of $\mathbb{Q}$.

We recall the action of $G_{1,\infty}^0$ on

$$\mathfrak{X} := \mathbb{R}^m \times \mathbb{R}_+^\times \quad (\mathbb{R}_+^\times \text{ is the set of positive real numbers})$$

and the action of $G_{2,\infty}^0$ on

$$\mathfrak{D} := \left\{ Z \in \mathbb{C}^{m+2} \,\middle|\, S_1[\mathrm{Im}(Z)] > 0, S_1(Y_0, \mathrm{Im}(Z)) > 0, Y_0 = \begin{pmatrix} 1 \\ 0_m \\ 1 \end{pmatrix} \right\}.$$

For $\mathbf{X} = (X, r) \in \mathfrak{X}$, put $\mathbf{X}^\sim := \begin{pmatrix} r - S[X]/2 \\ X \\ 1 \end{pmatrix} \in \mathbb{R}^{m+2}$. For $g_1 \in G_{1,\infty}^0$ and $\mathbf{X} \in \mathfrak{X}$, we define the action $g_1\langle\mathbf{X}\rangle \in \mathfrak{X}$ and the automorphy factor $j(g_1, \mathbf{X}) \in \mathbb{R}^\times$ by

$$g_1 \cdot \mathbf{X}^\sim = (g_1\langle\mathbf{X}\rangle)^\sim \cdot j(g_1, \mathbf{X}).$$

We fix a point $\mathbf{X}_0 = (0_m, 1) \in \mathfrak{X}$ and denote by $K_{1,\infty}$ the stabilizer subgroup of $\mathbf{X}_0$ in $G_{1,\infty}^0$. Clearly $K_{1,\infty}$ is a maximal compact subgroup of $G_{1,\infty}^0$ and $G_{1,\infty}^0/K_{1,\infty} \cong \mathfrak{X}$.

For $Z \in \mathfrak{D}$, put $Z^\sim := \begin{pmatrix} -S_1[Z]/2 \\ Z \\ 1 \end{pmatrix} \in \mathbb{C}^{m+4}$. For $g \in G_{2,\infty}^0$ and $Z \in \mathfrak{D}$, we define the action $g\langle Z\rangle \in \mathfrak{D}$ and the automorphy factor $J(g, Z) \in \mathbb{C}^\times$ by

$$g \cdot Z^\sim = (g\langle Z\rangle)^\sim \cdot J(g, Z).$$

We fix a point $Z_0 = iY_0 \in \mathfrak{D}$ and denote by $K_{2,\infty}$ the stabilizer subgroup of Z_0 in $G_{2,\infty}^0$. Clearly $K_{2,\infty}$ is a maximal compact subgroup of $G_{2,\infty}^0$ and $G_{2,\infty}^0/K_{2,\infty} \cong \mathfrak{D}$. We abbreviate $\prod_{p<\infty} K_{i,p}$ to $K_{i,f}$ and $K_{i,\infty}K_{i,f}$ to $K_{i,A}$ $(i = 1, 2)$.

Let P_1 be a maximal parabolic subgroup of G_1 defined by

$$P_{1,\mathbb{Q}} := \left\{ \begin{pmatrix} t_1 & * & * \\ & h_1 & * \\ & & t_1^{-1} \end{pmatrix} \in G_{1,\mathbb{Q}} \,\middle|\, t_1 \in \mathbb{Q}^\times, h_1 \in G_\mathbb{Q} \right\} \tag{1.1}$$

and let P_2 be a maximal parabolic subgroup of G_2 defined by

$$P_{2,\mathbb{Q}} := \left\{ \begin{pmatrix} t & * & * \\ & h & * \\ & & t^{-1} \end{pmatrix} \in G_{2,\mathbb{Q}} \,\middle|\, t \in \mathbb{Q}^\times, h \in G_{1,\mathbb{Q}} \right\}. \tag{1.2}$$

By the Iwasawa decomposition, each $g_1 \in G_{1,A}$ is written in the form

$$g_1 = \begin{pmatrix} t_1(g_1) & * & * \\ & h_1(g_1) & * \\ & & t_1(g_1)^{-1} \end{pmatrix} k_1(g_1), t_1(g_1) \in \mathbf{Q}_A^\times, h_1(g_1) \in G_A, k_1(g_1) \in K_{1,A},$$

and each $g \in G_{2,A}$ is written in the form

$$g = \begin{pmatrix} t(g) & * & * \\ & h(g) & * \\ & & t(g)^{-1} \end{pmatrix} k(g), t(g) \in \mathbf{Q}_A^\times, h(g) \in G_{1,A}, k(g) \in K_{2,A}.$$

For $s \in \mathbf{C}$, we define a function $\varphi(g_1; s)$ on $G_{1,A}$ by

$$\varphi(g_1; s) = |t_1(g_1)|_A^s.$$

For a non-negative even integer l, we define a function $f_l(g; s)$ on $G_{2,A}$ by

$$f_l(g; s) = |t(g)|_A^s J(k(g)_\infty, Z_0)^{-l},$$

where $|t|_A$ means the idele norm of $t \in \mathbf{Q}_A^\times$. Then the Eisenstein series on $G_{1,A}$ is defined by

$$\mathscr{E}(g_1, s) := \sum_{\gamma_1 \in P_{1,\mathbf{Q}} \backslash G_{1,\mathbf{Q}}} \varphi\left(\gamma_1 g_1; s + \frac{m}{2}\right), \tag{1.3}$$

which converges absolutely in a right half plane $\{s \in \mathbf{C} | \mathrm{Re}\, s > m/2\}$. The Eisenstein series on $G_{2,A}$ is defined by

$$E_l^{\mathrm{gr}}(g, s) := \sum_{\gamma \in P_{2,\mathbf{Q}} \backslash G_{2,\mathbf{Q}}} f_l\left(\gamma g; s + \frac{m+2}{2}\right), \tag{1.4}$$

which converges absolutely in a right half plane $\{s \in \mathbf{C} | \mathrm{Re}\, s > m/2 + 1\}$. We easily see that

$$\mathscr{E}(\gamma_1 g_1 k_1, s) = \mathscr{E}(g_1, s) \quad (\gamma_1 \in G_{1,\mathbf{Q}}, g_1 \in G_{1,A}, k_1 \in K_{1,A}), \tag{1.5}$$

$$E_l^{\mathrm{gr}}(\gamma g k, s) = E_l^{\mathrm{gr}}(g, s) J(k_\infty, Z_0)^{-l} \quad (\gamma \in G_{2,\mathbf{Q}}, g \in G_{2,A}, k \in K_{2,A}). \tag{1.6}$$

We prepare the following lemma (cf. [12, p29]).

LEMMA 1.1.

$$G_{1,A} = G_{1,\mathbf{Q}} G_{1,\infty}^0 K_{1,f}.$$

By Lemma 1.1, we easily see that

$$G_{2,A} = G_{2,\mathbf{Q}} G_{2,\infty}^0 K_{2,f}.$$

Therefore the values of $\mathscr{E}(g,s)$ [resp. $E_l(g,s)$] are determined by the restriction to $G^0_{1,\infty}$ [resp. $G^0_{2,\infty}$]. We define a discrete subgroup Γ of $G^0_{2,\infty}$ by

$$\Gamma := G_{2,\mathbf{Q}} \cap G^0_{2,\infty} K_{2,f}.$$

It is easily verified that

$$G_{2,\mathbf{Q}} = P_{2,\mathbf{Q}} \cdot \Gamma. \tag{1.7}$$

For $Z = g\langle Z_0 \rangle \in \mathfrak{D}$ $(g \in G^0_{2,\infty})$, we put

$$E_l(Z,s) := E_l^{\mathrm{gr}}(g,s) J(g, Z_0)^l. \tag{1.8}$$

Then by (1.7) we have

$$E_l(Z,s) = \left(\frac{S_1[\operatorname{Im} Z]}{2}\right)^{(2s-2l+m+2)/4} \sum_{\gamma \in (P_{2,\mathbf{Q}} \cap \Gamma) \backslash \Gamma} |J(\gamma, Z)|^{-s+l-m/2-1} J(\gamma, Z)^{-l}. \tag{1.9}$$

For $X \in \mathbf{Q}^m$ [resp. $X \in \mathbf{Q}^{m+2}$], put

$$n_1(X) = \begin{pmatrix} 1 & -{}^tXS & -S[X]/2 \\ & 1_m & X \\ & & 1 \end{pmatrix}, \qquad \bar{n}_1(X) = \begin{pmatrix} 1 & & \\ X & 1_m & \\ -S[X]/2 & -{}^tXS & 1 \end{pmatrix} \in G_1$$

$$\left[\text{resp. } n_2(X) = \begin{pmatrix} 1 & -{}^tXS_1 & -S_1[X]/2 \\ & 1_{m+2} & X \\ & & 1 \end{pmatrix}, \right.$$

$$\left. \bar{n}_2(X) = \begin{pmatrix} 1 & & \\ X & 1_{m+2} & \\ -S_1[X]/2 & -{}^tXS_1 & 1 \end{pmatrix} \in G_2 \right].$$

We embed G_1 to G_2 by

$$g_1 \mapsto \operatorname{diag}(1, g_1, 1) \quad (g_1 \in G_1).$$

We obtain the following Bruhat decomposition of G_1 and G_2.

LEMMA 1.2. (i)

$$G_{1,\mathbf{Q}} = P_{1,\mathbf{Q}} \coprod P_{1,\mathbf{Q}} w \{ n_1(X) | X \in \mathbf{Q}^m \}.$$

(ii)

$$G_{2,\mathbb{Q}} = P_{2,\mathbb{Q}} \coprod P_{2,\mathbb{Q}} w_1 \{n_2(X) | X \in \mathbb{Q}^{m+2}\} \coprod P_{2,\mathbb{Q}} w_2 \left\{ n_2\left(\begin{pmatrix} 0 \\ 0_m \\ x \end{pmatrix}\right) \middle| x \in \mathbb{Q} \right\}$$

$$\times \coprod P_{2,\mathbb{Q}} w_3 \left\{ n_2\left(\begin{pmatrix} y \\ 0_m \\ 0 \end{pmatrix}\right) n_1(X) \middle| y \in \mathbb{Q}, X \in \mathbb{Q}^m \right\},$$

$$w = \begin{pmatrix} & & 1 \\ & 1_m & \\ 1 & & \end{pmatrix}, \quad w_1 = \begin{pmatrix} & & 1 \\ & 1_{m+2} & \\ 1 & & \end{pmatrix}, \quad w_2 = \begin{pmatrix} J & & \\ & 1_m & \\ & & J \end{pmatrix},$$

$$w_3 = \begin{pmatrix} & & 1_2 \\ & 1_m & \\ 1_2 & & \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define a character χ of the adele ring $\mathbb{Q}_A$ by $\chi = \prod_v \chi_v$, where

$$\chi_v(x) = \begin{cases} e\,[\text{the fractional part of } -x] & \text{for } x \in \mathbb{Q}_p \text{ if } v = p, \\ e\,[x] & \text{for } x \in \mathbb{Q}_v = \mathbb{R} \text{ if } v = \infty \end{cases}.$$

We notice that χ is trivial on $\mathbb{Q}$. Now we normalize a Haar measure $dX = \prod_v dX_v$ on $\mathbb{Q}_A^m$ as

$$\int_{\mathbb{Z}_p^m} dX_p = 1, \quad \int_{\mathbb{Q}_A^m/\mathbb{Q}^m} dX = 1.$$

Then we note that dX_∞ is the ordinary Lebesgue measure. By the Fourier expansion of $\mathscr{E}(n_1(X)g_1, s)$ as a function of $X \in \mathbb{Q}_A^m$, we have

$$(1.10)\qquad \begin{aligned} \mathscr{E}(g_1, s) &= \sum_{\eta \in \mathbb{Q}^m} \mathscr{E}_\eta(g_1, s), \\ \mathscr{E}_\eta(g_1, s) &= \int_{\mathbb{Q}_A^m/\mathbb{Q}^m} \mathscr{E}(n_1(X)g_1, s)\chi(-S(\eta, X))dX. \end{aligned}$$

By (1.3) and Lemma 1.2(i), we have

$$(1.11)\qquad \begin{aligned} \mathscr{E}_\eta(g_1, s) &= \delta(\eta = 0)\varphi\left(g_1; s + \frac{m}{2}\right) \\ &+ \int_{\mathbb{Q}_A^m} \varphi\left(wn_1(X)g_1; s + \frac{m}{2}\right)\chi(-S(\eta, X))dX. \end{aligned}$$

We assume that $g_1 \in G^0_{1,\infty}$. Since

$$\mathscr{E}(n_1(X+U)g_1,s)=\mathscr{E}(n_1(X)g_1,s) \qquad \text{for } U\in L_p,$$

$\mathscr{E}_\eta(g_1,s)\neq 0$ only when $\eta\in L^*$. So we have

$$\mathscr{E}(g_1,s)=\sum_{\eta\in L^*}\mathscr{E}_\eta(g_1,s), \qquad g_1\in G^0_{1,\infty}. \tag{1.12}$$

Therefore we obtain the following proposition.

PROPOSITION 1.3. *Let s be a complex number with* $\mathrm{Re}\, s>m/2$. *For* $g_1=\mathrm{diag}(t,1_m,t^{-1})\in G^0_{1,\infty}$ *and* $X\in\mathbb{R}^m$, *we have*

$$\mathscr{E}(n_1(X)g_1,s)=\sum_{\eta\in L^*}\mathscr{E}_\eta(g_1,s)e[S(\eta,X)],$$

$$\mathscr{E}_\eta(g_1,s)=t^{s+m/2}\delta(\eta=0)+\mathscr{I}_\infty(t\eta;s)\mathscr{I}_f(\eta;s),$$

where

$$\mathscr{I}_f(\eta;s)=\prod_p\mathscr{I}_p(\eta;s),$$

$$\mathscr{I}_\infty(t,\eta;s)=t^{-s+m/2}\int_{\mathbb{R}^m}\varphi_\infty\left(\bar n_1(X);s+\frac{m}{2}\right)\chi_\infty(-S(t\eta,X))dX,$$

$$\mathscr{I}_p(\eta;s)=\int_{\mathbb{Q}_p^m}\varphi_p(\bar n_1(X);s+m/2)\chi_p(-S(\eta,X))dX.$$

In the same way, for $g\in G^0_{2,\infty}$, we have

$$\begin{aligned}E_l^{\mathrm{gr}}(g_1,s)&=\sum_{\eta\in L_1^*}E_{l,\eta}^{\mathrm{gr}}(g,s),\\ E_{l,\eta}^{\mathrm{gr}}(g,s)&=\int_{\mathbb{Q}_A^{m+2}/\mathbb{Q}^{m+2}}E_l^{\mathrm{gr}}(n_2(X)g,s)\chi(-S_1(\eta,X))dX.\end{aligned} \tag{1.13}$$

By Lemma 1.2(ii) we get

$$\begin{aligned}E_{l,\eta}^{\mathrm{gr}}(g,s)&=f_l\left(g;s+\frac{m}{2}+1\right)\delta(\eta=0)\\
&\quad+\int_{\mathbb{Q}_A^{m+2}}f_l\left(\bar n_2(X)g;s+\frac{m}{2}+1\right)\chi(-S_1(\eta,X))dX\\
&\quad+\int_{\mathbb{Q}^{m+2}\backslash\mathbb{Q}_A^{m+2}}\left\{\sum_{x\in\mathbb{Q}}f_l\left(u(x)n_2(X)g;s+\frac{m}{2}+1\right)\right.\\
&\quad\left.+\sum_{\substack{Y\in\mathbb{Q}^m\\ x\in\mathbb{Q}}}f_l\left(u(x)wn_1(Y)n_2(X)g;s+\frac{m}{2}+1\right)\right\}\chi(-S_1(\eta,X))dX,\end{aligned}$$

where we put

$$u(x) := w_2 n_2\left(\begin{pmatrix} 0 \\ 0_m \\ x \end{pmatrix}\right). \tag{1.14}$$

From Lemma 1.2(i), the third term is equal to

$$\begin{aligned}
&\int_{\mathbf{Q}^{m+2}\backslash\mathbf{Q}_A^{m+2}} \sum_{x\in\mathbf{Q}} \sum_{\gamma_1\in P_{1,\mathbf{Q}}\backslash G_{1,\mathbf{Q}}} f_l\Big(u(x)\gamma_1 n_2(X)g; s+\frac{m}{2}+1\Big)\chi(-S_1(\eta,X))dX \\
&= \sum_{\gamma_1\in P_{1,\mathbf{Q}}\backslash G_{1,\mathbf{Q}}} \int_{\mathbf{Q}^{m+2}\backslash\mathbf{Q}_A^{m+2}} \sum_{x\in\mathbf{Q}} f_l\Big(u(x)n_2(X)\gamma_1 g; s+\frac{m}{2}+1\Big)\chi(-S_1(\gamma_1\eta,X))dX.
\end{aligned}$$

We note the above series is well-defined (see (4.2), (4.4)).

Proposition 1.4. *Let l be a non-negative even integer and let s be a complex number with* $\operatorname{Re} s > m/2+1$. *For $g = \operatorname{diag}(t,h,t^{-1}) \in G_{2,\infty}^0$ and $X \in \mathbf{R}^{m+2}$, we have*

$$\begin{aligned}
E_l^{\mathrm{gr}}(n_2(X)g,s) &= \sum_{\eta\in L_1^*} E_{l,\eta}^{\mathrm{gr}}(g,s)e[S_1(\eta,X)], \\
E_{l,\eta}^{\mathrm{gr}}(g,s) &= t^{s+m/2+1}\delta(\eta=0) + I_{l,\infty}(g,\eta;s)I_f(\eta;s) + I_l'(g,\eta;s),
\end{aligned}$$

where

$$\begin{aligned}
I_{l,\infty}(g,\eta;s) &= t^{-s+m/2+1}\int_{\mathbf{R}^{m+2}} f_{l,\infty}\Big(\bar{n}_2(X); s+\frac{m}{2}+1\Big)e[-S_1(h^{-1}\eta t,X)]dX, \\
I_f(\eta;s) &= \prod_p I_p(\eta;s), \\
I_p(\eta;s) &= \int_{\mathbf{Q}_p^{m+2}} |t(\bar{n}_2(X))|_p^{s+m/2+1}\chi_p(-S_1(\eta,X))dX, \\
I_l'(g,\eta;s) &= \sum_{\gamma_1\in P_{1,\mathbf{Q}}\backslash G_{1,\mathbf{Q}}} F_l\Big(\gamma_1 g,\gamma_1\eta; s+\frac{m}{2}+1\Big), \\
F_l(g,\eta;s) &= \int_{\mathbf{Q}^{m+2}\backslash\mathbf{Q}_A^{m+2}} \sum_{x\in\mathbf{Q}} f_l(u(x)n_2(X)g;s)\chi(-S_1(\eta,X))dX.
\end{aligned}$$

2. Non-archimedean part

2.1. Results on non-archimedean part Let k be a non-archimedean local field with characteristic 0 and $\mathfrak{o}$ its maximal order. We fix a prime element p of k and denote by $\mathfrak{p} = (p)$ the maximal ideal of $\mathfrak{o}$. Let χ be a character of k trivial on $\mathfrak{o}$ and non-trivial on $\mathfrak{p}^{-1}$. We normalize the valuation $|| = ||_{\mathfrak{p}}$ of k so that

$|p| = q^{-1}$ where $q = \#(\mathfrak{o}/\mathfrak{p})$. Let S be a non-degenerate even integral symmetric matrix of rank m, where "even integral" means that $S = (s_{ij}) \in M_m(\mathfrak{o})$ and $s_{ii} \in 2\mathfrak{o}$. Put $L = \mathfrak{o}^m$ and $V = k^m$. Throughout this section we assume that S is maximal, namely, if M is a lattice containing L such that $\frac{1}{2}S[x] \in \mathfrak{o}$ for any $x \in M$, then $M = L$. We denote by $L^* = S^{-1}L$ the dual lattice of L and put

$$L' = \{x \in L^* \mid \tfrac{1}{2}S[x] \in \mathfrak{p}^{-1}\}.$$

Then L' is a lattice contained in Lp^{-1} and L'/L is a vector space over a finite field $\mathfrak{o}/\mathfrak{p}$. We denote its dimension by $\partial = \partial(S)$. We define the dual lattice of L'

$$L'^* := \{\eta \in V \mid S(\eta, X) \in \mathfrak{o} \text{ for all } X \in L'\}.$$

An element $\eta \in L^*$ is said to be primitive if $p^{-1}\eta$ is not in L^*. We denote by L^*_{prim} the set of primitive elements. As is well-known, taking a suitable $\mathfrak{o}$-basis of L, we may assume that

$$S = S_\nu = \begin{pmatrix} & & J_\nu \\ & S_0 & \\ J_\nu & & \end{pmatrix}, J_\nu = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad \text{(1 appears } \nu \text{ times)},$$

where S_0 is anisotropic and $\nu = \nu(S)$ is the Witt index of S. We denote by $n_0 = n_0(S)$ the rank of S_0, so $m = 2\nu + n_0$. Let G be the orthogonal group of S and put

$$K = G \cap GL_m(\mathfrak{o}).$$

When we need to emphasize the Witt index ν, we write ν as a suffix; G_ν, K_ν, V_ν, L_ν etc. For $X \in V_\nu$, we put

$$n_\nu(X) := \begin{pmatrix} 1 & -{}^tXS_\nu & -S_\nu[X]/2 \\ & 1_{n_0+2\nu} & X \\ & & 1 \end{pmatrix}, \bar{n}_\nu(X) := \begin{pmatrix} 1 & & \\ X & 1_{n_0+2\nu} & 0 \\ -S_\nu[X]/2 & -{}^tXS_\nu & 1 \end{pmatrix} \in G_{\nu+1}.$$

The main purpose of this section is to calculate the following integral

$$I(S, \eta; s) := \int_V |t_{\nu+1}(\bar{n}_\nu(X))|^{s+m/2} \chi(-S(\eta, X)) dX, \tag{2.1}$$

where we write

$$g = \begin{pmatrix} t_{\nu+1}(g) & * & * \\ 0 & * & * \\ 0 & 0 & t_{\nu+1}(g)^{-1} \end{pmatrix} k_{\nu+1}(g) \in P_{\nu+1}K_{\nu+1}.$$

We put $\eta = p^a\eta_0$, $\eta_0 \in L^*_{\mathrm{prim}}$. Since

$$I(S,\eta;s) = I(S,h\eta;s) \qquad \text{for any } h \in K,$$

if $\nu \geq 1$ we may assume that

$$(2.2)\quad \eta_0 = \begin{cases} \begin{pmatrix} p^{2f}\alpha_0 \\ p^f\beta_{\nu-1} \\ 1 \end{pmatrix} \in L^*_{\nu,\mathrm{prim}}, \beta_{\nu-1} = \begin{pmatrix} 0_{\nu-1} \\ \beta_0 \\ 0_{\nu-1} \end{pmatrix}, & S^{\sim} \text{ is maximal if } S[\eta] \neq 0, \\ \begin{pmatrix} 0 \\ 0_{n_0+2\nu-2} \\ 1 \end{pmatrix} \quad \text{if } S[\eta] = 0, & \end{cases}$$

where $S^{\sim} = \begin{pmatrix} S_{\nu-1} & -S_{\nu-1}\beta_{\nu-1} \\ -{}^t\beta_{\nu-1}S_{\nu-1} & -2\alpha_0 \end{pmatrix}$.

For our purpose, we define the local standard L-function of S after [6]:

$$(2.3)\qquad L_{\mathfrak{p}}(S;s) := \prod_{j=1}^{m-1} \zeta_{\mathfrak{p}}(s+j-m/2)B_S(s)\begin{cases} L_{\mathfrak{p}}(\chi_S,s) & \text{if } m\text{: even} \\ 1 & \text{if } m\text{: odd,} \end{cases}$$

where

$$(2.4)\qquad B_S(s) := \begin{cases} 1 & \text{if } \partial = 0 \text{ or } (n_0,\partial) = (2,1) \\ 1+q^{-s+1/2} & \text{if } (n_0,\partial) = (1,1) \\ (1+q^{-s+1})(1+q^{-s}) & \text{if } (n_0,\partial) = (2,2) \\ 1-q^{-s+1/2} & \text{if } (n_0,\partial) = (3,1) \\ (1+q^{-s+1/2})(1-q^{-s+1/2}) & \text{if } (n_0,\partial) = (3,2) \\ (1-q^{-s+1})(1-q^{-s}) & \text{if } (n_0,\partial) = (4,2) \end{cases}$$

and $\chi_S(\mathfrak{p})$ means the Legendre symbol corresponding to $k(\sqrt{(-1)^{m(m-1)/2}\det S})/k$.

When $\eta \in L^*$ is anisotropic, we denote by $\eta^{\perp}$ the orthogonal complement of η in V. There exists a maximal even integral symmetric matrix S_η of rank $m-1$ and $g \in M_{m-1}(\mathfrak{o})$ such that $S_\eta[g]$ is a matrix representation of $S|_{(\eta^{\perp}\cap L)}$. If S is anisotropic, any matrix representation of $S|_{(\eta^{\perp}\cap L)}$ is a maximal even integral symmetric matrix. We note that the isomorphic class of S_η (modulo $GL_{m-1}(\mathfrak{o})$) does not depend on the choice of S_η. We put

$$(2.5)\qquad \beta_{S,\eta} := \{q^{n_0} - q^{\partial+1} + q^{\partial'+(n_0-n_0'+1)/2} - q^{(n_0+n_0'-1)/2}\}/(q-1)$$

where $n_0' = n_0(S_\eta)$, $\partial' = \partial(S_\eta)$.

The following theorem is the main theorem in this section.

THEOREM 2.1. *The function* $I(S,\eta;s)$ *can be continued as a meromorphic function in* s *to the whole s-plane and written as follows*:

(i)

$$I(S,0;s)=\frac{L_{\mathfrak{p}}(S;s)}{L_{\mathfrak{p}}(S;s+1)}\begin{cases}1 & \text{if } m\text{: even}\\ \zeta_{\mathfrak{p}}(2s)/\zeta_{\mathfrak{p}}(2s+1) & \text{if } m\text{: odd.}\end{cases}$$

We put $\eta=p^a\eta_0$ *where* $\eta_0\in L^*_{\mathrm{prim}}$ *is as in* (2.2) *for* $\nu\geq 1$.

(ii) *If* $S[\eta]=0$,

$$I(S,\eta;s)=\frac{1}{\zeta_{\mathfrak{p}}(s-m/2)}\frac{L_{\mathfrak{p}}(S;s)}{L_{\mathfrak{p}}(S;s+1)}$$

$$\times\begin{Bmatrix}1 & \text{if } m:\text{even}\\ \zeta_{\mathfrak{p}}(2s)/\zeta_{\mathfrak{p}}(2s+1) & \text{if } m:\text{odd}\end{Bmatrix}\sum_{l=0}^{a}q^{(-s+m/2)l}.$$

(iii) *If* $S[\eta]\neq 0$,

$$I(S,\eta;s)=\frac{L_{\mathfrak{p}}(S_\eta;s+1/2)}{L_{\mathfrak{p}}(S;s+1)}\begin{Bmatrix}1 & \text{if } m:\text{even}\\ 1/\zeta_{\mathfrak{p}}(2s+1) & \text{if } m:\text{odd}\end{Bmatrix}\left|\frac{\det S}{\det S_\eta}S[\eta]\right|^{-s/2}g_{S,\mathfrak{p}}(\eta;s),$$

where $g_{S,\mathfrak{p}}(\eta;s)$ *is a polynomial in* q^s, q^{-s} *invariat under* $s\mapsto -s$. *Its explicit form is given as follows*:

(2.6) $g_{S,\mathfrak{p}}(\eta;s)$

$$:=\begin{cases}\dfrac{q^{(a+1)s}-q^{(-a-1)s}}{q^s-q^{-s}}+\delta(\eta_0\notin L_0'^*)q^{-n_0/2+\partial}\dfrac{q^{as}-q^{-as}}{q^s-q^{-s}} & \text{if } \nu=0,\\[2ex] \Big\{(q^s-q^{-n_0/2}\beta_{S,\eta}-q^{-s+\partial-1}\delta(\beta_0\notin L_0'^*))q^{(f+a)s}\displaystyle\sum_{k=0}^{a}q^{(-s+m/2-1)k} & \\ -(q^{-s}-q^{-n_0/2}\beta_{S,\eta}-q^{s+\partial-1}\delta(\beta_0\notin L_0'^*))q^{(-f-a)s}\displaystyle\sum_{k=0}^{a}q^{(s+m/2-1)k}\Big\} & \\ \times\dfrac{1}{q^s-q^{-s}} & \text{if } \nu\geq 1.\end{cases}$$

2.2. Proof of Theorem 2.1 In this subsection we give a proof of Theorem 2.1. The first part has been proved by Murase and Sugano (cf. [5, Theorem 1.9]). In the rest of this section we assume that $\eta\neq 0$.

When S is anisotropic, the Iwasawa decomposition for $\bar{n}_0(X)(X\in V-L)$ is

$$\bar{n}_0(X)=\begin{pmatrix}Z_X^{-1} & Z_X^{-1}{}^tXS & -1\\ 0 & 1_{n_0} & -X\\ 0 & 0 & Z_X\end{pmatrix}\begin{pmatrix}0 & 0 & -1\\ 0 & 1_{n_0}-Z_X^{-1}X^tXS & Z_X^{-1}X\\ -1 & -Z_X^{-1}{}^tXS & Z_X^{-1}\end{pmatrix},$$

where $Z_X = \frac{1}{2}S[X]$. Hence for any $\eta_0 \in L^*_{1,\mathrm{prim}}$ and a non-negative integer a we obtain

$$\begin{aligned}
I(S,p^a\eta_0;s) &= 1 + \int_{V_0-L_0} |\tfrac{1}{2}S[X]|^{-s-m/2}\chi(-p^aS(\eta_0,X))dX \\
&= (1-q^{-s-n_0/2})\left\{\begin{matrix} 1+q^{-s-n_0/2+\partial} & \text{if } \eta_0 \in L_0'^* \\ 1 & \text{if } \eta_0 \notin L_0'^* \end{matrix}\right\} \\
&\quad \times q^{-as}\left\{\sum_{j=0}^{a} q^{(-2j+a)s} + q^{-n_0/2+\partial}\delta(\eta_0 \notin L_0'^*)\sum_{j=0}^{a-1} q^{(-2j+a-1)s}\right\}.
\end{aligned}$$

This proves Theorem 2.1 in the case of $v = 0$.

Hereafter we assume that $v \geq 1$ and $\eta_0 \in L^*_{v,\mathrm{prim}}$ is as in (2.2). Let $\varphi_{i,j}$ be the characteristic function of $M_{i,j}(\mathfrak{o})$. We often omit the suffix i,j. For $g \in G_{v+1}$, it is easily seen that

$$\int_{k^\times} \varphi_{1,m}(t(0\ 0_m\ 1)g))|t|^{s+m/2}d^\times t = \zeta_{\mathfrak{p}}(s+m/2)|t_{v+1}(g)|^{s+m/2}, \tag{2.7}$$

where $d^\times t$ is the Haar measure normalized as $\int_{\mathfrak{o}^\times} d^\times t = 1$.

LEMMA 2.2. *We write*

$$\eta_0 = \begin{pmatrix} \alpha \\ 0_{v-1} \\ \beta \\ 0_{v-1} \\ 1 \end{pmatrix} \in L^*_{v,\mathrm{prim}}, \qquad (\alpha,\beta) \in \mathfrak{o} \times L_0^*.$$

Then we have

$$I(S_v,\eta_0;s) = \zeta_{\mathfrak{p}}\left(s+\frac{m}{2}\right)^{-1} I(S_0,(-\alpha,-\beta);s),$$

where

$$I(S_0,(-\alpha,-\beta);s) := 1 + \int_{k-\mathfrak{o}} dx \int_{L_0} dX_0 |x^{-1}|^{s-n_0/2+1}\chi(x(\alpha+S_0(\beta,X_0)-\tfrac{1}{2}S_0[X_0])).$$

PROOF. We put $I_v = \zeta_{\mathfrak{p}}\left(s+\frac{m}{2}\right)I(S_v,\eta_0;s)$. By (2.7) we note

$$\begin{aligned}
I_v &= \int_V dX \int_{k^\times} d^\times t\,\varphi(t(0\ 0_m\ 1)\bar{n}(X))\chi(-S(\eta,X))|t|^{s+m/2}. \\
&= \int_{\mathfrak{o}^{v-1}} d\mathbf{x} \int_{k^{v-1}} d\mathbf{y} \int_k dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\,\varphi(tx)\varphi(t^tX_0S_0)\varphi(ty)\varphi(t\mathbf{y}) \\
&\quad \times \varphi({}^t\mathbf{x}\mathbf{y} + t(xy+\tfrac{1}{2}S_0[X_0]))\chi(\alpha x + S_0(\beta,X_0)+y)|t|^{s+n_0/2+1}.
\end{aligned}$$

If $v \geq 2$, we deform the above expression as follows:

$$I_v = \int_{\mathfrak{o}} dx_{v-1} \int_k dy_{v-1} \int_{\mathfrak{o}^{v-2}} d\mathbf{x} \int_{k^{v-2}} d\mathbf{y} \int_k dx \int_{V_0} dX_0 \int_k dy$$
$$\times \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(t\,{}^tX_0S_0)\varphi(ty)\varphi(ty_{v-1})\varphi(t\mathbf{y})$$
$$\times \varphi(x_{v-1}y_{v-1} + {}^t\mathbf{x}\mathbf{y} + t(xy + \tfrac{1}{2}S_0[X_0]))\chi(\alpha x + S_0(\beta, X_0) + y)|t|^{s+n_0/2+1}$$
$$= \int_{\mathfrak{o}^{v-2}} d\mathbf{x} \int_{k^{v-2}} d\mathbf{y} \int_k dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(t\,{}^tX_0S_0)\varphi(ty)\varphi(t\mathbf{y})$$
$$\times \varphi({}^t\mathbf{x}\mathbf{y} + t(xy + \tfrac{1}{2}S_0[X_0]))\chi(\alpha x + S_0(\beta, X_0) + y)|t|^{s+n_0/2+1}$$
$$+ \int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_{\mathfrak{o}^{v-2}} d\mathbf{x} \int_{k^{v-2}} d\mathbf{y} \int_k dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}}$$
$$\times d^\times t\, \varphi(tx)\varphi(t\,{}^tX_0S_0)\varphi(ty)\varphi(ty_{v-1})\varphi(t\mathbf{y})\varphi(x_{v-1}y_{v-1} + {}^t\mathbf{x}\mathbf{y} + t(xy + \tfrac{1}{2}S_0[X_0]))$$
$$\times \chi(\alpha x + S_0(\beta, X_0) + y)|t|^{s+n_0/2+1}.$$

We prove that the second term in the last expression vanishes. Let $f(t)$ [resp. $z(t)$] be the $\mathbb{C}$-valued [resp. k-valued] continuous function on $k^\times$. Then we have

$$\int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_k dx \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(ty)\varphi(ty_{v-1})$$
$$\times \varphi(x_{v-1}y_{v-1} + txy + z(t))\chi(\alpha x + y)f(t)$$
$$= \int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_{k-\mathfrak{o}} dx \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(ty)\varphi(ty_{v-1})$$
$$\times \varphi(x_{v-1}y_{v-1} + txy + z(t))\chi(\alpha x + y)f(t)$$
$$= \int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_{k-\mathfrak{o}} dx \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx^{-1}y)\varphi(tx^{-1}y_{v-1})$$
$$\times \varphi(x_{v-1}y_{v-1} + ty + z(tx^{-1}))\chi(\alpha x + y)f(tx^{-1})$$
$$= \int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_{k-\mathfrak{o}} dx \int_k dy \int_{\mathfrak{o}^\times} d^\times t\, \varphi(x^{-1}y)\varphi(tx^{-1}y_{v-1})$$
$$\times \varphi(x_{v-1}y_{v-1} + y + z(tx^{-1}))\chi(\alpha x + t^{-1}y)f(tx^{-1})$$
$$= \int_{\mathfrak{o}} dx_{v-1} \int_{k-\mathfrak{o}} dy_{v-1} \int_{k-\mathfrak{o}} dx \int_k dy \int_{\mathfrak{o}^\times} d^\times t\, \varphi(x^{-1}y - x^{-1}x_{v-1}y_{v-1})\varphi(x^{-1}y_{v-1})$$
$$\times \varphi(y + z(tx^{-1}))\chi(\alpha x + t^{-1}(y - x_{v-1}y_{v-1}))f(tx^{-1})$$
$$= 0.$$

This means that $I_\nu = I_{\nu-1} = \cdots = I_1$. Hence we only have to calulate I_1. By similar arguments as above, we get

$$\begin{aligned}
I_1 = &\int_{\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(t\,{}^tX_0S_0)\varphi(ty)\varphi(t(xy+\tfrac{1}{2}S_0[X_0])) \\
&\times \chi(\alpha x + S_0(\beta, X_0) + y)|t|^{s+n_0/2+1} \\
&+ \int_{k-\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(tx)\varphi(t\,{}^tX_0S_0)\varphi(ty)\varphi(t(xy+\tfrac{1}{2}S_0[X_0])) \\
&\times \chi(\alpha x + S_0(\beta, X_0) + y)|t|^{s+n_0/2+1} \\
= &\int_{\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(t\,{}^tX_0S_0)\varphi(y)\varphi(xy + t\tfrac{1}{2}S_0[X_0]) \\
&\times \chi(S_0(\beta, X_0) + t^{-1}y)|t|^{s+n_0/2} \\
&+ \int_{k-\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(t\,{}^t(x^{-1}X_0)S_0)\varphi(tx^{-1}y) \\
&\times \varphi(t(y + x^{-1}\tfrac{1}{2}S_0[X_0])\chi(\alpha x + S_0(\beta, X_0) + y)|x^{-1}|^{s+n_0/2+1}|t|^{s+n_0/2+1} \\
= &\int_{\mathfrak{o}} dx \int_{V_0} dX_0 \int_{\mathfrak{o}} dy \int_{\mathfrak{o}^\times} d^\times t\, \varphi(t\,{}^tX_0S_0)\varphi(t\tfrac{1}{2}S_0[X_0])\chi(S_0(\beta, X_0)) \\
&+ \int_{k-\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy \int_{k^\times \cap \mathfrak{o}} d^\times t\, \varphi(t\,{}^tX_0S_0)\varphi(t(x^{-1}y - \tfrac{1}{2}S_0[X_0]))\varphi(ty) \\
&\times \chi(\alpha x + xS_0(\beta, X_0) - x\tfrac{1}{2}S_0[X_0] + y)|x^{-1}|^{s-n_0/2+1}|t|^{s+n_0/2+1} \\
= 1 &+ \int_{k-\mathfrak{o}} dx \int_{V_0} dX_0 \int_k dy\, \varphi({}^tX_0S_0)\varphi(x^{-1}y - \tfrac{1}{2}S_0[X_0])\varphi(y) \\
&\times \chi(x(\alpha + S_0(\beta, X_0) - \tfrac{1}{2}S_0[X_0]) + y)|x^{-1}|^{s-n_0/2+1} \\
= 1 &+ \int_{k-\mathfrak{o}} dx \int_{L_0} dX_0 |x^{-1}|^{s-n_0/2+1}\chi(x(\alpha + S_0(\beta, X_0) - \tfrac{1}{2}S_0[X_0])),
\end{aligned}$$

and our lemma is proved. ■

The function $I(S_0, (-\alpha, -\beta); s)$ in Lemma 2.2 coincides with the function $I(S_0, (-\alpha, -\beta); q^{-s})$ in the notation of [12, (2.21)] and this function is calculated explicitly in [12, Proposition 2.14]. Therefore we have proved Theorem 2.1 for any $\eta_0 \in L^*_{\mathrm{prim}}$.

We now consider the general η.

Lemma 2.3. *We assume $\eta_0 \in L^*_{\mathrm{prim}}$ and $S[\eta_0] = 0$. Then for any $a \geq 0$, we* *have*

$$I(S_\nu,p^a\eta_0;s)=I(S_\nu,\eta_0;s)\sum_{l=0}^{a}q^{(-s+m/2-1)l}. \tag{2.8}$$

Proof. Using (2.7), we have

$$\begin{aligned}
\zeta_{\mathfrak{p}}\Big(s+\frac{m}{2}\Big)I(S_\nu,p^a\eta_0;s) &= \int_{\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\int_{k^\times\cap\mathfrak{o}}d^\times t\,\varphi(t\,{}^tXS_{\nu-1})\varphi(ty)\\
&\quad\times\varphi(t(xy+\tfrac{1}{2}S_{\nu-1}[X]))\chi(p^ay)|t|^{s+m/2}\\
&\quad+\int_{k-\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\int_{k^\times\cap\mathfrak{o}}d^\times t\,\varphi(tx)\\
&\quad\times\varphi(t\,{}^tXS_{\nu-1})\varphi(ty)\varphi(t(xy+\tfrac{1}{2}S_{\nu-1}[X]))\chi(p^ay)|t|^{s+m/2}\\
&=\int_{\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\int_{k^\times\cap\mathfrak{o}}d^\times t\,\varphi(t\,{}^tXS_{\nu-1})\varphi(y)\\
&\quad\times\varphi(xy+t\tfrac{1}{2}S_{\nu-1}[X])\chi(p^at^{-1}y)|t|^{s+m/2-1}\\
&\quad+\int_{k-\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\int_{k^\times\cap\mathfrak{o}}d^\times t\,\varphi(tx^{-1}\,{}^tXS_{\nu-1})\\
&\quad\times\varphi(tx^{-1}y)\varphi(t(y+x^{-1}\tfrac{1}{2}S_{\nu-1}[X]))\chi(p^ay)|x^{-1}|^{s+m/2}|t|^{s+m/2}\\
&=\sum_{l=0}^{a}\int_{V_{\nu-1}}\varphi(p^l\,{}^tXS_{\nu-1})\varphi(p^l\tfrac{1}{2}S_{\nu-1}[X])q^{(-s-m/2+1)l}dX\\
&\quad+\sum_{l=0}^{a}\int_{k-\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\int_{p^l\mathfrak{o}^\times}d^\times t\\
&\quad\times\varphi(p^lx^{-1}\,{}^tXS_{\nu-1})\varphi(p^lx^{-1}y)\varphi(p^l(y+x^{-1}\tfrac{1}{2}S_{\nu-1}[X]))\\
&\quad\times\chi(p^ay)|x^{-1}|^{s+m/2}q^{(-s-m/2)l}\\
&=\sum_{l=0}^{a}\int_{V_{\nu-1}}\varphi({}^tXS_{\nu-1})\varphi(p^{-l}\tfrac{1}{2}S_{\nu-1}[X])q^{(-s+m/2-1)l}dX\\
&\quad+\sum_{l=0}^{a}\int_{k-\mathfrak{o}}dx\int_{V_{\nu-1}}dX\int_k dy\,\varphi(p^l\,{}^tXS_{\nu-1})\\
&\quad\times\varphi(p^lx^{-1}(y-x\tfrac{1}{2}S_{\nu-1}[X]))\varphi(p^ly)\\
&\quad\times\chi\Big(p^a\Big(y-\frac{x}{2}S_{\nu-1}[X]\Big)\Big)|x^{-1}|^{s-m/2+2}q^{(-s-m/2)l}
\end{aligned}$$

$$= \sum_{l=0}^{a} q^{(-s+m/2-1)l} \int_{L_{\nu-1}} \delta(\tfrac{1}{2} S_{\nu-1}[X] \in \mathfrak{p}^l) dX$$

$$+ \sum_{l=0}^{a} \int_{k-\mathfrak{o}} dx \int_{V_{\nu-1}} dX \int_{\mathfrak{o}} dy\, \varphi({}^tXS_{\nu-1})$$

$$\times \varphi(x^{-1}y - p^{-l}x\tfrac{1}{2}S_{\nu-1}[X])\varphi(y)$$

$$\times \chi(p^{a-l}y - p^{a-2l}x\tfrac{1}{2}S_{\nu-1}[X])|x^{-1}|^{s-m/2+2} q^{(-s+m/2-1)l}$$

$$= \sum_{l=0}^{a} q^{(-s+m/2-1)l} \left\{ \int_{L_{\nu-1}} \delta(\tfrac{1}{2} S_{\nu-1}[X] \in \mathfrak{p}^l) dX \right.$$

$$+ \int_{k-\mathfrak{o}} dx \int_{L_{\nu-1}} dX\, \delta(\tfrac{1}{2} S_{\nu-1}[X] \in \mathfrak{p}^l)$$

$$\left. \times \chi(-p^{a-2l}x\tfrac{1}{2}S_{\nu-1}[X])|x^{-1}|^{s-m/2+2} \right\}$$

$$= \sum_{l=0}^{a} q^{(-s+m/2-1)l} \sum_{\lambda=l}^{\infty} \int_{L_{\nu-1}} \delta(\tfrac{1}{2} S_{\nu-1}[X] \in p^\lambda \mathfrak{o}^\times) dX$$

$$\times \left\{ 1 + \int_{k-\mathfrak{o}} |x^{-1}|^{s-m/2+2} \chi(p^{a-2l+\lambda}) dx \right\}$$

$$= \frac{\zeta_{\mathfrak{p}}\left(s - \frac{m}{2} + 1\right)}{\zeta_{\mathfrak{p}}\left(s - \frac{m}{2} + 2\right)} \sum_{l=0}^{a} q^{(-s+m/2-1)l}$$

$$\times \sum_{\lambda=l}^{\infty} (1 - q^{(-s+m/2-1)(a-2l+\lambda+1)})$$

$$\times \int_{L_{\nu-1}} \delta(\tfrac{1}{2} S_{\nu-1}[X] \in p^\lambda \mathfrak{o}^\times) dX.$$

To emphasize the primitivity we put

$$J_a := \frac{\zeta_{\mathfrak{p}}\left(s + \frac{m}{2}\right)\zeta_{\mathfrak{p}}\left(s - \frac{m}{2} + 2\right)}{\zeta_{\mathfrak{p}}\left(s - \frac{m}{2} + 1\right)} I(S_\nu, p^a \eta_0; s).$$

Then our task is to prove

$$J_a = J_0 \sum_{l=0}^{a} q^{(-s+m/2-1)l}. \tag{2.9}$$

For this purose, we introduce some notations:

$$f_{\nu-1}(T) := \sum_{\lambda=0}^{\infty} T^{\lambda} v_{\nu-1}(\lambda), \qquad v_{\nu-1}(\lambda) := \int_{L_{\nu-1}} \delta(\tfrac{1}{2} S_{\nu-1}[X] \in \mathfrak{p}^{\lambda}) dX.$$

Using the above notations, we express J_a as follows:

$$\begin{aligned}
J_a &= \sum_{l=0}^{a} \left\{ q^{(-s+m/2-1)l} \sum_{\lambda=l}^{\infty} (v_{\nu-1}(\lambda) - v_{\nu-1}(\lambda+1))(1 - q^{(-s+m/2-1)(a-2l+\lambda+1)}) \right\} \\
&= \sum_{l=0}^{a} \left\{ q^{(-s+m/2-1)l} \left\{ \sum_{\lambda=l}^{\infty} v_{\nu-1}(\lambda) - \sum_{\lambda=l}^{\infty} v_{\nu-1}(\lambda+1) \right\} \right. \\
&\quad - q^{(-s+m/2-1)(a-l+1)} \sum_{\lambda=l}^{\infty} v_{\nu-1}(\lambda) q^{(-s+m/2-1)\lambda} \\
&\quad \left. + q^{(-s+m/2-1)(a-l)} \sum_{\lambda=l}^{\infty} v_{\nu-1}(\lambda+1) q^{(-s+m/2-1)(\lambda+1)} \right\} \\
&= \sum_{l=0}^{a} \left\{ q^{(-s+m/2-1)l} v_{\nu-1}(l) + \sum_{\lambda=0}^{l-1} q^{(-s+m/2-1)(a-l+\lambda+1)} v_{\nu-1}(\lambda) \right. \\
&\quad - q^{(-s+m/2-1)(a-l+1)} f_{\nu-1}(q^{-s+m/2-1}) \\
&\quad \left. - \sum_{\lambda=0}^{l} q^{(-s+m/2-1)(a-l+\lambda)} v_{\nu-1}(\lambda) + q^{(-s+m/2-1)(a-l)} f_{\nu-1}(q^{-s+m/2-1}) \right\} \\
&= \left\{ (1 - q^{-s+m/2-1}) f_{\nu-1}(q^{-s+m/2-1}) \sum_{l=0}^{a} q^{(-s+m/2-1)(a-l)} + \sum_{l=0}^{a} q^{(-s+m/2-1)l} v_{\nu-1}(l) \right. \\
&\quad \left. + \sum_{l=0}^{a} \sum_{\lambda=0}^{l-1} q^{(-s+m/2-1)(a-l+\lambda+1)} v_{\nu-1}(\lambda) - \sum_{l=0}^{a} \sum_{\lambda=0}^{l} q^{(-s+m/2-1)(a-l+\lambda)} v_{\nu-1}(\lambda) \right\} \\
&= \zeta_{\mathfrak{p}}\left(s - \frac{m}{2} + 1\right)^{-1} f_{\nu-1}(q^{-s+m/2-1}) \sum_{l=0}^{a} q^{(-s+m/2-1)l}.
\end{aligned}$$

This means that J_a satisfies (2.9) ■

By Lemma 2.2, [12, Proposition 2.14], and Lemma 2.3, we obtain Theorem 2.1(ii).

In the rest of this section we assume η_0 is anisotropic. To emphasize the conductor and the primitivity we write $\eta_{f,0} = \eta_0$ and write $\eta_{f,a} = p^a \eta_{f,0}$. Let $\mathscr{H}_\nu = \mathscr{H}(G_\nu, K_\nu)$ be the Hecke algebra of the pair (G_ν, K_ν) i.e.

$$\begin{aligned}
\mathscr{H}(G_\nu, K_\nu) &:= \mathscr{H}_\nu \\
&= \{ f : G_\nu \to \mathbb{C} \mid f(u_1 g u_2) = f(g) \text{ for } u_1, u_2 \in K_\nu, \operatorname{supp} f \text{ is compact} \}.
\end{aligned}$$

For $0 \leq r \leq \nu$, we put

$$c_\nu^{(r)} := \mathrm{diag}(p1_r, 1_{n_0+2\nu-2r}, p^{-1}1_r),$$

$$C_\nu^{(r)} := K_\nu c_\nu^{(r)} K_\nu = \{g \in G_\nu \mid p \cdot g \in M_{n_0+2\nu}(\mathfrak{o}), \mathrm{rank}_{\mathfrak{o}/\mathfrak{p}}(pg) = r\}.$$

It is well known that $\mathscr{H}_\nu$ is generated by $C_\nu^{(r)}$ $(0 \leq r \leq \nu)$ (cf. [8]). For the sake of simplicity, we put

$$A = q^{-(n_0+2\nu)}\{C_{\nu+1}^{(1)} - (q^\partial - 1 + q^2 f_{\nu-1,1} + q^{\partial+1} - q)\},$$

$$B = q^{-(2n_0+4\nu-1)}\{C_{\nu+1}^{(2)} - \{(q^\nu - 1)(q^{n_0+\nu-1} + q^\partial) + (q^\partial - 1)(q^2 f_{\nu-1,1} + q^{\partial+1} - q)$$
$$+ f_{\nu-1,1}(q^4 f_{\nu-1,2} + q^{\partial+3} - q^2)\}\} - q^{-(2\nu+n_0-1)}(q f_{\nu-1,1} + q^\partial - 1)A + 2q^{-(2\nu+n_0)},$$

where $f_{\nu,j} = q^{j-1}(q^{\nu-j+1} - 1)(q^{\nu-j+n_0} + q^\partial)/(q^j - 1)$ (cf. [12, (7.44)]).

For $t \in k^\times$ and $g \in G_\nu$, we put

$$(t, g) = \begin{pmatrix} t & & \\ & g & \\ & & t^{-1} \end{pmatrix} \in G_{\nu+1}.$$

Let $\eta \in L_\nu^*$ be anisotropic. We denote by $\mathscr{W}_\eta^{\mathscr{F}}$ the space of functions W on $G_{\nu+1}$ satisfying

$$W(n_\nu(X)(1,h)gu) = \chi(S_\nu(\eta, X))W(g) \tag{2.10}$$

for any $X \in V_\nu$, $u \in K_{\nu+1}$ and $h \in G_\nu$ such that $h\eta = h$. The Hecke algebra $\mathscr{H}_{\nu+1}$ acts on $\mathscr{W}_\eta^{\mathscr{F}}$ by

$$W * \phi(g) = \int_{G_{\nu+1}} W(gu)\phi(u^{-1})du \qquad (\phi \in \mathscr{H}_{\nu+1}, W \in \mathscr{W}_\eta^{\mathscr{F}}),$$

where we normalize the measure so that the volume of $K_{\nu+1}$ is 1. It is easily seen that

$$\Phi_\eta(g) := \int_{V_\nu} |t_{\nu+1}(\bar{n}_\nu(X)g)|^{s+n_0/2+\nu}\chi(-S_\nu(\eta, X))dX$$

belongs to $W_\eta^{\mathscr{F}}$. For $f \geq 0$ and $a \in \mathbf{Z}$, we put

$$\Phi_{f,a} := \Phi_{\eta_{0,0}}((p^{f+a}, M_f)) = q^{(s-n_0/2-\nu)(f+a)}\Phi_{\eta_{f,a}}(1),$$

where

$$M_f = \begin{pmatrix} p^{-f} & & \\ & 1_{n_0+2\nu-2} & \\ & & p^f \end{pmatrix}.$$

Note that $\Phi_{f,a} = 0$ for negative a.

Lemma 2.4. *Let $\eta \in L_\nu^*$ be anisotropic. The function Φ_η is a simultaneous eigen function of $\mathscr{H}_{\nu+1}$. The eigenvalue $\lambda(A)$ [resp. $\lambda(B)$] of A [resp. B] is*

$$\lambda(A) = q^{-(n_0+2\nu)}\{q^{-s+n_0/2+\nu} + q^{s+n_0/2+\nu} + q^{n_0+2\nu-1} + q\}$$

$$[\text{resp. } \lambda(B) = q^{-(2n_0+4\nu-1)}\{(q^{-s+n_0/2+\nu} + q^{s+n_0/2+\nu})(q^{2\nu+n_0-2} + 1) + 2q^{n_0+2\nu-1}\}].$$

Proof. Let $\phi \in \mathscr{H}_{\nu+1}(G_{\nu+1}, K_{\nu+1})$ be the characteristic function on $K_{\nu+1}hK_{\nu+1} = \coprod_{i\in I} h_i K_{\nu+1}$. Then we have easily

$$\Phi_\eta * \phi(g) = \left(\sum_{i\in I} |t_{\nu+1}(h_i)|^{s+n_0/2+\nu}\right)\Phi_\eta(g) \qquad \text{for } g \in G_{\nu+1}.$$

Hence we know Φ_η is the simultaneous eigen function of $\mathscr{H}_{\nu+1}$. Using the explicit coset decomposotion of $C_{\nu+1}^{(1)}$ and $C_{\nu+1}^{(2)}$ (cf. [12, Lemma 7.1]), we have eigenvalues $\lambda(A)$ and $\lambda(B)$. ■

Proposition 2.5. *Let $\eta \in L_\nu^*$ be anisotropic. The function Φ_η satisfies the following additional relation:*

$$\Phi_{\eta_{f,a}} = \sum_{t=0}^{a} q^{(-s+m/2-1)t}\Phi_{\eta_{f+a-t,0}} \qquad \textit{for } a, f \geq 0. \tag{2.11}$$

Proof. Lemma 2.4 implies

$$0 = (\Phi * \{(q^{-1} + q^{-(n_0+2\nu-1)})A + (q^{-2} + q^{-(2n_0+4\nu-2)}) - B\})_{f,a}.$$

Hence, by [12, Corollary 7.6], we have

(2.12)

$$\begin{aligned}
0 = {} & q^{-(4\nu+2n_0-1)}\{q\Phi_{f-1,a+1} - q\Phi_{f,a} - \Phi_{f-1,a} + \Phi_{f,a-1}\} \\
& + q^{-(2\nu+n_0+1)}\{q(\Phi_{f-1,a+1} - \Phi_{f,a}) - q(\Phi_{f,a} - \Phi_{f+1,a-1}) \\
& + (\Phi_{f,a-1} - \Phi_{f+1,a-2}) - q^2(\Phi_{f-1,a+2} - \Phi_{f,a+1})\} \\
& + q^{-1}\{\Phi_{f,a+1} - \Phi_{f+1,a} - q^{-1}(\Phi_{f,a} - \Phi_{f+1,a-1})\} \\
& + \delta(f=0)\{(q^{\nu-1}\beta_{S_\nu,\eta} + \rho_\eta)\{q^{-(2\nu+n_0+1)}(q\Phi_{0,a} - q\Phi_{1,a-1}) \\
& - q^2\Phi_{0,a+1} + q^2\Phi_{1,a}) + q^{-(n_0+4\nu-1)}(q\Phi_{0,a} - q\Phi_{1,a-1} - \Phi_{0,a-1} + \Phi_{1,a-2})\} \\
& + q^{-(n_0+2\nu+1)}(q\Phi_{0,a} - q^2\Phi_{0,a+1}) + q^{-(2n_0+4\nu-1)}(q\Phi_{0,a} - \Phi_{0,a-1})\} \\
& + \delta(f=1)\rho_\eta\{q^{-(n_0+2\nu+1)}(q\Phi_{0,a+1} - q\Phi_{1,a} - q^2\Phi_{0,a+2} + q^2\Phi_{1,a+1})\} \\
& + q^{-(n_0+4\nu-1)}(q\Phi_{0,a+1} - q\Phi_{1,a} - \Phi_{0,a} + \Phi_{1,a-1})\} \\
& + \delta(a=f=0)\{-q^{-(2n_0+4\nu-1)}(q^{\nu-1}\beta_{S_\nu,\eta} + \rho_\eta)qW_{0,0}\} \\
& + \delta(a=0)q^{-(n_0+2\nu+1)}q\Phi_{f,0}
\end{aligned}$$

where we put $\rho_\eta = q^{\partial-1}\,\delta$ $(\eta \notin L'^*_\nu)$. Using (2.12), we obtain

$$\text{(2.13)}\qquad \begin{cases} \Phi_{0,1} - \Phi_{1,0} = q^{-1}\Phi_{0,0} & \text{for } (f,a) = (0,0) \\ \Phi_{0,2} - \Phi_{1,1} = q^{-2}\Phi_{0,0} & \text{for } (f,a) = (0,1) \\ \Phi_{1,1} - \Phi_{2,0} = q^{-1}\Phi_{1,0} & \text{for } (f,a) = (1,0) \\ \Phi_{0,3} - \Phi_{1,2} = q^{-3}\Phi_{0,0} & \text{for } (f,a) = (0,2) \\ \Phi_{1,2} - \Phi_{2,1} = q^{-2}\Phi_{1,0} & \text{for } (f,a) = (1,1) \\ \Phi_{2,1} - \Phi_{3,0} = q^{-1}\Phi_{2,0} & \text{for } (f,a) = (2,0). \end{cases}$$

We assume that the following equations are valid for $3 \le l \le L$:

$$\text{(2.14)}\qquad \Phi_{n,l-n} - \Phi_{n+1,l-n-1} = q^{-l+n}\Phi_{n,0} \qquad (0 \le n \le l-1).$$

Then we can prove that (2.14) is valid for $l = L+1$ in the same way as (2.13). By the induction on l we have proved the following relation:

$$\Phi_{f,a} - \Phi_{f+1,a-1} = q^{-a}\Phi_{f,0} \qquad \text{for } a, f \ge 0.$$

Note that this relation is equivalent to

$$\Phi_{f,a} = \sum_{t=0}^{a} q^{-t}\Phi_{f+a-t,0}. \qquad \text{for } a, f \ge 0. \qquad \blacksquare$$

By Lemma 2.2, [12, Proposition 2.14], and Proposition 2.5, we obtain Theorem 2.1(iii) in the case of $\nu \ge 1$. Therefore we have proved Theorem 2.1 completely.

REMARK. The following proposition gives another proof of Theorem 2.1 (iii) in the case of $\nu \ge 1$ and $\eta \in L^*_{\mathrm{prim}}$.

PROPOSITION 2.6.

$$\Phi_{\eta_f,0} = q^{-fs}\left\{\frac{q^{(f+1)s} - q^{-(f+1)s}}{q^s - q^{-s}} - q^{-n_0/2}\beta_{S_\nu,\eta}\,\frac{q^{fs} - q^{-fs}}{q^s - q^{-s}} \right.$$
$$\left. - q^{\partial-1}\delta(\beta_0 \in L'^*_0)\,\frac{q^{(f-1)s} - q^{-(f-1)s}}{q^s - q^{-s}}\right\}\Phi_{\eta_0,0}$$

PROOF. By Lemma 2.4 and [12, Corollary 7.8], we have

$$\lambda(A)\Phi_{f,0} = \Phi_{f+1,0} + \{q^{-1} + q^{-(n_0+2\nu-1)}(1 + \delta(f=0)q^{\nu-1}\beta_{S_\nu,\eta})\}\Phi_{f,0}$$
$$+ q^{-(n_0+2\nu)}(1 + \delta(f=1)\rho_\eta)\Phi_{f-1,0} \qquad \text{for } f \ge 0.$$

Since $\Phi_{f,a} = q^{(-s-n_0/2-\nu)(f+a)}\Phi_{\eta_{f,a}}$, we obtain

$$\Phi_{\eta_{f+1,0}} = (1 + q^{2s} - q^{-s-n_0/2}\delta(f = 0)\beta_{S_\nu,\eta})\Phi_{\eta_{f,0}} - q^{-2s}\{1 + \delta(f = 1)\rho_\eta\}\Phi_{\eta_{f-1,0}}.$$

This recurrence formula can be easily solved. ∎

3. Archimedean part

3.1. Hypergeometric functions In this subsection, we summarize some properties of hypergeometric functions studied in Shimura [10]. We put

$$\mathscr{P} := \{X \in \mathbb{R}^{m+2} | S_1[X] > 0, S_1(X, Y_0) > 0\}.$$

For $h \in \mathbb{R}^{m+2}$ and $g \in \mathscr{P}$, we define the eigenvalues of h relative to g by the roots of the quadratic equation

$$t^2 - S_1(h,g)t + S_1[h]S_1[g]/4 = 0.$$

Notice that the above quadratic equation has only real roots, since signature of S_1 is $(1, m+1)$. We then put (cf [10, (4.1)])

$$(3.1)\quad \begin{cases} \delta_+(h,g) = \text{the product of all positive eigenvalues of } h \text{ relative to } g, \\ \delta_-(h,g) = \delta_+((-h),g), \\ \delta(h,g) = \delta_+(h,g)\delta_-(h,g), \\ \tau(h,g) = \text{the sum of all absolute values of nonzero eigenvalues of } h \\ \qquad\qquad \text{relative to } g, \\ \mu(h,g) = \text{the smallest absolute value of nonzero eigenvalues of } h \\ \qquad\qquad \text{relative to } g \text{ if } h \neq 0; \mu(h,g) = 1 \text{ if } h = 0 \\ \lambda(h,g) = \text{the largest absolute value of nonzero eigenvalues of } h \\ \qquad\qquad \text{relative to } g \text{ if } h \neq 0; \mu(h,g) = 1 \text{ if } h = 0. \end{cases}$$

We write $\mu(h) = \mu(h, Y_0)$ and $\lambda(h) = \lambda(h, Y_0)$. Set

$$(3.2)\quad \xi(g,h;\alpha,\beta) = \int_{\mathbb{R}^{m+2}} e[\tfrac{1}{2}S_1(h,X)](\tfrac{1}{2}S_1[X + i\cdot g])^{-\alpha}(\tfrac{1}{2}S_1[X - i\cdot g])^{-\beta}dX$$

for $(g,h) \in \mathscr{P} \times \mathbb{R}^{m+2}$ and $(\alpha,\beta) \in \mathbb{C}^2$. In [10], Shimura studied a function $\omega(g,h;\alpha,\beta)$ defined for $(g,h,\alpha,\beta) \in \mathscr{P} \times \mathbb{R}^{m+2} \times \mathbb{C}^2$ which is holomorphic in (α,β) and satisfies

$$(3.3)\quad \xi(g,h;\alpha,\beta) = |\det S_1|^{-1/2} 2^{-\alpha-\beta+m+2} i^{2\beta-2\alpha} S_1[g]^{-\alpha-\beta+m/2+1} \omega(2\pi g,h;\alpha,\beta)$$

$$\times \begin{cases} 2^{m+3}\pi^{2\alpha-m/2}\Gamma(\alpha)^{-1}\Gamma(\alpha-m/2)^{-1}|S_1[h]S_1[g]|^{\alpha-m/2-1} \\ \qquad \text{if } S_1[h]>0, S_1(Y_0,h)>0, \\ 2^{m+3}\pi^{2\beta-m/2}\Gamma(\beta)^{-1}\Gamma(\beta-m/2)^{-1}|S_1[h]S_1[g]|^{\beta-m/2-1} \\ \qquad \text{if } S_1[h]>0, S_1(Y_0,h)<0, \\ 2^{\alpha+\beta+m/2+2}\pi^{\alpha+\beta-m/2}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\delta_+(h,g)^{\alpha-m/4-1}\delta_-(h,g)^{\beta-m/4-1} \\ \qquad \text{if } S_1[h]<0, \\ 2^{\alpha+m/2+3}\pi^{\alpha-m/2+1}\Gamma(\alpha+\beta-m/2-1)\Gamma(\alpha)^{-1} \\ \qquad \times\Gamma(\beta)^{-1}\Gamma(\alpha-m/2)^{-1}|S_1(h,g)|^{\alpha-m/2-1} \qquad \text{if } S_1[h]=0, S_1(h,Y_0)>0, \\ 2^{\beta+m/2+3}\pi^{\beta-m/2+1}\Gamma(\alpha+\beta-m/2-1)\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} \\ \qquad \times\Gamma(\alpha-m/2)^{-1}|S_1(h,g)|^{\beta-m/2-1} \qquad \text{if } S_1[h]=0, S_1(h,Y_0)<0, \\ 2\pi^{m/2+2}\Gamma(\alpha+\beta-m/2-1)\Gamma(\alpha+\beta-m-1)\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} \\ \qquad \times\Gamma(\alpha-m/2)^{-1}\Gamma(\beta-m/2)^{-1} \qquad \text{if } h=0. \end{cases}$$

The following theorem is one of the main results of [10].

LEMMA 3.1 (Shimura [10] Theorem 4.1). *The function ω satisfies*

$$(3.4)\quad \omega(g,h;\alpha,\beta) = \begin{cases} \omega(g,h;m/2+1-\beta,m/2+1-\alpha) & \text{if } h=0 \text{ or } S_1[h]\neq 0, \\ \omega(g,h;m+1-\beta,m+1-\alpha) & \text{if } S_1[h]=0. \end{cases}$$

If (α,β) stays in a compact subset T of $\mathbf{C}^2$, then

$$(3.5)\qquad |\omega(g,h;\alpha,\beta)| \leq Ae^{-\tau(h,g)/2}(1+\mu(h,g)^{-B}),$$

where A and B are positive constants depending only on T and S_1.

We denote by $W_{\kappa,\mu}(z)$ the classical Whittaker function

$$(3.6)\qquad W_{\kappa,\mu}(z) = \frac{z^{\kappa}e^{-z/2}}{\Gamma(\mu+1/2-\kappa)}\int_0^{\infty} t^{\mu-\kappa-1/2}e^{-t}\left(1+\frac{t}{z}\right)^{\mu+\kappa-1/2}dt$$
$$(\mathrm{Re}(\mu+1/2-\kappa)>0, |\arg z|<\pi),$$

which is continued to the whole $\mathbf{C}^2$ as a holomorphic function in (κ,μ) and satisfies $W_{\kappa,\mu}=W_{\kappa,-\mu}$. By [10, (4.29)] if $h\in\mathbf{R}^{m+2}$ and $S_1[h]=0$ we have

$$(3.7)\quad \omega(g,h;\alpha,\beta) = 2^{-m-3}\pi^{m/2}|S_1(h,g)|^{(\beta-\alpha)/2}W_{(\alpha-\beta)/2,(\alpha+\beta-1)/2}(|S_1(h,g)|).$$

The following lemma is well-known (cf. [10]).

LEMMA 3.2. *The function $W_{\kappa,\mu}$ satisfies*

$$|z^{-\kappa} W_{\kappa,\mu}(z)| \leq Ae^{-z/2}(1+z^{-B}) \qquad \text{for } z > 0,$$

if (κ, μ) stays in a compact subset T of $\mathbb{C}^2$, where A and B are positive constants depending only on T.

3.2. Calculation of $\mathscr{I}_\infty(t, \eta; s)$ As is well-known, taking a suitable $\mathbb{R}$-basis of $\mathbb{R}^m$, we may assume that

$$S = \operatorname{diag}(-2a_1 \ldots -2a_m), \qquad a_i > 0 \ (0 \leq i \leq m).$$

We assume that $t > 0$, $\eta \in L^*$ and $\operatorname{Re} s > m/2$. We calculate

$$\mathscr{I}_\infty(t, \eta; s) = t^{-s+m/2} \int_{\mathbb{R}^m} |t_1(\bar{n}_1(X))|^{s+m/2} e[-S(t\eta, X)]dX. \tag{3.8}$$

PROPOSITION 3.3. *Let s be a complex number with $\operatorname{Re} s > m/2$. For $t > 0$, we have the followings:*
(i) *When $\eta = 0$, we have*

$$\mathscr{I}_\infty(t, 0; s) = t^{-s+m/2} |\det S|^{-1/2} (2\pi)^{m/2} \frac{\Gamma(s)}{\Gamma(s+m/2)}.$$

(ii) *When $0 \neq \eta \in L^*$, we have*

$$I_\infty(t, \eta; s) = |\det S|^{-1/2} \frac{2^{(2s+2m-1)/4} \pi^{s+m/2} t^{s-1/2} |S[\eta]|^{(2s-1)/4}}{\Gamma(s+m/2)} W_{0,s}(8\pi t \sqrt{|\tfrac{1}{2} S[\eta]|}).$$

PROOF. Since $\mathscr{I}_\infty(t, \eta; s) = \mathscr{I}_\infty(t, h\eta; s)$ for $h \in G_\infty^0$, we may assume that

$$\eta = \begin{pmatrix} 0_{m-1} \\ a_m^{-1/2} N_\eta \end{pmatrix},$$

where we put $N_\eta = \sqrt{|\frac{1}{2} S[\eta]|}$. To obtain an explicit description of $|t_1(\bar{n}_1(X))|^{s+m/2}$, we take a decomposition

$$\bar{n}_1(X) = n_1(X') \begin{pmatrix} y' & & \\ & h & \\ & & y'^{-1} \end{pmatrix} k' \in P_{1,\infty} K_{1,\infty}.$$

For $\mathbf{X}_0 = (0_m, 1)$, we have

$$j(\bar{n}_1(x), \mathbf{X}_0) = 1 - \tfrac{1}{2} S[X], \qquad j\left(n_1(X') \begin{pmatrix} y' & & \\ & h & \\ & & y'^{-1} \end{pmatrix} k', \mathbf{X}_0 \right) = y'^{-1} j(k', \mathbf{X}_0).$$

Since $j(k', \mathbf{X}_0)^2 = 1$, we obtain

$$|y'| = (1 - \tfrac{1}{2}S[X])^{-1}.$$

Therefore we have

$$\begin{aligned}\mathscr{I}_\infty(t, \eta; s) &= t^{-s+m/2} \int_{\mathbb{R}^m} (1 - \tfrac{1}{2}S[X])^{-s-m/2} e[-S(\eta, X)]dX \\ &= t^{-s+m/2} \left|\det \frac{S}{2}\right|^{-1/2} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} (1 + \|\mathbf{x}\|^2 + x^2)^{-s-m/2} e[-2tN_\eta x] d\mathbf{x}\, dx,\end{aligned}$$

where we put $\|\mathbf{x}\| = ({}^t\mathbf{x}\mathbf{x})^{1/2}$.

We assume that $\eta = 0$. Making use of the formula

$$\int_{\mathbb{R}^m} (1 + \|\mathbf{x}\|^2)^{-s-m/2} d\mathbf{x} = \pi^{m/2} \frac{\Gamma(s)}{\Gamma(s + m/2)},$$

we get the first assertion of our proposition.

We assume that $\eta \neq 0$. By the change of polar coordinates

$$\begin{aligned}(3.9)\quad & \int_{\mathbb{R}^m} \int_{\mathbb{R}} (1 + \|\mathbf{x}\|^2 + x^2)^{-s-m/2} e[-2tN_\eta x] d\mathbf{x}\, dx \\ &= \Omega_{m-2} \cdot \int_{\mathbb{R}} \int_0^\infty (1 + x^2 + r^2)^{-s-m/2} r^{m-2} dr e[-2tN_\eta x] dx \\ &= \Omega_{m-2} \int_0^\infty (1 + r^2)^{-s-m/2} r^{m-2} dr \int_{\mathbb{R}} (1 + x^2)^{-s-1/2} e[-2tN_\eta x] dx \\ &= 2^{-1}\Omega_{m-2} \cdot \int_0^\infty (1 + r)^{-s-m/2} r^{(m-3)/2} dr \int_{\mathbb{R}} (1 + x^2)^{-s-1/2} e[-2tN_\eta x] dx \\ &= \pi^{(m-1)/2} \frac{\Gamma(s + 1/2)}{\Gamma(s + m/2)} \cdot \int_{\mathbb{R}} (1 + x^2)^{-s-1/2} e[-2tN_\eta x] dx,\end{aligned}$$

where Ω_{m-2} is the volume of the $m-2$ dimensional unit sphere. As is well-known, the last integral in (3.9) becomes as follows (cf. [9], [10]):

$$\int_{\mathbb{R}} (1 + x^2)^{-s-1/2} e[-2tN_\eta x] dx = \frac{t^{s-1/2} 2^{s-1/2} \pi^{s+1/2} N_\eta^{s-1/2}}{\Gamma(s + 1/2)} W_{0,s}(8\pi t N_\eta). \quad \blacksquare$$

3.3. Calculation of $I_{l,\infty}(g, \eta; s)$ In this section we assume that l is a non-negative even integer, $\mathrm{Re}\, s > m/2 + 1$ and $g = \mathrm{diag}(t, h, t^{-1}) \in G^0_{2,\infty}$. We calculate the integral

$$I_{l,\infty}(g, \eta; s) = t^{-s+m/2+1} \int_{\mathbb{R}^{m+2}} f_{l,\infty}\left(\bar{n}_2(X); s + \frac{m}{2} + 1\right) e[-S_1(h^{-1}\eta t, X)]dX.$$

For this purpose, we introduce several polynomials in s

$$
(3.10)\qquad
\begin{aligned}
P_l(s) &:= P_l^{(+)}(s)P_l^{(-)}(s),\\
P_l^{(+)}(s) &:= \prod_{j=0}^{l/2-1}((2s+m+2)/4+j),\\
P_l^{(-)}(s) &:= \prod_{j=0}^{l/2-1}((2s-m+2)/4+j)
\end{aligned}
$$

and we put

$$
(3.11)\quad Q_{l,\eta}(s) := \begin{cases}
P_l(-s)\cdot P_l(s)^{-1} & \text{if } \eta = 0\\
(-1)^{l/2}P_l^{(-)}(-s)\cdot P_l(s)^{-1} & \text{if } S_1[\eta]=0, S_1(\eta,Y_0)>0\\
P_l(-s)\cdot P_l^{(+)}(s)^{-1} & \text{if } S_1[\eta]=0, S_1(\eta,Y_0)<0\\
P_l(s)^{-1} & \text{if } S_1[\eta]>0, S_1(\eta,Y_0)>0\\
P_l(-s) & \text{if } S_1[\eta]>0, S_1(\eta,Y_0)<0\\
(-1)^{l/2}P_l^{(-)}(-s)\cdot P_l^{(+)}(s)^{-1} & \text{if } S_1[\eta]<0.
\end{cases}
$$

The next proposition is the main result in this subsection.

Proposition 3.4. *Let l be a non-negative even integer and let s be a complex number with* $\mathrm{Re}\, s > m/2+1$. *For*

$$
g = \mathrm{diag}(t,h,t^{-1}) \in G_{2,\infty}^0 \qquad \textit{and} \qquad hY_0t = Y,
$$

we have the following.

(i) *If* $\eta = 0$,

$$
\begin{aligned}
I_{l,\infty}(g,0;s) &= |\det S_1|^{-1/2}2^{(-6s+m+6)/4}\pi^{m/2+2}S_1[Y]^{(-2s+m+2)/4}\\
&\quad\times Q_{l,\eta}(s)\,\frac{\Gamma(s)\Gamma(s-m/2)}{\Gamma((2s+m+2)/4)^2\Gamma((2s-m+2)/4)^2}.
\end{aligned}
$$

(ii) *If* $\eta\in L_1^*$ *and* $S_1[\eta]=0$,

$$
\begin{aligned}
I_{l,\infty}(g,\eta;s) &= |\det S_1|^{-1/2}2^{-s+1}\pi^{(2s+m+6)/4}\left(\frac{S_1[Y]}{2|S_1(Y,\eta)|}\right)^{(-2s+m+2)/4}\\
&\quad\times Q_{l,\eta}(s)\,\frac{\Gamma(s)}{\Gamma((2s+m+2)/4)^2\Gamma((2s-m+2)/4)^2}\\
&\quad\times W_{\pm l/2,(2s-m)/4}(4\pi|S_1(Y,\eta)|) : S_1(\eta,Y_0)\gtrless 0.
\end{aligned}
$$

(iii) *If* $\eta \in L_1^*$ *and* $S_1[\eta] > 0$,

$$I_{l,\infty}(g,\eta;s) = |\det S_1|^{-1/2} 2^{(-2s+3m\pm 4l+10)/4} \pi^{s\pm l+1} S_1[Y]^{\pm l/2} S_1[\eta]^{(2s-m\pm 2l-2)/4}$$
$$\times Q_{l,\eta}(s) \frac{1}{\Gamma((2s+m+2)/4)\Gamma((2s-m+2)/4)}$$
$$\times \omega(2\pi Y, 2\eta; (2s+m+2l+2)/4, (2s+m-2l+2)/4) : S_1(\eta, Y_0) \gtrless 0.$$

(iv) *If* $\eta \in L_1^*$ *and* $S_1[\eta] < 0$,

$$I_{l,\infty}(g,\eta;s)$$
$$= |\det S_1|^{-1/2} 2^{(-2s+5m+14)/4} \pi^{s+1} S_1[Y]^{m/4} S_1[\eta]^{(s-1)/2} \delta_+(Y,\eta)^{l/2} \delta_-(Y,\eta)^{-l/2}$$
$$\times Q_{l,\eta}(s) \frac{1}{\Gamma((2s+m+2)/4)^2}$$
$$\times \omega(2\pi Y, 2\eta; (2s+m+2l+2)/4, (2s+m-2l+2)/4).$$

Notice that

$$\Gamma\left(\frac{2s+m+2l+2}{4}\right) = P_l^{(+)}(s)\Gamma\left(\frac{2s+m+2}{4}\right),$$
$$\Gamma\left(\frac{2s+m-2l+2}{4}\right) = \frac{(-1)^{l/2}}{P_l^{(-)}(-s)}\Gamma\left(\frac{2s+m+2}{4}\right),$$
$$\Gamma\left(\frac{2s-m+2l+2}{4}\right) = P_l^{(-)}(s)\Gamma\left(\frac{2s-m+2}{4}\right),$$
$$\Gamma\left(\frac{2s-m-2l+2}{4}\right) = \frac{(-1)^{l/2}}{P_l^{(+)}(-s)}\Gamma\left(\frac{2s-m+2}{4}\right).$$

Hence Proposition 3.4 follows from the next Lemma, (3.3) and (3.7).

LEMMA 3.5. *Notation being as above,*

$$I_{l,\infty}(g,\eta;s) = (\tfrac{1}{2}S_1[\eta])^{(2s+m+2)/4}\xi\left(Y, 2\eta; \frac{2s+m+2l+2}{4}, \frac{2s+m-2l+2}{4}\right).$$

PROOF. By means of the similar method in the proof of Proposition 3.3, comparing the automorphy factor of $\bar{n}_2(X)$ and the Iwasawa decomposition of $\bar{n}_2(X)$, we get

$$1 - \tfrac{1}{2}S_1[X] - iS_1(X, Y_0) = t_\infty(\bar{n}_2(X))^{-1} J(k_\infty(\bar{n}_2(X)), Z_0).$$

Since $|J(k_\infty(\bar{n}_2(X)),Z_0)|=1$, we have

(3.12) $$t_\infty(\bar{n}_2(X))=\{(-1+\tfrac{1}{2}S_1[X])^2+(S_1(X,Y_0))^2\}^{-1/2},$$

(3.13) $$\begin{aligned} &J(k_\infty(\bar{n}_2(X)),Z_0)^{-l} \\ &\quad=\{(-1+\tfrac{1}{2}S_1[X])^2+(S_1(X,Y_0))^2\}^{l/2}(-1+\tfrac{1}{2}S_1[X]+iS_1(X,Y_0))^{-l}. \end{aligned}$$

By (3.12) and (3.13) we obtain

$$\begin{aligned} &f_{l,\infty}\left(\bar{n}_2(X);s+\frac{m}{2}+1\right) \\ &\quad=t_\infty(\bar{n}_2(X))^{s+m/2+1}J(k_\infty(\bar{n}_2(X)),Z_0)^{-l} \\ &\quad=(\tfrac{1}{2}S_1[X+iY_0])^{(-2s-m-2l-2)/4}(\tfrac{1}{2}S_1[X-iY_0])^{(-2s-m+2l-2)/4}. \end{aligned}$$

Since $t=(\frac{1}{2}S_1[Y])^{1/2}$ and $Y=hY_0t$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{m+2}}f_{l,\infty}\left(\bar{n}_2(X);s+\frac{m}{2}+1\right)e[-S_1(h^{-1}\eta t,X)]dX \\ &\quad=(\tfrac{1}{2}S_1[Y])^s\xi\left(Y,2\eta;\frac{2s+m+2l+2}{4},\frac{2s+m-2l+2}{4}\right). \end{aligned}$$ ∎

4. Calculation of $I_l'(g,\eta;s)$

In this section we assume that l is a non-negative even integer, $\operatorname{Re}s>m/2+1$ and

$$g=\begin{pmatrix} t & & \\ & h & \\ & & t^{-1}\end{pmatrix}\in G_{2,\infty}^0,\qquad hY_0t=Y.$$

Let us calculate

$$I_l'(g,\eta;s)=\sum_{\gamma_1\in P_{1,\mathbb{Q}}\backslash G_{1,\mathbb{Q}}}F_l\left(\gamma_1 g,\gamma_1\eta;s+\frac{m}{2}+1\right),$$

where

$$F_l(g,\eta;s)=\int_{\mathbb{Q}^{m+2}\backslash\mathbb{Q}_A^{m+2}}\sum_{x\in\mathbb{Q}}f_l(u(x)n_2(X)g;s)\chi(-S_1(\eta,X))dX,$$

and $u(x)$ is defined in (1.14). We fix a $\gamma_1\in P_{1,\mathbb{Q}}\backslash G_{1,\mathbb{Q}}$ and write

$r_1\eta = \begin{pmatrix} a \\ B \\ c \end{pmatrix}$. Since

$$u(x)n_2\left(\begin{pmatrix} x_0 \\ X_0 \\ z_0 \end{pmatrix}\right) = n_1(X_0)n_2\left(\begin{pmatrix} x_0 \\ 0_m \\ 0 \end{pmatrix}\right)u(x+z_0),$$

we have

$$(4.1) \qquad F_l(\gamma_1 g, \gamma_1\eta; s) = \delta(c=0)\delta(B=0)\int_{\mathbf{Q}_A} f_l(u(x)\gamma_1 g; s)\chi(ax)dx.$$

Therefore we know that $I_l'(g, \eta; s) \neq 0$ only when $\eta \in L_1^*$ and $S_1[\eta] = 0$. The following proposition is the main result in this section.

PROPOSITION 4.1. *Let the notation be the same as above. Let $\eta \in L_1^*$ be isotropic.*
(i) *If $\eta = 0$,*

$$I_l'(g, 0; s) = (\tfrac{1}{2}S_1[Y])^{1/2}\frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)}\frac{\xi(s+m/2)}{\xi(s+m/2+1)} \times \mathscr{E}(h, s).$$

(ii) *When $\eta \neq 0$, we take a positive integer A so that $A^{-1}\eta$ is primitive in L_1^*. Then we have*

$$\begin{aligned} I_l'(g, \eta; s) = {} & (\tfrac{1}{2}S_1[Y])^{(2s+m+2)/4}|S_1(Y, \eta)|^{(-2s-m-2)/4} \\ & \times \xi(s+m/2+1)^{-1}\sigma_{s+m/2}(A) \\ & \times \left\{\begin{array}{ll} (-1)^{l/2}P_l^{(+)}(s)^{-1}W_{l/2,(2s+m)/4}(4\pi|S_1(Y,\eta)|) & \text{if } S_1(Y_0,\eta) > 0 \\ P_l^{(-)}(-s)W_{-l/2,(2s+m)/4}(4\pi|S_1(Y,\eta)|) & \text{if } S_1(Y_0,\eta) < 0 \end{array}\right\}, \end{aligned}$$

where $\sigma_s(A) = \sum_{r|A} r^s$.

PROOF. Since $J(k_{1,\infty}(\gamma_1 h), Z_0) = 1$ and

$$(4.2) \qquad u(x)n_1(Y) = n_1(-xY)n_2\left(\begin{pmatrix} -xS_0[Y]/2 \\ Y \\ 0 \end{pmatrix}\right)u(x),$$

we have

$$(4.3) \qquad f_l(u(x)\gamma_1 g; s) = f_l(u(x)\ \mathrm{diag}(t, t_1(\gamma_1 h), 1_m, t_1(\gamma_1 h)^{-1}, t^{-1}); s).$$

(i) By (4.1), (4.3) and

$$(4.4)\qquad u(x)\,\mathrm{diag}(t, t_1(\gamma_1 h), 1_m, t_1(\gamma_1 h)^{-1}, t^{-1})$$
$$= \mathrm{diag}(t_1(\gamma_1 h), t, 1_m, t^{-1}, t_1(\gamma_1 h)^{-1})u\Big(\frac{t_1(\gamma_1 h)}{t}\, x\Big),$$

we have

$$\begin{aligned} I_l'(g,0;s) &= \sum_{\gamma_1 \in P_{1,\mathbf{Q}}\backslash G_{1,\mathbf{Q}}} |t_1(\gamma_1 h)|_A^{s+m/2+1} \int_{\mathbf{Q}_A} f_l\Big(u\Big(\frac{t_1(\gamma_1 h)}{t}\, x\Big); s+\frac{m}{2}+1\Big)dx \\ &= t \sum_{\gamma_1 \in P_{1,\mathbf{Q}}\backslash G_{1,\mathbf{Q}}} |t_1(\gamma_1 h)|_A^{s+m/2} \int_{\mathbf{Q}_A} f_l\Big(u(x); s+\frac{m}{2}+1\Big)dx \\ &= t\mathscr{E}(h,s)\int_{\mathbf{Q}_A} f_l\Big(u(x); s+\frac{m}{2}+1\Big)dx. \end{aligned}$$

We now calculate the local integral. First we consider the non-archimedean part. Since

$$(4.5)\qquad \begin{pmatrix} 0 & 1 \\ 1 & -x \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & -1 \end{pmatrix} \qquad \text{for } x \neq 0,$$

we get

$$\begin{aligned} \int_{\mathbf{Q}_p} |t_p(u(x))|_p^{s+m/2+1}\, dx &= 1 + \int_{\mathbf{Q}_p - \mathbf{Z}_p} |x^{-1}|^{s+m/2+1} dx \\ &= \frac{\zeta_p(s+m/2)}{\zeta_p(s+m/2+1)}. \end{aligned}$$

Second we consider the archimedean part. By means of similar method in the proof of Proposition 3.3, comparing the automorphy factor of $u(x)$ and its Iwasawa decomposition, we get

$$x + i = t_\infty(u(x))^{-1} J(k_\infty(u(x)), Z_0).$$

Because of $|J(k_\infty(u(x)), Z_0)| = 1$, we have

$$\begin{aligned} f_{l,\infty}\Big(u(x); s+\frac{m}{2}+1\Big) &= t_\infty(u(x))^{s+m/2+1} J(k_\infty(u(x)), Z_0)^{-l} \\ &= (x+i)^{(-2s-m-2l-2)/4}(x-i)^{(-2s-m+2l-2)/4}. \end{aligned}$$

Hence we get

$$\int_{\mathbb{R}} f_{l,\infty}\Big(u(x); s+\frac{m}{2}+1\Big)dx = \int_{\mathbb{R}} (x+i)^{(-2s-m-2l-2)/4}(x-i)^{(-2s-m+2l-2)/4}dx$$
$$= 2^{-s-m/2+1}\pi \frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)} \frac{\Gamma(s+m/2)}{\Gamma((2s+m+2)/4)^2}.$$

Therefore we know

$$I_l'(g,0;s) = t2^{-s-m/2+1}\pi \frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)} \frac{\Gamma(s+m/2)}{\Gamma((2s+m+2)/4)^2}$$
$$\times \frac{\zeta_p(s+m/2)}{\zeta_p(s+m/2+1)} \times \mathscr{E}(h,s).$$

(ii) There exists a $\gamma_1 \in P_{1,\mathbb{Q}}\backslash G_{1,\mathbb{Q}}$ uniquely such that

$$\gamma_1\eta = \begin{pmatrix} \lambda \\ 0_m \\ 0 \end{pmatrix}, \qquad \lambda \neq 0.$$

We take γ_1 so that $\lambda = 1$. Therefore we only have to calculate the following integral (cf. (4.1)):

$$I'(g,\eta;s) = F_l(\gamma_1 g, \gamma_1\eta; s)$$
$$= \int_{\mathbb{Q}_A} f_l\Big(u(x)\gamma_1 g; s+\frac{m}{2}+1\Big)\chi(-x)dx.$$

First we consider the non-archimedean parts. When we write

$$\gamma_1 = \begin{pmatrix} t_1(\gamma_1) & * & * \\ & \beta_1(\gamma_1) & * \\ & & t_1(\gamma_1)^{-1} \end{pmatrix} k_{1,p} \in P_{1,p}K_{1,p},$$

we obtain $\begin{pmatrix} t_1(\gamma_1)^{-1} \\ 0_m \\ 0 \end{pmatrix} = k_{1,p}\eta$. We put $a = \mathrm{ord}_p(A)$. Since the $p^{-a}k_{1,p}\eta$ is primitive in $L_{1,p}^*$, there exists a $\lambda_0 \in \mathbb{Z}_p^\times$ such that $t_1(\gamma_1)^{-1} = p^a\lambda_0$. By (4.3), (4.4) and (4.5), we have

$$\int_{\mathbb{Q}_p} |t_p(u(x)\gamma_1)|_p^{s+m/2+1}\chi_p(-x)dx$$
$$= |t_1(\gamma_1)|_p^{s+m/2+1} \int_{\mathbb{Q}_p} |u(t_1(\gamma_1)x)|^{s+m/2+1}\chi_p(-p^a t_1(\gamma_1)\lambda_0 x)dx$$
$$= |t_1(\gamma_1)|_p^{s+m/2} \int_{\mathbb{Q}_p} |u(x)|_p^{s+m/2+1}\chi_p(-p^a\lambda_0 x)dx$$

$$= |t_1(\gamma_1)|_p^{s+m/2}\left\{1+\int_{\mathbf{Q}_p-\mathbf{Z}_p}|x^{-1}|_p^{s+m/2+1}\chi_p(-p^a\lambda_0 x)dx\right\}$$

$$= |p^{-a}|_p^{s+m/2}(1-p^{-s-m/2-1})\sum_{t=0}^{a}p^{(-s-m/2)t}$$

$$= (1-p^{-s-m/2-1})\sum_{t=0}^{a}p^{(s+m/2)t}.$$

Next we consider the archimedean part. Since the $(m+2)$-th component of $\gamma_1 Y$ is

$$^t(\gamma_1 Y)\begin{pmatrix}0\\0_m\\1\end{pmatrix} = S_1(\gamma_1 Y,\gamma_1\eta) = S_1(Y,\eta),$$

we have

$$\begin{aligned}J(u(x)\gamma_1 g, Z_0) &= t^{-1}(x+iS_1(Y,\eta))\\ &= t_\infty(u(x)\gamma_1 g)^{-1}J(k_\infty(u(x)\gamma_1 g),Z_0).\end{aligned}$$

We know that

$$\begin{aligned}&f_l\Big(n(x)\gamma_1 g; s+\frac{m}{2}+1\Big)\\ &= t_\infty(u(x)\gamma_1 g)^{s+m/2+1}J(k_\infty(u(x)\gamma_1 g),Z_0)^{-l}\\ &= t^{s+m/2+1}(x+iS_1(Y,\eta))^{(-2s-m-2l-2)/4}(x-iS_1(Y,\eta))^{(-2s-m+2l-2)/4}\end{aligned}$$

and we get

$$\begin{aligned}&\int_{\mathbf{R}} f_{l,\infty}\Big(u(x)\gamma_1 g; s+\frac{m}{2}+1\Big)e[-x]dx\\ &= t^{s+m/2+1}\int_{\mathbf{R}}(x+iS_1(Y,\eta))^{(-2s-m-2l-2)/4}(x-iS_1(Y,\eta))^{(-2s-m+2l-2)/4}e[-x]dx.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}I'(g,\eta;s) &= (\tfrac{1}{2}S_1[Y])^{(2s+m+2)/4}\zeta(s+m/2+1)^{-1}\sigma_{s+m/2}(A)\\ &\quad\times\int_{\mathbf{R}}(x+iS_1(Y,\eta))^{(-2s-m-2l-2)/4}(x-iS_1(Y,\eta))^{(-2s-m+2l-2)/4}e[-x]dx.\end{aligned}$$

As is well-known (cf. [10]), the last integral becomes

$$(-1)^{l/2}\frac{\pi^{(2s+m+2)/4}|S_1(Y,\eta)|^{(-2s-m-2)/4}}{\Gamma((2s+m+2\pm 2l)/4)}W_{\pm l/2,(2s+m)/4}(4\pi|S_1(Y,\eta)|)$$

$$\text{for } S_1(Y,\eta)\gtrless 0$$

and this proves the assertion (ii). ■

5. Eisenstein series on $O(1, m+1)$

5.1. Standard L-function Let Q be a maximal even integral symmetric matrix of rank m. We assume that $Q < 0$ or assume that sinature of Q is $(1, m-1)$ $(m \geq 2)$. Then we define the (global) standard L-function attached to the constant function by

$$L(Q; s) := \prod_{p<\infty} L_p(Q; s) \qquad (s \in \mathbf{C}),$$

where $L_p(Q; s)$ is the local standard L-function normalized in (2.3). As the gamma factor, we take

$$(5.1) \quad L_\infty(Q; s) := \left\{ \begin{array}{ll} 1 & \text{if } Q < 0 \\ 2^{-m/2+2}\pi^{1/2} \dfrac{\Gamma((2s-m+2)/4)}{\Gamma((2s+m)/4)} & \text{if } \operatorname{sgn}(Q) = (1, m-1) \end{array} \right\}$$
$$\times (2\pi)^{-[m/2]s} \prod_{j=1}^{[m/2]} \Gamma(s-j+m/2) \left\{ \begin{array}{ll} |\det Q|^{s/2} & \text{if } m \text{ is even} \\ |2^{-1}\det Q|^{s/2} & \text{if } m \text{ is odd.} \end{array} \right.$$

Put

$$(5.2) \qquad \xi(Q; s) := L_\infty(Q; s) L_f(Q; s) \qquad (\text{cf. [6]}).$$

The function $\xi(Q; s)$ is continued to $\mathbf{C}$ as a meromorphic function of s and invariant under $s \mapsto 1-s$. If $m = 1$, $\xi(Q; s)$ is entire and does not vanish at $s = 1/2$. If $m \geq 2$, $\xi(Q; s)$ is holomorphic except for possible poles at $s = m/2 - k$ $(0 \leq k \leq m-1, k \in \mathbf{Z})$ and has a simple pole at $s = m/2$.

Let $\eta^\perp$ be the orthogonal complement of $\eta \in L^*$ in V. There exists a maximal even integral symmetric matrix Q_η of rank $m-1$ and $g \in M_{m-1}(\mathbf{Z})$ such that $Q_\eta[g]$ is a matrix representation of $Q|_{(\eta^\perp \cap L)}$. We note that the determinant of Q_η does not depend on the choice of Q_η.

5.2. Eisenstein series Since $\mathscr{I}_p(\eta; s) = I(S, \eta; s)$ in the notation of (2.1), we can write $\mathscr{I}_f(\eta; s)$ explicitly (cf. Theorem 2.1).

PROPOSITION 5.1. *Let s be a complex number with* $\operatorname{Re} s > m/2$.

(i)

$$\mathscr{I}_f(0; s) = |\det S|^{1/2} (2\pi)^{-m/2} \frac{\Gamma(s+m/2)}{\Gamma(s)} \frac{\xi(S; s)}{\xi(S; s+1)} \left\{ \begin{array}{ll} 1 & \text{if } m \text{ is even} \\ \dfrac{\xi(2s)}{\xi(2s+1)} & \text{if } m \text{ is odd} \end{array} \right.$$

(ii) *If* $0 \neq \eta \in L^*$,

$$\mathscr{I}_f(\eta; s) = |\det S|^{1/2} 2^{(-2s-m-1)/4} \pi^{(-2s-m+[(m-1)/2])/2} |S[\eta]|^{-s/2} |\det S_\eta|^{-1/4} \Gamma(s+m/2)$$

$$\times \frac{\xi(S_\eta; s+1/2)}{\xi(S; s+1)} g_S(\eta; s) \begin{cases} 1 & \text{if } m \text{ is even} \\ \xi(2s+1)^{-1} & \text{if } m \text{ is odd,} \end{cases}$$

where $g_S(\eta; s) := \prod_p g_{S,p}(\eta; s)$ *is a finite product of polynomials in* p^s *and* p^{-s} *defined in* (2.6).

We define the normalized Eisenstein series $\mathscr{E}^*(g_1, s)$ by

$$\mathscr{E}^*(g_1, s) = \xi(S; s+1) \mathscr{E}(g_1, s) \begin{cases} 1 & \text{if } m \text{ is even} \\ \xi(2s+1) & \text{if } m \text{ is odd} \end{cases} \quad (g_1 \in G_{1,A}).$$

By Proposition 3.3 and Proposition 5.1, we obtain the Fourier expansion of $\mathscr{E}^*(g_1, s)$ explicitly.

Theorem 5.2. *Let* s *be a complex number with* $\operatorname{Re} s > m/2$. *For* $g_1 = \operatorname{diag}(t, 1_m, t^{-1}) \in G^0_{1,\infty}$ *and* $X \in \mathbb{R}^m$, *the normalized the Eisenstein series has the following expansion*

$$\mathscr{E}^*(n_1(X) g_1, s) = \sum_{\eta \in L^*} \mathscr{E}_\eta(g_1, s) e[S(\eta, X)],$$

where

$$\mathscr{E}^*_0(g_1, s) = t^{s+m/2} \xi(S; s+1) \begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s+1) & \text{if } m \text{ is odd} \end{Bmatrix}$$
$$+ t^{-s+m/2} \xi(S; s) \begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s) & \text{if } m \text{ is odd} \end{Bmatrix}$$

and for $0 \neq \eta \in L^*$,

$$\mathscr{E}^*_\eta(g_1, s) = t^{(m-1)/2} |2^{-m+2} \pi^{-2[(m-1)/2]} S[\eta] \det S_\eta|^{-1/4} \xi(S_\eta; s+1/2)$$
$$\times g_S(\eta; s) W_{0,s}(8\pi t \sqrt{|\tfrac{1}{2} S[\eta]|}).$$

The rest of this section will be devoted to the proof of the continuation and the functional equation of the normalized Eisenstein series. On each Fourier coefficient we obtain the following proposition.

Proposition 5.3.. *Let the notation be the same as in Theorem* 5.2.
(i) *The Fourier coefficient* $\mathscr{E}^*_\eta(g_1, s)$ *has a meromorphic continuation in* s *to the whole* s*-plane and is invariant under* $s \mapsto -s$.

(ii) *For an arbitrary $s_0 \in \mathbf{C}$, there exist $\delta > 0$ and $0 \leq \tau \in \mathbf{Z}$ depending only on S and s_0 such that*

$$(s-s_0)^{\tau}\mathscr{E}^*_{\eta}(g_1, s)$$

is holomorphic in s on $U_\delta(s_0) = \{s \in \mathbf{C} \| s - s_0| \leq \delta\}$.
(iii) *Let s_0, δ and τ be as above. Given $\rho > 0$, there exist positive constants $c_1, \dots, c_4$ depending only on S, ρ, δ and τ such that*

$$|(s-s_0)^{\tau}\mathscr{E}^*_{\eta}(g_1, s)| \leq c_1 t^{c_2} e^{-c_3\sqrt{|S[\eta]|}} |S[\eta]|^{c_4}$$

for $t \geq \rho$, $s \in U_\delta(s_0)$ and $0 \neq \eta \in L^*$.

PROOF. The assertions (i) and (ii) are easily seen from Theorem 5.2. Since $g_{S,p}(\eta; s)$ is a polynomial in p^s and p^{-s} whose degree depends only on a, f and S in the notation of (2.6), for any compact subset T of $\mathbf{C}$, there exist two positive constants A and B depending only on T and S such that

$$|g_S(\eta; s)| \leq A|S[\eta]|^B \qquad \text{for any } s \in T,\ \eta \in L^*.$$

We note that

$$|\det S_\eta| \leq |S[\eta] \det S| \qquad \text{for } \eta \in L^*.$$

Therefore, by Lemma 3.2 and Theorem 5.2, we obtain the assertion (iii). ■

We now apply Proposition 5.3 to Theorem 5.2. For an arbitrary $s_0 \in \mathbf{C}$, we take $\delta > 0$ and $0 \leq \tau \in \mathbf{Z}$ as in Proposition 5.3(ii). For given $\rho > 0$, there exist positive constants $c_1, \dots, c_6$ depending only on S, ρ, δ and τ such that

$$\begin{aligned}\sum_{\eta \in L^*} |(s-s_0)^{\tau}\mathscr{E}^*_{\eta}(g_1, s)| &\leq c_1 t^{c_2}\left\{1 + \sum_{0 \neq \eta \in L^*} e^{-c_3\sqrt{|S[\eta]|}}|S[\eta]|^{c_4}\right\} \\ &\leq c_5 t^{c_6}.\end{aligned}$$

for $t \geq \rho, s \in U_\delta(s_0)$. Therefore we have the following theorem.

THEOREM 5.4. *The normalized Eisenstein series $\mathscr{E}^*(g_1, s)$ $(g_1 \in G_{1,A})$ has a meromorphic continuation in s to the whole s-plane and is invariant under $s \mapsto -s$. Furthermore, it is holomorphic except for possible simple poles at $s = m/2 - k$ $(0 \leq k \leq m, k \in \mathbf{Z})$ and the residue at $s = m/2$ is given by*

$$\operatorname{Res}_{s=m/2} \mathscr{E}^*(g, s) = \begin{cases} \operatorname{Res}_{s=m/2} \xi(S; s) & \text{if } m \text{ is even} \\ \operatorname{Res}_{s=m/2} \xi(S; s)\xi(2s) & \text{if } m \text{ is odd.} \end{cases}$$

6. Eisenstein series on $O(2, m+2)$

6.1. Real analytic Eisenstein series Since $I_p(\eta;s) = I(S_1,\eta;s)$ in the notation of (2.1), we can write $I_f(\eta;s)$ explicitly (cf. Theorem 2.1).

PROPOSITION 6.1. *Let s be a complex number such that* $\mathrm{Re}\, s > m/2+1$.
(i)

$$I_f(0;s) = 2^{2s-m/2-2}\pi^{-m/2-2}|\det S_1|^{1/2}\frac{\Gamma((2s+m+2)/4)^2\Gamma((2s-m+2)/4)^2}{\Gamma(s)\Gamma(s-m/2)}$$

$$\times\frac{\xi(S_1;s)}{\xi(S_1;s+1)}\begin{cases} 1 & \text{if } m \text{ is even} \\ \dfrac{\xi(2s)}{\xi(2s+1)} & \text{if } m \text{ is odd.}\end{cases}$$

(ii) *If $0 \neq \eta \in L_1^*$, $S_1[\eta] = 0$ and $A^{-1}\eta$ (A is a positive integer) is primitive in L_1^*,*

$$\mathscr{I}_f(\eta;s) = 2^{s-1}\pi^{(-2s-m-6)/4}|\det S_1|^{1/2}\frac{\Gamma((2s+m+2)/4)^2\Gamma((2s-m+2)/4)}{\Gamma(s)}$$

$$\times\,\xi(s-m/2)^{-1}\frac{\xi(S_1;s)}{\xi(S_1;s+1)}\left\{\begin{matrix} 1 & \text{if } m \text{ is even} \\ \dfrac{\xi(2s)}{\xi(2s+1)} & \text{if } m \text{ is odd}\end{matrix}\right\}\sigma_{-s+m/2}(A).$$

(iii) *If $\eta \in L_1^*$ and $S_1[\eta] \neq 0$,*

$$I_f(\eta;s) = 2^{(2s-m+1)/4}\pi^{(-2s-[m/2]-2)/2}|S_1[\eta]|^{-s/2}|\det S_1|^{1/2}|\det S_{1,\eta}|^{-1/4}$$

$$\times\left\{\begin{matrix} \Gamma((2s+m+2)/4)\Gamma((2s-m+2)/4) & \text{if } S_1[\eta] > 0 \\ 2^{(m-3)/2}\pi^{-1/2}\Gamma((2s+m+2)/4)^2 & \text{if } S_1[\eta] < 0\end{matrix}\right\}$$

$$\times\frac{\xi(S_{1,\eta};s+1/2)}{\xi(S_1;s+1)}\left\{\begin{matrix} 1 & \text{if } m \text{ is even} \\ \xi(2s+1)^{-1} & \text{if } m \text{ is odd,}\end{matrix}\right\}g_{S_1}(\eta;s),$$

where $g_{S_1}(\eta;s) := \prod_p g_{S_{1,p}}(\eta;s)$ is a finite product of polynomials in p^s, p^{-s} defined in (2.6).

We normalize the Eisenstein series $E_l(Z,s)$ as follows:

$$E_l^*(Z,s) := P_l(s)\xi(S_1;s+1)E_l(Z,s)\begin{cases} 1 & \text{if } m \text{ is even} \\ \xi(2s+1) & \text{if } m \text{ is odd}\end{cases}$$

where $P_l(s)$ is the polynomial in s defined in (3.10). By Proposition 3.4 and Proposition 6.1, we obtain the Fourier expansion of $E_l^*(Z,s)$ explicitly.

THEOREM 6.2. *Let l be a non-negative even integer and let s be a complex number with* $\mathrm{Re}\, s > m/2+1$. *For $X+iY \in \mathfrak{D}$, $g\langle Z_0\rangle = X+iY$ ($g \in G_{2,\infty}^0$), the* *normalized Eisenstein series* $E_l^*(X+iY,s)$ *has the following expansion*

$$E_l^*(X+iY,s)=\sum_{\eta\in L_l^*} a_l^*(Y,\eta;s)e[S_1(\eta,X)],$$

where the Fourier coefficient $a_l^*(Y,\eta;s)$ *is given as follows*:
(i) *When* $\eta=0$,

$$\begin{aligned}
a_l^*(Y,0;s) &= (\tfrac{1}{2}\,S_1[Y])^{(2s+m-2l+2)/4}P_l(s)\xi(S_1;s+1)\begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s+1) & \text{if } m \text{ is odd}\end{Bmatrix} \\
&+ (\tfrac{1}{2}\,S_1[Y])^{(-2s+m-2l+2)/4}P_l(-s)\xi(S_1;s)\begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s) & \text{if } m \text{ is odd}\end{Bmatrix} \\
&+ (\tfrac{1}{2}\,S_1[Y])^{(-2l+2)/4}P_l^{(-)}(s)P_l^{(-)}(-s)\xi(s-m/2+1)\xi(s+m/2) \\
&\times \mathscr{E}^*(h(g),s).
\end{aligned}$$

(ii) *When* $S_1[\eta]=0$, $S_1(\eta,Y_0)\gtrless 0$ *and* $A^{-1}\eta$ (A *is a positive integer*) *is primitive in* L_1^*,

$$\begin{aligned}
&a_l^*(Y,\eta;s) \\
&= (\tfrac{1}{2}\,S_1[Y])^{(-2s+m-2l+2)/4}|S_1(\eta,Y)|^{(2s-m-2)/4}Q_{l,\eta}^*(s)\xi(s-m/2)^{-1}\xi(S_1;s) \\
&\times \begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s) & \text{if } m \text{ is odd}\end{Bmatrix} W_{\pm l/2,(2s-m)/4}(4\pi|S_1(\eta,Y)|)\sigma_{-s+m/2}(A) \\
&+ (\tfrac{1}{2}\,S_1[Y])^{(2s+m-2l+2)/4}|S_1(\eta,Y)|^{(-2s-m-2)/4}Q_{l,\eta}^*(-s)\xi\left(s+\frac{m}{2}+1\right)^{-1}\xi(S_1;s+1) \\
&\times \begin{Bmatrix} 1 & \text{if } m \text{ is even} \\ \xi(2s+1) & \text{if } m \text{ is odd}\end{Bmatrix} W_{\pm l/2,(2s+m)/4}(4\pi|S_1(\eta,Y)|)\sigma_{s+m/2}(A).
\end{aligned}$$

(iii) *When* $S_1[\eta]>0$ *and* $S_1(\eta,Y_0)\gtrless 0$,

$$\begin{aligned}
a_l^*(Y,\eta;s) &= (\tfrac{1}{2}\,S_1[Y])^{-l/2}2^{(2m\pm 4l+11)/4}\pi^{(-[m/2]\pm 2l)/2}S_1[Y]^{\pm l/2} \\
&\times S_1[\eta]^{(-m\pm 2l-2)/4}|S_{1,\eta}|^{-1/4}Q_{1,\eta}^*(s)\xi(S_{1,\eta};s+1/2)g_{S_1}(\eta;s) \\
&\times \omega(2\pi Y,2\eta;(2s+m+2l+2)/4,(2s+m-2l+2)/4).
\end{aligned}$$

(iv) *When* $S_1[\eta]<0$,

$$\begin{aligned}
a_l^*(Y,\eta;s) &= (\tfrac{1}{2}\,S_1[Y])^{-l/2}2^{(6m+9)/4}\pi^{(-[m/2]-1)/2}S_1[Y]^{m/4}S_1[\eta]^{-1/2}|S_{1,\eta}|^{-1/4} \\
&\times \delta_+(\eta,Y)^{l/2}\delta_-(\eta,Y)^{-l/2}Q_{l,\eta}^*(s)\xi(S_{1,\eta};s+1/2)g_{S_1}(\eta;s) \\
&\times \omega(2\pi Y,2\eta;(2s+m+2l+2)/4,(2s+m-2l+2)/4).
\end{aligned}$$

Here $g_{S_1}(\eta; s) := \prod_p g_{S_{1,p}}(\eta; s)$ *is a finite product of polynomials defined in* (2.6) *and we put* $Q^*_{l,\eta}(s) := P_l(s) Q_{l,\eta}(s)$ (*cf.* (3.10), (3.11)).

To prove the analytic continuation and the functional equation of $E^*_l(Z, s)$ (Theorem 6.4 below), we consider analytic properties of each Fourier coefficient.

Proposition 6.3. (i) *The Fourier coefficient* $a^*_l(Y, \eta; s)$ *has a meromorphic continuation in* s *to the whole s-plane and is invariant under* $s \mapsto -s$.

(ii) *For an arbitrary* $s_0 \in \mathbf{C}$, *there exist* $\delta > 0$ *and* $0 \le \tau \in \mathbf{Z}$ *depending only on* S_1 *and* s_0 *such that*

$$(s - s_0)^\tau a^*_l(Y, \eta; s)$$

is holomorphic in s *on* $U_\delta(s_0) = \{s \in \mathbf{C} \mid |s - s_0| \le \delta\}$ *and is real analytic in* $(Y, s) \in \mathscr{P} \times \mathbf{C}$.

(iii) *Let* s_0, δ *and* τ *be as above. There exist positive constants* $c_1, \dots, c_{10}$ *depending only on* S_1, δ *and* τ, *such that*

$$|(s - s_0)^\tau a^*_l(Y, \eta; s)| \le c_1 (\lambda(Y)^{c_2} + \mu(Y)^{-c_2}) \|\eta\|^{c_3} e^{-c_4 \mu(Y) \|\eta\|} \quad \text{if } S_1[\eta] = 0$$

$$|(s - s_0)^\tau a^*_l(Y, \eta; s)| \le c_5 S_1[Y]^{c_6} |S_1[\eta]|^{c_7} e^{-c_8 \tau(\eta, Y)} (\lambda(Y)^{c_9} + \mu(Y)^{-c_9}) \|\eta\|^{c_{10}} \qquad \text{if } S_1[\eta] \ne 0$$

for $s \in U_\delta(s_0)$ *and* $\eta \in L^*_1$.

Proof. The assertions (i) and (ii) are easily seen from Theorem 5.4 and Theorem 6.2. We shall prove (iii). Since there are only finitely many terms with $\|\eta\| < 1$, it is sufficient to consider the terms with $\|\eta\| \ge 1$. First we consider the case of $S_1[\eta] = 0$. By Lemma 3.2 and Theorem 6.2, there exist positive constants A, B and C depending only on S_1, δ and τ such that

$$|(s - s_0)^\tau a^*_l(Y, \eta; s)| \le A e^{-B S_1(\eta, Y)} (S_1(\eta, Y)^{-C} + S_1(\eta, Y)^C)$$

for $s \in U_\delta(s_0)$ and $\eta \in L^*_l$. By [10, p. 299–p. 300] we have

$$|S_1(\eta, Y)| \le D \lambda(Y) \|\eta\|, \qquad |S_1(\eta, Y)|^{-1} \le E \mu(Y)^{-1} \|\eta\|^F,$$
$$|S_1(\eta, Y)| \ge G \mu(Y) \|\eta\|,$$

with positive constants D, E, F, G independent of Y and η. This proves the assertion in the cace of $S_1[\eta] = 0$.

Next we consider the case of $S_1[\eta] \ne 0$. For any compact subset T of $\mathbf{C}$, there exist two positive constants A and B depending only on T and S_1 such that

$$|g_{S_1}(\eta; s)| \le A |S_1[\eta]|^B \qquad \text{for any } s \in T,\ \eta \in L^*_1$$

(cf. proof of Proposition 5.3). We note that

$$|\det S_{1,\eta}| \leq |S_1[\eta] \det S_1| \qquad \text{for } \eta \in L_1^*.$$

Notice

$$\delta_+(\eta, Y) \leq A\lambda(Y)\|\eta\|, \qquad \delta_+(\eta, Y)^{-1} \leq B\mu(Y)^{-1}\|\eta\|^C \qquad \text{if } S_1[\eta] < 0,$$
$$\mu(\eta, Y)^{-1} \leq D\mu(Y)^{-1}\|\eta\|^E,$$

with positive constants A, B, C, D and E independent of Y and η (cf. [10, p. 299–p. 300]). Therefore, by Lemma 3.1 and the above facts, we can easily prove the assertion in this case. ∎

We now apply Proposition 6.3 to Theorem 6.2. For an arbitrary $s_0 \in \mathbf{C}$, we take $\delta > 0$ and $0 \leq \tau \in \mathbf{Z}$ as in Proposition 6.3(ii). Given $\rho > 0$, there exist positive constants $c_1, \ldots, c_8$ depending only on S_1, ρ, δ and τ, such that

$$\sum_{\eta \in L_l^*} |(s - s_0)^\tau a_l^*(Y, \eta; s) e[S_1(\eta, X)]|$$
$$\leq c_1\lambda(Y)^{c_2}\left\{1 + \sum_{\substack{0 \neq \eta \in L_l^* \\ S_1[\eta]=0}} \|\eta\|^{c_3} e^{-c_4\|\eta\|} + \sum_{\substack{\eta \in L_l^* \\ S_1[\eta] \neq 0}} |S_1[\eta]|^{c_5}\|\eta\|^{c_6} e^{-c_8\tau(\eta, Y)}\right\}$$

for $\mu(Y) \geq \rho$, $s \in U_\delta(s_0)$. By the inequality

$$\tau(\eta, Y) \geq \sqrt{S_1[Y]|S_1[\eta]|} \quad \text{and} \quad \tau(\eta, Y) \geq A\mu(Y)\|\eta\|$$

with positive constant A independent of Y and η (cf. [10, p. 300]), there exists a positive constant C depending only on ρ and S_1 such that

$$e^{-\tau(\eta, Y)} < e^{-C\sqrt{|S_1[\eta]|}} \cdot e^{-C\|\eta\|}.$$

Hence Schwarz' inequality gives

$$\sum_{\eta \in L_l^*} |(s - s_0)^\tau a_l^*(Y, \eta; s) e[S_1(\eta, X)]| \leq c_9\lambda(Y)^{c_{10}}$$

for $\mu(Y) \geq \rho$ and $s \in U_\delta(s_0)$, where positive constants c_9, c_{10} depend only on S_1, ρ, δ and τ. Thus we have the following theorem.

THEOREM 6.4. *The Eisenstein series $E_l^*(Z, s)$ has a meromorphic continuation in s to the whole s-plane and is invariant under $s \mapsto -s$. Furthermore, it is holomorphic except for possible simple poles at $s = m/2 + 1 - k$ $(0 \leq k \leq m + 2, k \in \mathbf{Z})$.*

6.2. Holomorphic Eisenstein series In this section we consider the holomorphic Eisenstein series on $\mathfrak{D}$. We denote by $M_l(\Gamma)$ the space of holomorphic automorphic forms on $\mathfrak{D}$ of weight l with respect to Γ. For $l > m+2$, we put

$$E_l(Z) := E_l(Z, l - m/2 - 1) = \sum_{\gamma \in (P_{2,\mathbf{Q}} \cap \Gamma) \backslash \Gamma} J(\gamma, Z)^{-l}.$$

Since the above series is absolutely convergent, we know $E_l(Z) \in M_l(\Gamma)$. The convergence of (1.9) at $s = l - m/2 - 1$ is not guaranteed if $l \leq m+2$. However, as in Shimura [11], we can construct the holomorphic Eisenstein series of smaller weights.

THEOREM 6.5. *We define*

$$E_l(Z) := E_l(Z, l - m/2 - 1) \qquad \text{for } l > (m+4)/2.$$

Then $E_l(Z)$ is a holomorphic function in Z on $\mathfrak{D}$ i.e. $E_l(Z) \in M_l(\Gamma)$. Moreover if m is even and χ_{S_1} is non-trivial, then $E_l(Z) := E_l(Z, 1) \in M_l(\Gamma)$ for $l = (m+4)/2$.

PROOF. For $X + iY = g\langle Z_0 \rangle$ and $g \in G^0_{2,\infty}$, we write

$$E_l(X + iY, s) = \sum_{\eta \in L_l^*} a_l(Y, \eta; s) e[S_1(\eta, X)].$$

Then by Theorem 6.2 we have

(6.1)

$$\begin{aligned} a_l(Y, 0; s) &= \left(\tfrac{1}{2} S_1[Y]\right)^{(2s+m-2l+2)/4} \\ &\quad + (S_1[Y])^{(-2s+m-2l+2)/4} 2^{(-2s-m+2l+2)/4} \pi^{1/2} |\det S_1|^{-1/2} \frac{B_{S_1}(-s)}{B_{S_1}(-s-1)} \\ &\quad \times \frac{P_l(-s)}{P_l(s)} \frac{1}{\Gamma\left(\frac{2s+m+2}{4}\right) \Gamma\left(\frac{2s-m+2}{4}\right)} \frac{\xi\left(s - \frac{m}{2}\right)}{\xi\left(s + \frac{m}{2} + 1\right)} \\ &\quad \times \begin{cases} \dfrac{|d(S_1)|^{1/2} \Gamma\left(\frac{s+1+\delta_{S_1}}{2}\right) \Gamma(s)}{\Gamma\left(\frac{s+\delta_{S_1}}{2}\right)} \dfrac{\xi(\chi_{S_1}, s)}{\xi(\chi_{S_1}, s+1)} & m : \text{even} \\ \Gamma\left(s + \frac{1}{2}\right) \dfrac{\xi(2s)}{\xi(2s+1)} & m : \text{odd} \end{cases} \end{aligned}$$

$$+\, t_1(h(g))^{s+m/2}(\tfrac{1}{2}S_1[Y])^{(-l+1)/2}\frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)}\frac{\xi\left(s+\frac{m}{2}\right)}{\xi\left(s+\frac{m}{2}+1\right)}$$

$$+\, t_1(h(g))^{-s+m/2}(S_1[Y])^{(-l+1)/2}2^{(-2s+l+1)/2}\pi^{1/2}|\det S|^{-1/2}\frac{B_S(-s)}{B_S(-s-1)}$$

$$\times\frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)}\frac{1}{\Gamma\left(\frac{2s+m+2}{4}\right)\Gamma\left(\frac{2s-m+2}{4}\right)}\frac{\xi\left(s-\frac{m}{2}+1\right)}{\xi\left(s+\frac{m}{2}+1\right)}$$

$$\times\begin{cases}\dfrac{|d(S)|^{1/2}\Gamma\left(\frac{s+1+\delta_{S_1}}{2}\right)\Gamma(s)}{\Gamma\left(\frac{s+\delta_{S_1}}{2}\right)}\dfrac{\xi(\chi_S,s)}{\xi(\chi_S,s+1)} & m:\text{even}\\[2ex] \Gamma\left(s+\frac{1}{2}\right)\dfrac{\xi(2s)}{\xi(2s+1)} & m:\text{odd}\end{cases}$$

$$+\, t_1(h(g))^{(m-1)/2}\sum_{\eta\in L^*\setminus\{0\}}a_{S,l}(\eta;s)\frac{P_l^{(-)}(-s)}{P_l^{(+)}(s)}\frac{B_{S_\eta}\left(-s-\frac{1}{2}\right)}{B_S(-s-1)}$$

$$\times\frac{1}{\xi\left(s+\frac{m}{2}+1\right)}\frac{1}{\Gamma\left(\frac{2s+m+2}{4}\right)}W_{0,s}\left(8\pi\alpha_1(\beta)\sqrt{|\tfrac{1}{2}S_1[Y]|}\right)$$

$$\times\begin{cases}\Gamma\left(\frac{s+1+\delta_S}{2}\right)\dfrac{1}{\xi(\chi_S,s+1)} & m:\text{even}\\[2ex] \dfrac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{2s+1+\delta_{S_\eta}}{4}\right)}\dfrac{\xi\left(\chi_{S_\eta},s+\frac{1}{2}\right)}{\xi(2s+1)} & m:\text{odd,}\end{cases}$$

where $B_S(s)=\prod_{p<\infty}B_{S,p}(S)$ ($B_{S,p}(s)$ is defined by (2.4)),

$$\delta_{S_1}=\delta_S=\begin{cases}0 & m\equiv 0 \pmod 4\\ 1 & m\equiv 2 \pmod 4\end{cases},\qquad \delta_{S_\eta}=\begin{cases}0 & m\equiv 1 \pmod 4\\ 1 & m\equiv 3 \pmod 4\end{cases},$$

$d(S_1)$ denotes the discriminant of the quadratic field $\mathbf{Q}\left(\sqrt{(-1)^{(m+2)(m+1)/2}\det S_1}\right)$ and $a_S(\eta;s)$ is an entire function in s which does not have any zero. If $l>m/2+2$ and $l\neq m+1$, then every term of (6.1) vanishes at $s=l-m/2-1$ except for the first term. If $l=m+1$, then we have

$$a_l(Y, 0; s)|_{s=m/2} = 1 + S_1[Y]^{-m/2} 2\pi^{1/2} |\det S_1|^{-1/2} \frac{\Gamma\left(\frac{m+1}{2}\right)\xi(m)}{m!\Gamma\left(\frac{1}{2}\right)\xi(m+1)^2}$$
$$\times \left(\frac{B_{S_1}(-s)}{B_{S_1}(-s-1)} P_l^{(-)}(-s)\xi\left(s - \frac{m}{2}\right)\right)\Bigg|_{s=m/2}$$
$$+ S_1[Y]^{-m/2} 2\pi^{1/2} |\det S|^{-1/2} \frac{\Gamma\left(\frac{m+1}{2}\right)\xi(m)}{m!\Gamma\left(\frac{1}{2}\right)\xi(m+1)^2}$$
$$\times \left(\frac{B_S(-s)}{B_S(-s-1)} P_l^{(-)}(-s)\xi\left(s - \frac{m}{2} + 1\right)\right)\Bigg|_{s=m/2}$$
$$= 1.$$

Hence we have $a_l(Y, 0; l - m/2 - 1) = 1$ for $l > m/2 + 2$. If χ_{S_1} is non-trivial, $\xi(\chi_{S_1}, s)$ is an entire function and $a_l(Y, 0; 1) = 1$ for $l = m/2 + 2$.

In the same way, if $S_1[\eta] < 0$ or $S_1(\eta, Y_0) < 0$, we have $a_l(Y, \eta; l - m/2 - 1) = 0$ for $l > (m+3)/2$. We assume that $S_1[\eta] \geq 0$ and $S_1(\eta, Y_0) > 0$. Notice that

$$W_{l/2,(l-1)/2} = (4\pi|S_1(Y, \eta)|)^{l/2} e[i|S_1(\eta, Y)|] \qquad \text{if } S_1[\eta] = 0,\ S_1(Y_0, \eta) > 0,$$

$$\omega(2\pi, 2\eta; l, 0) = 2^{-m-2} e[S_1(\eta, iY)] \qquad \text{if } S_1[\eta] > 0,\ S_1(Y_0, \eta) > 0.$$

Hence we can write

$$a_l(Y, \eta; l - m/2 - 1) e[S_1(\eta, X)] = a_l(\eta) e[S_1(\eta, Z)] \qquad \text{for } l > (m+3)/2,$$

where $a_l(\eta)$ does not depend on Z.

Here we set

$$\text{(6.2)} \qquad \hat{g}_l(\eta) := \left|\frac{\det S_1}{\det S_{1,\eta}} S_1[\eta]\right|^{(2l-m-2)/4} g_{S_1}(\eta; l - m/2 - 1)$$
$$= \prod_p \hat{g}_{l,p}(\eta).$$

To write $\hat{g}_{l,p}(\eta)$ more explicitly, we take a positive integer A so that $A^{-1}\eta$ is primitive in L_1^* and put

$$a_p = \mathrm{ord}_p(A) \quad \text{and} \quad \frac{1}{2}\,\mathrm{ord}_p\left(\frac{\det S_1}{\det S_{1,\eta}} S_1[\eta]\right) = a_p + f_p.$$

We can write $\hat{g}_{l,p}(\eta)$ as follows:

(i) If $\nu_p = 0$

$$\hat{g}_{l,p}(\eta) = \sum_{t=0}^{a_p} p^{(2l-m-2)t} + p^{-n_{0,p}/2+\partial_p}\delta(\eta_{0,p} \notin L'^{*}_{0,p}) \sum_{t=0}^{a_p-1} p^{(2l-m-2)t+l-m/2-1}$$

(ii) If $\nu_p \geq 1$,

$$\begin{aligned}\hat{g}_{l,p}(\eta) = &\sum_{k=0}^{a_p} \sum_{t=0}^{f_p+a_p-k} p^{(2l-m-2)t+(l-1)k} \\ &- p^{-n_{0,p}/2}\beta_{S_{1,\eta,p}} \sum_{k=0}^{a_p} \sum_{t=0}^{f_p+a_p-k-1} q^{(2l-m-2)t+(l-1)k+l-m/2-1} \\ &- p^{\partial_p-1}\delta(\beta_{0,p} \notin L'^{*}_{0,p}) \sum_{k=0}^{a_p} \sum_{t=1}^{f_p+a_p-k-1} p^{(2l-m-2)t+(l-1)k}.\end{aligned}$$

Here $\beta_{S_{1,\eta,p}}$ defined by (2.5) and $\beta_{0,p}$ is as in (2.2). We note that the case (i) does not occur, since the $\mathbf{Q}$-rank of S_1 is 1. We give an explicit formula for the Fourier coefficients of the holomorphic Eisenstein series by Theorem 6.2 and Theorem 6.5.

THEOREM 6.6. *Let l be an even integer. We assume that $l \geq m/2+2$ if m is even and χ_{S_1} is non-trivial, $l > m/2+2$ otherwise. The Fourier coefficients of the holomorphic Eisenstein series*

$$E_l(Z) = 1 + \sum_{\substack{\eta \in L_l^* \\ S_1[\eta] \geq 0, S_1(\eta, Y_0) > 0}} a_l(\eta) e[S_1(\eta, Z)]$$

is given as follows:

(i) *When $S_1[\eta] = 0$ and $A^{-1}\eta$ (A is a positive integer) is primitive in L_1,*

$$a_l(\eta) = -\frac{2l}{B_l}\sigma_{l-1}(A).$$

(ii) *When $S_1[\eta] > 0$,*

$$a_l(\eta) = \frac{B_{S_{1,\eta}}(-l+(m+1)/2)}{B_{S_1}(-l+m/2)}\hat{g}_l(\eta)$$

$$\left\{\begin{array}{ll} (-1)^{[(m+2)/4]}2^{-l+m/2+3}l\left(l-\dfrac{m}{2}\right)\dfrac{1}{B_l B_{l-m/2,\chi_{S_1}}}|\det S_{1,\eta}|^{l-m/2-1}\left|\dfrac{d(S_1)}{\det S_1}\right|^{l-(m+1)/2} & \\ \hfill \textit{if } m \textit{ is even} & \\ -(-1)^{[(m+2)/4]}2^{l-(m-3)/2}l\,\dfrac{B_{l-(m+1)/2,\chi_{S_{1,\eta}}}}{B_l B_{2l-m-1}}\left|\dfrac{\det S_{1,\eta}}{d(S_{1,\eta})}\right|^{l-m/2-1}|\det S_1|^{-l+(m+1)/2} & \\ \hfill \textit{if } m \textit{ is odd} & \end{array}\right\},$$

where B_n [*resp.* $B_{n,\chi}$] *is the n-th Bernoulli* [*resp. generalized Bernoulli*] *number* (*for the definition see* [3, p. 89, 94]).

On the Fourier coefficient of $E_l(Z)$, the following corollary is obtained.

COROLLARY 6.7. *The Fourier coefficient* $a_l(\eta)$ *is a rational number. More precisely there exists a constant* $C \in \mathbf{Z} - \{0\}$ *depending only on* S_1 *and* l *such that* $Ca_l(\eta) \in \mathbf{Z}$ *for all* $\eta \in L_1^*$.

REMARK. When $l > m+2$, an explicit formula for the Fourier coefficients of the holomorphic Eisenstein series is given also by theta lifting of Jacobi form (cf. [7]).

7. Eisenstein series on $O(2, m+2)$ in the case of $\mathbf{Q}$-rank 1

To complete our results we consider the Eisenstein series on $O(2,m+2)$ in the case of $\mathbf{Q}$-rank 1.

7.1. Definition of Eisenstein series Let $S_1 \in M_{m+2}(\mathbf{Q})$ be an even integral anisotropic symmetric matrix of signature $(1,m+1)$ and assume that S_1 is maximal. Since S_1 is isotropic for $m \geq 3$, we may consider the case of $m = 1$ or 2. We denote by G_1 the orthogonal group of S_1 and by G_2 the orthogonal group of

$$S_2 = \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix}.$$

Put $L_1 = \mathbf{Z}^{m+2}$, $L_1^* = S_1^{-1}L_1$. We define the maximal compact subgroups $K_{1,p} := G_{1,p} \cap GL_{m+2}(\mathbf{Z}_p)$ and $K_{2,p} := G_{2,p} \cap GL_{m+4}(\mathbf{Z}_p)$. We fix a point $Z_0 = iY_0$ such that $S_1[Y_0] = 2$. We define the action of $G_{2,\infty}^0$ on

$$\mathfrak{D} := \{Z \in \mathbf{C}^{m+2} | S_1[\mathrm{Im}(Z)] > 0, S_1(Y_0, \mathrm{Im}(Z)) > 0\}$$

by

$$g \cdot Z^{\sim} = (g\langle Z \rangle)^{\sim} \cdot J(g,Z), Z^{\sim} := \begin{pmatrix} -S_1[Z]/2 \\ Z \\ 1 \end{pmatrix} \in \mathbf{C}^{m+4} \quad (g \in G_{2,\infty}^0, Z \in \mathfrak{D}).$$

We denote by $K_{2,\infty}$ the stabilizer subgroup of Z_0 in $G_{2,\infty}^0$. Clearly $K_{2,\infty}$ is a maximal compact subgroup of $G_{2,\infty}^0$ and $G_{2,\infty}^0/K_{2,\infty} \cong \mathfrak{D}$. Let l be a non-negative even integer. We define the Eisenstein series of weight l with respect to Γ on $\mathfrak{D}$ by

$$(7.1)\quad E_l(Z,s) = \left(\frac{S_2[\mathrm{Im}\,Z]}{2}\right)^{(2s-2l+m+2)/4} \sum_{\gamma\in(P_{2,\mathbb{Q}}\cap\Gamma)\backslash\Gamma} |J(\gamma,Z)|^{-s+l-m/2-1}J(\gamma,Z)^{-l},$$

which converges absolutely in a right half plane $\{s\in\mathbb{C}|\mathrm{Re}\,s > m/2+1\}$.

Since the $\mathbb{Q}$-rank of S_1 is 0, $S_1[\eta]\neq 0$ for all $\eta\neq 0$ and the Bruhat decomposition of $G_{2,\mathbb{Q}}$ is given by

$$(7.2)\qquad G_{2,\mathbb{Q}} = P_{2,\mathbb{Q}} \coprod P_{2,\mathbb{Q}}w_1\{n_2(X)|X\in\mathbb{Q}^{m+2}\},$$

where $n_2(\cdot)$ and w_1 is same as in §1. We easily see that all properties in §3 also hold for this case. Therefore all the necessary calculations to obtain the Fourier expansion of $E_l(Z,s)$ explicitly have be done in §2 and §3.

7.2. Main theorem We put

$$E_l^*(Z,s) := P_l(s)\xi(S_1;s+1)E_l(Z,s)\cdot\begin{cases}1 & \text{if } m \text{ is even}\\ \xi(2s+1) & \text{if } m \text{ is odd.}\end{cases}$$

THEOREM 7.1. *Let l be a non-negative even integer and let s be a complex number with* $\mathrm{Re}\,s > m/2+1$. *For $X+iY\in\mathfrak{D}$, the normalized Eisenstein series $E_l^*(X+iY,s)$ has the following expansion*

$$E_l^*(X+iY,s) = \sum_{\eta\in L_1^*} a_l^*(Y,\eta;s)e[S_1[\eta,X)],$$

where the Fourier coefficient $a_l^*(Y,\eta;s)$ is given as follows:
(i) *When $\eta=0$,*

$$a_l^*(Y,0;s) = (\tfrac{1}{2}S_1[Y])^{(2s+m-2l+2)/4}P_l(s)\xi(S_1;s+1)\cdot\left\{\begin{matrix}1 & \text{if } m \text{ is even}\\ \xi(2s+1) & \text{if } m \text{ is odd}\end{matrix}\right\}$$
$$+ (\tfrac{1}{2}S_1[Y])^{(-2s+m-2l+2)/4}P_l(-s)\xi(S_1;s)\cdot\left\{\begin{matrix}1 & \text{if } m\text{is even}\\ \xi(2s) & \text{if } m \text{ is odd}\end{matrix}\right\}.$$

(ii) *When $S_1[\eta]>0$ and $S_1(\eta,Y_0)\gtrless 0$,*

$$a_l^*(Y,\eta;s) = (\tfrac{1}{2}S_1[Y])^{-l/2}2^{(2m\pm 4l+11)/4}\pi^{(-[m/2]\pm 2l)/2}S_1[Y]^{\pm l/2}$$
$$\times S_1[\eta]^{(-m\pm 2l-2)/4}|S_{1,\eta}|^{-1/4}Q_{l,\eta}^*(s)\xi(S_{1,\eta};s+1/2)g_{S_1}(\eta;s)$$
$$\times\,\omega(2\pi Y,2\eta;(2s+m+2l+2)/4,(2s+m-2l+2)/4).$$

(iii) *When* $S_1[\eta] < 0$,

$$a_l^*(Y,\eta;s) = (\tfrac{1}{2}S_1[Y])^{-l/2}2^{(6m+9)/4}\pi^{(-[m/2]-1)/2}S_1[Y]^{m/4}S_1[\eta]^{-1/2}|S_{1,\eta}|^{-1/4}$$
$$\times \delta_+(Y,\eta)^{l/2}\delta_-(Y,\eta)^{-l/2}Q_{l,\eta}^*(s)\xi(S_{1,\eta};s+1/2)g_{S_1}(\eta;s)$$
$$\times \omega(2\pi Y, 2\eta; (2s+m+2l+2)/4, (2s+m-2l+2)/4).$$

THEOREM 7.2. *The Eisenstein series $E_l^*(Z,s)$ has a meromorphic continuation in s to the whole s-plane and is invariant under $s \mapsto -s$. Furthermore, it is holomorphic except for possible simple poles at $s = m/2+1-k$ $(0 \le k \le m+2, k \in \mathbf{Z})$.*

The convergence of (7.1) at $s = l - m/2 - 1$ is not guaranteed if $l \le m+2$. However, as in Shimura [11], we can construct the holomorphic Eisenstein series of smaller weights. Notice that the number of primes such that $S_{1,p}$ is anisotropic over $\mathbf{Q}_p$ is even if $m = 1$.

THEOREM 7.3. *We define*

$$E_l(Z) := E_l(Z, l-m/2-1) \qquad \textit{for } l \ge (m+3)/2.$$

Then $E_l(Z)$ is a holomorphic function in Z on $\mathfrak{D}$ i.e. $E_l(Z) \in M_l(\Gamma)$.

THEOREM 7.4. *Let l be an even integer with $l \ge (m+3)/2$. The holomorphic Eisenstein series $E_l(Z)$ has the following Fourier expansion*

$$E_l(Z) = 1 + \sum_{\substack{\eta \in L_l^* \\ S_1[\eta]>0, S_1(\eta,Y_0)>0}} a_l(\eta)e[S_1[\eta,Z)],$$

$$a_l(\eta) = \frac{B_{S_{1,\eta}}(-l+(m+1)/2)}{B_{S_1}(-l+m/2)}\hat{g}_l(\eta)$$

$$\begin{cases} (-1)^{[(m+2)/4]}2^{-l+m/2+3}l\left(l-\dfrac{m}{2}\right)\dfrac{1}{B_lB_{l-m/2,\chi_{S_1}}}|\det S_{1,\eta}|^{l-m/2-1}\left|\dfrac{d(S_1)}{\det S_1}\right|^{l-(m+1)/2} \\ \hfill \textit{if m is even} \\ -(-1)^{[(m+2)/4]}2^{l-(m-3)/2}l\,\dfrac{B_{l-(m+1)/2,\chi_{S_{1,\eta}}}}{B_lB_{2l-m-1}}\left|\dfrac{\det S_{1,\eta}}{d(S_{1,\eta})}\right|^{l-m/2-1}|\det S_1|^{-l+(m+1)/2} \\ \hfill \textit{if m is odd} \end{cases},$$

where $\hat{g}_l(\eta)$ is defined by (6.2).

REMARK. When $m = 1$, the algebraic group G_2 is isogenous to a quaternion unitary group of degree 2 and this Eisenstein series is the one studied in [1].

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Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima, 739-8526, *Japan*