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A remark on homology localization

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ABSTRACT. A. K. Bousfield [1], [2] introduced the notion of localization of spaces and spectra with respect to homology functor h and proved the exsitence theorem. In this note we introduce a variation of this notion, (h, n)-localization, which interpolates a contractible space or spectrum pt and the localization $L_h(X)$ of the original space or spectrum X and prove the existence theorem along the arguments of [1], [2].

1. Statement of results

Let $\mathscr{C}, \mathscr{S}, \tilde{\mathscr{C}}, \tilde{\mathscr{S}}$ denote the categories of CW-complexes, CW-spectra and their homotopy categories respectively.

DEFINITION. Let $\mathscr{A}, \mathscr{B}$ be categories and $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ a functor.

i) $C \in Ob(\mathscr{A})$ is called *$\mathscr{F}$-local* if $f^* : \mathscr{A}(B, C) \to \mathscr{A}(A, C)$ is bijective for any $A, B \in Ob(\mathscr{A})$ and any $f : A \to B$ such that $\mathscr{F}(f)$ is an isomorphism.

ii) A morphism $g : A \to C$ is called an *$\mathscr{F}$-localization map* of A if C is $\mathscr{F}$-local and $\mathscr{F}(g)$ is an isomorphism. In this case C is called an *$\mathscr{F}$-localization* of A.

Let h be a generalized homology functor and n an integer. Let $\alpha = (h, n)$ be the functor defined by $\alpha(X) = (h_k(X)|k < n)$ from $\mathscr{D}(= \tilde{\mathscr{C}}$ *or* $\tilde{\mathscr{S}})$ to $\{(A_k|k < n); A_k \in Ab\}$, *where Ab is the category of abelian groups. Then we can prove the following.*

THEOREM 1. *Let h be a generalized homology functor which is representable by a spectrum and $\alpha = (h, n)$ the functor above for an integer n. Then it holds that*

i) *Any object $X \in Ob(\mathscr{D})$ has an α-localization map.*

ii) *Let $f : X \to Y$ and $g : X \to Z$ be α-localization maps of X. Then there exisits a map $k : Y \to Z$ such that $k \circ f \simeq g$. Moreover, such a map k is always an isomorphism in the category $\mathscr{D}$.*

Note that (h, n)-localization of X is unique up to homotopy by this theorem. This may be called also *half h-localization* and denoted by $L_h^n(X)$.

REMARK. Let $h_2 = H_2(-, \mathbf{Z} \oplus \mathbf{Z}_2)$. Then there is no h_2-localization map of the real projective plane RP^2 for the functor $h_2 : \tilde{\mathscr{C}} \to Ab$.

PROOF. Assume that $f : RP^2 \to Y$ is an h_2-localization map. Since the map $S^m \to pt$ is an h_2-isomorphism for $m \neq 2$, Y should be an Eilenberg-MacLane complex $K(G, 2)$. Moreover $G = 0$ since $H_2(RP^2, \mathbf{Z}) = 0$. This contradicts the fact that $H_2(RP^2, \mathbf{Z}_2) \neq 0$.

2. Proof of Theorem 1

We shall prove theorem 1 only for the case of CW complexes or more precisely simplicial sets. The case of CW-spectra can be proved similarly and moreover we can give a sligtly clearer proof by using the additive and triangulable properties. If X is a simplicial set, $\# X$ denotes the cardinality of the set of non-degenerate simplices of X. Let γ be a fixed cardinal number greater than the cardinality of $\sum_{k<n} \# h_k(pt)$ and $\aleph_0$. A map $f : X \to Y$ is called an α-isomorphism, if $\alpha(f)$ is an isomorphism. An α-isomorphism $f : X \to Y$ is called a *versal α-isomorphism* (resp. *versal α-epimorphism*) if 'Z is α-local' is equivalent to '$f^* : [Y, Z] \to [X, Z]$ is bijective (resp. surjective)' for any Z.

PROPOSITION 1. *Let X, Y, Z be simplicial sets with $X, Z \subset Y$. If the inclusion maps $X \subset Y$ and $X \cap Z \subset Z$ induce α-isomorphisms, then the inclusion map $X \cup Z \subset Y$ also induces an α-isomorphism.*

PROOF. This follows from the five lemma applied to the Mayer-Vietoris exact sequence:

$$\begin{array}{ccccccccc} h_k(X \cap Z) & \to & h_k(X) \oplus h_k(Z) & \to & h_k(X \cup Z) & \to & h_{k-1}(X \cap Z) & \to & h_{k-1}(X) \oplus h_{k-1}(Z) \\ \cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ h_k(Z) & \to & h_k(Y) \oplus h_k(Z) & \to & h_k(Y) & \to & h_{k-1}(Z) & \to & h_{k-1}(Y) \oplus h_{k-1}(Z) \end{array}$$

for $k < n$.

PROPOSITION 2. *Let X, Y be simplicial sets with $X \subset Y$. If the inclusion map $f : X \to Y$ induces an α-isomorphism, then the inclusion map $X \times [0, 1] \cup Y \times \{0, 1\} \subset Y \times [0, 1]$ also induces an α-isomorphism.*

Proof is similar to the above.

PROPOSITION 3. *Let X, Y, Z be simplicial sets such that $X, Z \subset Y$, $\# Z < \gamma$ and the inclusion map $f : X \to Y$ is an α-isomorphism. Then there is a subcomplex W of Y such that $Z \subset W$, $\# W < \gamma$ and the two inclusion maps $g : X \cap W \to W$ and $g' : X \cup W \to Y$ are α-isomorphism.*

PROOF. We shall construct a tower $(W_m | m = 0, 1, 2, 3, \dots)$ such that i) $W_0 = Z$, ii) $W_0 \subset W_1 \subset W_2 \subset \cdots \subset Y$, iii) $\# W_m < \gamma$ for $m \geqq 0$ and iv) for every homology class $a \in h_k(W_m \cap X)$ with $k < n$ which vanishes in W_m vanishes in $W_{m+1} \cap X$. First put $W_0 = Z$. Assume that W_m is defined. Since the inclusion map $X \subset Y$ is an α-isomorphism, for every homology class $a \in h_k(W_m \cap X)$ with $k < n$ which vanishes in W_m, there is a finite subcomplex $W(a, m)$ of X such that a vanishes in $W(a, m) \cup (W_m \cap X)$. And also for every homology class $b \in h_k(W_m)$, there are finite subcomplex $W'(b, m)$ of X and finite subcomplex $W''(b, m)$ of Y such that b is realized in $W'(b, m)$ and the realizatios coincide with each other in $W''(b, m)$. Let W_{m+1} be the union of all such $W(a, m)$'s, $W'(b, m)$'s, $W''(b, m)$'s and W_m, then we get a desired $W = \bigcup_{m=0}^{\infty} W_m$ by Proposition 1, since the inclusion $g : X \cap W \to W$ is an α-isomorphism by the construction and $h_k(W) = \varinjlim h_k(W_m)$.

PROPOSITION 4. *There exists a simplicial pair $P \subset Q$ which is a versal α-epimorophism.*

PROOF. Let C be the set of all non isomorphic simplicial pairs $X_\lambda \subset Y_\lambda$ such that the inclusion map induces an α-isomorphism and $\# Y_\lambda < \gamma$. Write that $C = \{X_\lambda \subset Y_\lambda | \lambda \in \Lambda\}$. Then the pair $P = \bigcup_{\lambda \in \Lambda} X_\lambda \subset Q = \bigcup_{\lambda \in \Lambda} Y_\lambda$, where $\bigcup$ means disjoint union, is a desired pair. In fact, let $X \subset Y$ be any α-isomorphism, Z any space such that induced map $[Q, Z] \to [P, Z]$ is surjective and $f : X \to Z$ any map. Then by Proposition 3, we can extend f to Y by a transfinite induction as in Lemma 11.3 of [1]. By Proposition 2 this gives also uniqueness of the extension up to homotopy.

From now on we shall fix this *versal* α-epimorphism and $p : P \to Q$ denotes the inclusion map.

PROOF of THEOREM 1. Let X be any simplicial set. We construct a tower $(A_\xi | \xi :$ ordinal numbers) by the transfinite induction as follows. Let $A_0 = X$, and suppose that A_ξ is defined. Let $M_\xi = \mathrm{CMap}(P, A_\xi)$ where $\mathrm{CMap}(K, L)$ denotes the set of all maps from K to L, and define $A_{\xi+1}$ as the push out in the following push out square diagram in the category of simplicial sets:

$$\begin{array}{ccc} \bigcup_{m \in M_\xi} P & \xrightarrow{\ \psi\ } & \bigcup_{m \in M_\xi} Q \\ \varphi \Big\downarrow & & \Big\downarrow \\ A_\xi & \subset & A_{\xi+1}, \end{array}$$

where φ is the composition of the disjoint uion $\bigcup_{m \in M_\xi} m$ and the codiagonal (=folding) map $\bigcup_{m \in M_\xi} A_\xi \to A_\xi$ and ψ is the disjoint $\bigcup_{m \in M_\xi} p$. We see that

$A_\xi \to A_{\xi+1}$ is an α-isomorphism, because ψ is an α-isomorphism in the following diagram $(k < m)$

$$\begin{array}{ccccccc} & & h_k(\bigcup_{m\in M_\xi} P) & \longrightarrow & h_k(\bigcup_{m\in M_\xi} Q) & & \\ & \overset{0}{\nearrow} & \downarrow & & \downarrow & \searrow & \\ h_k(C_\psi) & & & & & & h_{k-1}(C_\psi) = 0 \\ & \searrow & & & & \nearrow & \\ & & h_k(A_\xi) & \longrightarrow & h_k(A_{\xi+1}) & & \end{array}$$

for $k < n$. For the limit ordinal, define $A_\zeta = \bigcup_{\xi<\zeta} A_\xi$. Note that for any ordinals ξ, ζ with $\xi < \zeta$, the inclusion map $A_\xi \subset A_\zeta$ in an α-isomorphism because $h_k(A_\zeta) = \varinjlim_{\xi<\zeta} h_k(A_\xi)$. Let κ be the smallest ordinal with cardinality greater than #P. Then any map $k : P \to A_\kappa$ passes A_ξ for some $\xi < \kappa$, hence k is extendable to a map from Q which passes $A_{\xi+1}$. Moreover this is shown to be unique up to homotopy by using Proposition 2. Therefore A_κ is α-local because $P \subset Q$ is versal α-epimorphism. Uniqueness of (h, n)-localization follows by the definition itself.

COROLLARY. *Let h_* be a generalized homology functor which is representable by a spectrum. Then we can construct the following natural sequence*:

$$L_h(X) \to \cdots \to L_h^{n+1}(X) \to L_h^n(X) \to \cdots \to \{pt\}.$$

EXAMPLES. Hereafter we consider in the stable category $\mathscr{S}$ or $\tilde{\mathscr{S}}$.

1. If h is the stable homotopy functor $\pi, L_\pi(X) = X$ and $L_\pi^n(X)$ is obtained from X by killing homotopy of $\dim \geqq n$. Moreover if h is connective, $\mathrm{hocolim}_{n\to-\infty} L_h^n(X)$ is contractible.

2. If h is periodic, for example $h = K$, the complex K-homology theory, $L_h^n(X) = L_h(X)$ for any n.

3. In the case $h_* = (\pi \oplus K)_*$, we obtain an interesting sequence which interpolates X and $L_k(X)$. $\mathrm{holim}_{n\to\infty} L_h^n(X) = X$ since h contains π as a factor. Let $A = M(\mathbf{Z}_p)$ be the Moore spectrum mod p (p:prime), $f : \Sigma^* A \to A$ be the Adams'K-equivalence map and B be the cofiber of f. Then the map $B \to \{pt\}$ is an (h, O)-isomorphism. This implies that $\mathrm{hocolim}_{n\to-\infty} L_h^n(X) = L_K(X)$.

References

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