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Liouville-Picard theorem in harmonic spaces

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ABSTRACT. An extended version of the classical Liouville-Picard theorem for the harmonic functions in $\mathbb{R}^n$ is considered in the context of biharmonic functions in a Brelot harmonic space with a symmetric Green kernel.

1. Introduction

In [3] is considered a Liouville-Picard type theorem for superharmonic functions in $\mathbb{R}^n$, $n \geq 2$. A simple special case of this theorem shows that if $s \geq 0$ is superharmonic in $\mathbb{R}^n$, $n = 3$ or 4, and $\Delta^2 s \geq 0$ then s is a constant and this result is not true if $n \geq 5$.

Since for a superharmonic function s in a domain ω in $\mathbb{R}^n$, $n \geq 2$, the condition $\Delta^2 s \geq 0$ is equivalent to saying that Δs is subharmonic ≤ 0 in ω, the above special case can be formulated as follows: there exist p and q, potentials > 0 in $\mathbb{R}^n$ such that $\Delta q = -p$ if and only if $n \geq 5$. This shows a variation in the study of potential theory in $\mathbb{R}^n$, $n \geq 3$, depending on n, even though (symmetric) Green kernels can be defined in all these spaces.

In this note, we obtain some results which reflect this variation. With a view to introduce only the essential assumptions in the proofs, we have chosen to work in a Brelot harmonic space possessing a symmetric Green kernel [2]. Another advantage is that some of these results, proved earlier in a Riemannian manifold [5] but not meaningful in a Riemann surface because the Laplacian is not invariant under a parametric change, have a general validity.

2. Preliminaries

Let Ω be a Brelot harmonic space with a countable base, having potentials > 0 and satisfying the axiom of proportionality; then, Mme. R. M. Hervé has proved that there exists a Green function $G(x, y)$ on Ω which is assumed here to be symmetric; it is also assumed that the constants are harmonic in Ω. (The terms are explained in F. Y. Maeda [2], p. 35).

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Some examples of such spaces Ω are: any domain in $\mathbb{R}^n$, $n \geq 3$; any domain in $\mathbb{R}^2$ whose complement is not locally polar; hyperbolic Riemannian manifolds; and hyperbolic Riemann surfaces.

Let λ be a fixed Radon measure on Ω such that every superharmonic function in Ω is locally λ-integrable. For example, if Ω_n is a regular exhaustion of Ω containing a point z, that is $z \in \Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ and $\Omega = \bigcup \Omega_n$ and if $\rho_z^{\Omega_n}$ is the harmonic measure on $\partial\Omega_n$ and if $e_n \geq 0$ is a sequence of numbers, take $\lambda = \sum e_n \rho_z^{\Omega_n}$. If $\Omega = \mathbb{R}^n$, λ can be taken as the Lebesque measure and if Ω is a Riemannian manifold or a Riemann surface, λ can be taken as the volume or the surface measure.

For a given locally λ-integrable function f on Ω, if $u(x) = \int G(x,y)f(y)\,d\lambda(y)$ is a well-defined difference of two potentials in Ω, we write $Lu = -f$.

DEFINITION 2.1. Ω is said to be a bipotential space if and only if there exist p and q, potentials > 0 is Ω such that $Lq = -p$.

Recall that for a given nonempty set A in Ω, R_1^A stands for the infimum of all superharmonic functions $s > 0$ in Ω such that $s \geq 1$ on A; and $\hat{R}_1^A$ is its lower semicontinuous regularization, namely $\hat{R}_1^A(x) = \liminf_{y \to x} R_1^A(y)$ for each x in Ω.

3. Characterization of bipotential spaces

Given a measure $\mu \geq 0$ on Ω, we know (pp. 67–68 [2]) that $u(x) = \int G(x,y)\,d\mu(y)$ is a potential in Ω if $u(x)$ is finite at some point; and that is so if μ has compact support or more generally in our context if $\mu(\Omega)$ is finite. This situation is made precise in the following theorem:

THEOREM 3.1. *Given a measure $\mu \geq 0$ on Ω, $\int G(x,y)\,d\mu(y)$ is a potential in Ω if and only if for a non-empty open set ω, $\int \hat{R}_1^{\omega}(y)\,d\mu(y)$ is finite.*

PROOF. 1) Let $\int \hat{R}_1^{\omega}(y)\,d\mu(y)$ be finite. If k is an outerregular compact set such that $\phi \neq \mathring{k} \subset k \subset \omega$, we have $\int \hat{R}_1^{k}(y)\,d\mu(y) < \infty$.

Fix $x_0 \in \mathring{k}$. Then $G(x_0,y) = B_k G(x_0,y)$ for $y \in \Omega\backslash k$, where $B_k f$ stands for the Dirichlet solution in $\Omega\backslash k$ with boundary values f on ∂k and 0 at infinity. Let $a \leq G(x_0,y) \leq b$ for every $y \in \partial k$. Then $a\hat{R}_1^k(y) \leq G(x_0,y) \leq b\hat{R}_1^k(y)$ in $\Omega\backslash k$. Hence $\int_{\Omega\backslash k} G(x_0,y)\,d\mu(y)$ is finite for any fixed $x_0 \in \mathring{k}$.

Now $p(x) = \int_k G(x,y)\,d\mu(y)$ is a potential in Ω. If $x_0 \in \mathring{k}$ had been chosen so that $p(x_0) < \infty$, we would have $\int G(x_0,y)\,d\mu(y) < \infty$. Hence $u(x) = \int_\Omega G(x,y)\,d\mu(y)$ is a potential in Ω.

2) Conversely, suppose $\int G(x,y)\,d\mu(y)$ is a potential. Then for an outerregular compact set k and some $x_0 \in \mathring{k}$, $\int G(x_0,y)\,d\mu(y) < \infty$. Hence if

$G(x_0, y) \geq a$ on ∂k, $a \int_{\Omega \setminus k} \hat{R}_1^k(y)\, d\mu(y) < \infty$ and of course $\int_{\overset{\circ}{k}} \hat{R}_1^k(y) d\mu(y) \leq \int_{\overset{\circ}{k}} d\mu(y) < \infty$. Thus, $\int_\Omega \hat{R}_1^k(y)\, d\mu(y) < \infty$ which implies $\int \hat{R}_1^k(y)\, d\mu(y) < \infty$. Writing $\overset{\circ}{k} = \omega$, we have $\int \hat{R}_1^\omega\, d\mu(y) < \infty$.

Remark. The above theorem in particular states that there exists a potential u in Ω such that $Lu = -1$, that is $u(x) = \int G(x,y)\, d\lambda(y)$, if and only if $\int \hat{R}_1^\omega(y)\, d\lambda(y)$ is finite for some nonempty open set ω. For a proof of a related result in the context of a hyperbolic Riemannian manifold Ω, see Theorem 4.1 [5], p. 336.

Corollary 3.2. (Proposition 4.7 [2]). Let $\mu \geq 0$ be a measure such that $\mu(\Omega)$ is finite. Then $\int G(x,y)\, d\mu(y)$ is a potential.

The following proposition is proved in [4] (Theorem 3.1); but the proof there is a little involved whereas here it is obtained as a simple consequence of the above theorem. Also included is a corollary which slightly improves on the result that a positive harmonic function in $\mathbb{R}^n$ is a constant.

Proposition 3.3. *Given a measure $\mu \geq 0$ on $\mathbb{R}^n$, $n \geq 3$, $\int \frac{1}{|x-y|^{n-2}}\, d\mu(y)$ is a potential if and only if $\int_{|y| \geq 1} |y|^{-n+2}\, d\mu(y)$ is finite.*

Proof. 1) $\int_{|y| \geq 1} |y|^{-n+2}\, d\mu(y)$ is finite means that $\int \hat{R}_1^\omega(y)\, d\mu(y)$ is finite where ω is the unit ball.

2) Conversely let $\int \frac{1}{|x-y|^{n-2}}\, d\mu(y)$ be a potential. Then for some x_0, $\int |y - x_0|^{-n+2}\, d\mu(y)$ is finite. Hence if R is large and $|y| \geq R$, $|y - x_0| \leq 2|y|$ which implies that $\int_{|y| \geq R} |y|^{-n+2}\, d\mu(y)$ is finite and consequently $\int_{|y| \geq 1} |y|^{-n+2}\, d\mu(y)$ is finite.

For the following corollary, recall [1] that a superharmonic function u in $\mathbb{R}^n$, $n \geq 2$, is said to be admissible if and only if u has a harmonic minorant outside a compact set; and if u is admissible and μ is the measure associated with its local Riesz representation, then $\int_{|y| \geq 1} |y|^{-n+2}\, d\mu(y)$ is finite for all $n \geq 2$.

Corollary 3.4. Let u be an admissible superharmonic function in $\mathbb{R}^n$, *$n \geq 2$, such that $\Delta^2 u \leq 0$ and $\liminf_{|x| \to \infty} \frac{u(x)}{|x|} \geq 0$. Then u is a constant.*

Proof. Let $s = -\Delta u$. Then s is a subharmonic function ≥ 0 in $\mathbb{R}^n$. Since u is admissible and since the measure associated with u in a local Riesz representation is proportional to the measure with density $-\Delta u$, $\int_{|y| \geq 1} |y|^{-n+2} s(y)\, dy$ is finite. This means that if B is the unit sphere in $\mathbb{R}^n$ and

σ_n is the area of ∂B, $\int_{r=1}^{\infty}\int_{\partial B} r^{-n+2}s(r,w)r^{n-1}\,dr\,dw < \infty$. Since s is a subharmonic function ≥ 0 in $\mathbb{R}^n$, we have the following inequalities:

$$\infty > \int_{r=1}^{\infty} r\sigma_n \left[\frac{1}{\sigma_n}\int_{\partial B} s(r,w)\,dw\right] dr$$
$$\geq \int_{1}^{\infty} r\sigma_n s(o)\,dr.$$

This is possible if and only if $s(o) = 0$. o being arbitrary, $s \equiv 0$ in $\mathbb{R}^n$ and hence u is harmonic. Then the assumption on the behaviour of u at infinity implies [3] that u is a constant.

THEOREM 3.5. *Ω is a bipotential space (Definition 2.1) if and only if for a nonempty open set w, $\int(\hat{R}_1^w)^2\,d\lambda < \infty$.*

PROOF. 1) Let $\int(\hat{R}_1^w)^2\,d\lambda < \infty$. Choose an outerregular compact set k, $\mathring{k} \subset k \subset w$. Then $\int(\hat{R}_1^k)^2\,d\lambda < \infty$. Therefore by Theorem 3.1, $u(x) = \int G(x,y)\hat{R}_1^k(y)\,d\lambda(y)$ is a potential. Since $\hat{R}_1^k$ is a potential and $Lu = -\hat{R}_1^k$, Ω is a bipotential space.

2) Conversely, let Ω be a bipotential space. That is, by definition, there exists a potential p in Ω such that $\int G(x,y)p(y)\,d\lambda(y)$ is a potential and hence for a nonempty open set w, $\int \hat{R}_1^w(y)p(y)\,d\lambda(y)$ is finite. Choose an outerregular compact set k such that $\phi \neq \mathring{k} \subset k \subset w$. Then for a constant $c(=\min_k p)$, $p(x) \geq c\hat{R}_1^{\mathring{k}}$ in Ω and $\hat{R}_1^k \leq \hat{R}_1^w$. Consequently, $\int(\hat{R}_1^k)^2\,d\lambda < \infty$ and hence $\int(\hat{R}_1^{\mathring{k}})^2\,d\lambda < \infty$.

REMARK. If Ω is a bipotential space, for any compact set e in Ω, $\int_\Omega(\hat{R}_1^e)^2\,d\lambda < \infty$. For, since Ω is a bipotential space, there is a potential $p > 0$ in Ω such that $\int G(x,y)p(y)\,d\lambda(y)$ is also a potential. Hence, given any nonpolar compact set e (if e is polar, note $\hat{R}_1^e \equiv 0$), there is some $x_0 \in e$ such that $\int G(x_0,y)p(y)\,d\lambda(y) < \infty$. Since e is compact, there are constants a and b such that $\hat{R}_1^e(y) \leq aG(x_0,y)$ and also $\hat{R}_1^e(y) \leq bp(y)$ for all $y \in \Omega$. Consequently, $\int_\Omega(\hat{R}_1^e)^2\,d\lambda < \infty$.

COROLLARY 3.6. Let Ω be a harmonic space with a symmetric Green kernel $G(x,y)$. Then Ω is a bipotential space if and only if for some (and hence every) potential $p > 0$ with compact support in Ω, there exists a potential u in Ω such that $Lu = -p$, that is $u(x) = \int G(x,y)p(y)\,d\lambda(y)$. In particular, for any fixed z in a bipotential space Ω, there exists a potential $u_z(x)$ such that $Lu_z(x) = -G(z,x)$.

PROOF. Let A be the compact (harmonic) support of p and let k be an outerregular compact set such that $A \subset \mathring{k} \subset k$. Since $p = B_k p$ in $\Omega\backslash k$, if $0 < a \leq p \leq b$ on ∂k, we have $a\hat{R}_1^k \leq p \leq b\hat{R}_1^k$ in $\Omega\backslash k$.

1) Let Ω be a bipotential space. Then, by the Remark above,

$$\int_{\Omega\setminus k} \hat{R}_1^k(y)p(y)\,d\lambda(y) \le b \int_{\Omega\setminus k} (\hat{R}_1^k)^2\,d\lambda(y) < \infty.$$

Moreover, since the potential p is locally λ-integrable,

$$\int_k \hat{R}_1^k(y)p(y)\,d\lambda(y) \le \int_k p(y)\,d\lambda(y) < \infty.$$

Hence, $\int_\Omega \hat{R}_1^{\mathring{k}}(y)p(y)\,d\lambda(y) < \infty$ and by Theorem 3.1, $\int G(x,y)p(y)\,d\lambda(y)$ is a potential.

2) Conversely, if $\int G(x,y)p(y)\,d\lambda(y)$ is a potential, then Ω is a bipotential space by definition.

COROLLARY 3.7. (p. 306 [5]) Let Ω be a harmonic space with a symmetric Green kernel $G(x,y)$. Then Ω is a bipotential space if and only if for some (and hence every) $z \in \Omega$, $\int_{\Omega\setminus V} G^2(z,y)\,d\lambda(y) < \infty$, where $V(\neq \Omega)$ is any neighbourhood of z.

PROOF. Since the given condition is equivalent to saying that $\int_{\Omega\setminus k}(\hat{R}_1^k)^2\,d\lambda$ is finite for an outerregular compact set k such that $z \in \mathring{k}$, the corollary follows from Theorem 3.5.

COROLLARY 3.8. If $f \in L^2(\lambda)$ in a bipotential space Ω, then $\int G(x,y)f(y)\,d\lambda(y)$ is well-defined as a difference of two potentials.

PROOF. Since Ω is a bipotential space, there exists a nonempty open set w such that $\int(\hat{R}_1^w)^2\,d\lambda < \infty$. Now $\left(\int(\hat{R}_1^w)|f|\,d\lambda\right)^2 \le \left(\int(\hat{R}_1^w)^2\,d\lambda\right)\cdot \left(\int|f|^2\,d\lambda\right) < \infty$. Hence by Theorem 3.1, $\int G(x,y)|f(y)|\,d\lambda(y)$ is a potential and the corollary follows.

REMARK. A result similar to Corollary 3.8 is proved (p. 251 [5]) in the context of a hyperbolic Riemannian manifold satisfying a stronger condition that it is in $\tilde{O}_{QP}$, that is a manifold having positive quasiharmonic functions. Corresponding to this stronger condition, we give the following definition in a harmonic space.

DEFINITION 3.9. A harmonic space Ω is said to be a strongly bipotential space if and only if for a nonempty open set w, $\int(\hat{R}_1^w)\,d\lambda < \infty$.

REMARKS. 1) In a harmonic space Ω with a symmetric Green kernel, the following four conditions are equivalent:

a) Ω is a strongly bipotential space.

b) $\int G(x,y)\,d\lambda(y)$ is a potential in Ω.

c) $\int p(y)\,d\lambda(y) < \infty$ for any potential with a compact harmonic support in Ω.

d) For some (and hence every) $z \in \Omega$, $\int G(z,y)\,d\lambda(y)$ is finite.

2) Since $(\hat{R}_1^w)^2 \leq \hat{R}_1^w$, a strongly bipotential space is a bipotential space. $\mathbb{R}^n$, $n \geq 5$, are bipotential spaces but not strongly bipotential.

3) The Riemannian manifolds in $\tilde{O}_{QP}$, that is those having positive quasiharmonic functions (p. 73 [5]) are examples of strongly bipotential spaces.

4) A particular form of Corollary 3.4 gives a slight extension of the classical Liouville-Picard theorem in $\mathbb{R}^n$, namely: If $u \geq 0$ is a superharmonic function in $\mathbb{R}^n$, $n \geq 3$, and if Δu is a constant then u is a constant.

Corresponding to the above result one can formulate the Liouville-Picard theorem for a harmonic space as follows: In a harmonic space with a symmetric Green kernel, if $u \geq 0$ is a potential for which Lu is a constant, then $u \equiv 0$.

The above discussion shows that such a Liouville-Picard theorem is valid in a harmonic space Ω if and only if Ω is not a strongly bipotential space.

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