HIROSHIMA MATH. J.
28 (1998), 437–461

On the vector field problem for product manifolds

Bernard JUNOD and Ueli SUTER

(Received August 18, 1997)

ABSTRACT. Let $Span(M)$ be the largest number of linearly independent tangent vector fields on the manifold M. In this paper we establish a criterion giving an upper bound for $Span(M)$ when M is a product of stably complex manifolds. We obtain explicit upper bounds and exact values of $Span(M)$ in some special cases, such as products of lens spaces, products of quaternionic spherical space forms and products of Dold manifolds.

1. Introduction

Let M be a smooth, closed (i.e. compact and without boundary), connected manifold, we denote $Span(M)$, the largest number of everywhere linearly independent tangent vector fields on M. Finding $Span(M)$ is a classical problem in differential topology. This problem was solved when M is a sphere by A. Hurwitz, J. Radon and J. F. Adams (see [11], [20] and [1]). For spherical space forms, J. C. Becker has calculated $Span(M)$ in [6]. For more details about the present state of the question, the reader may consult the survey paper of J. Korbaš and P. Zvengrowski [17].

In this paper we shall study $Span(M)$ for M being a product of two stably complex manifolds M_1 and M_2. In other words, we suppose that the stable class of the tangent bundle τ_{M_i} of M_i carries a complex structure for $i = 1, 2$. We shall prove the following criterion for $Span(M)$ in the framework of complex K-theory.

THEOREM 1.1. *Let M_i be a smooth, closed and connected stably complex m_i-manifold and let $y_i \in \widetilde{KU}(M_i)$ be the stable class represented by the tangent bundle τ_{M_i}, $(i = 1, 2)$. If $Span(M_1 \times M_2) = m_1 + m_2 - k$, then the following relation is valid in $KU^0(M_1) \otimes KU^0(M_2)$,*

$$2^{n-1}\gamma_{1/2}(y_1) \otimes \gamma_{1/2}(y_2) \equiv 0 \pmod{2^{n-j-1}},$$

1991 *Mathematics Subject Classification:* 57R25, 55N15.

Key words and phrases: Tangent vector fields, vector bundles, geometric dimension, stably complex manifolds, complex projective space, lens space, quaternionic spherical space forms, Dold manifolds.

where $2n > m_1 + m_2$, $j = \left[\frac{k}{2}\right]$ *and* γ_t *is the formal power series associated to Atiyah's* γ^i*-operations in* KU*-theory.*

REMARK. At this point we should explain the meaning of the term $\gamma_{1/2}(x)$. In general, for $x \in KU(X)$ the expression $\gamma_{1/2}(x)$ does not make sense in $KU(X)$, but multiplied by a sufficiently high power of 2 it does. Explicitely, if $dim(X) \leq 2m+1$ we define $2^m\gamma_{1/2}(x) \in KU(X)$ by

$$2^m\gamma_{1/2}(x) = \sum_{i=0}^{m} 2^{m-i}\gamma^i(x).$$

Throughout this paper we will adopt this convention. Note that the exponential property of γ_t implies

$$2^m\gamma_{1/2}(x+y) = 2^m\gamma_{1/2}(x)\gamma_{1/2}(y) = \sum_{r=0}^{m}\sum_{i=0}^{r} 2^{m-r}\gamma^i(x)\gamma^{r-i}(y)$$

In particular we shall consider the case where M is a product of lens spaces $L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})$, or a product of quaternionic spherical space forms $N^{n_1}(m_1) \times N^{n_2}(m_2)$. We obtain the following results, where $\nu_2(n)$ is the exponent of 2 in the prime factor decomposition of n.

THEOREM 1.2. *For all positive integers* n_1 *and* n_2, *if* m_1 *and* m_2 *are large enough, we have*

$$Span(L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})) = 2\nu_2(n_1+1) + 2\nu_2(n_2+1) + 2.$$

Precisely, the above result is valid when:

1) $n_i + 1 = 2^{s_i}(2u_i+1)$ *with* $u_i \geq 1$ *and* $m_i \geq [log_2 n_i] + 2^{s_1} + 2^{s_2}$, $(i = 1, 2)$,

2) $n_1 + 1 = 2^{s_1}(2u_1+1)$ *with* $u_1 \geq 1$, $n_2 + 1 = 2^{s_2}$ *and* $m_i \geq [log_2 n_i] + \min\left(n_2 + 2^{s_1}, n_1 + 3\left[\frac{n_2}{4}\right] + 4\right)$, $(i = 1, 2)$,

3) $n_i + 1 = 2^{s_i}$ *and* $m_i \geq [log_2 n_i] + \min\left(n_2 + 3\left[\frac{n_1}{4}\right] + 4, n_1 + 3\left[\frac{n_2}{4}\right] + 4\right)$, $(i = 1, 2)$.

If m_1 and m_2 are small, the best results we know are those of M. Yasuo in [24].

THEOREM 1.3. *For all positive integers* n_i, *if* $m_i > [log_2 n_i] + \nu_2(n_1+1) + \nu_2(n_2+1) + 4$, $(i = 1, 2)$, *we have*

$$Span(N^{n_1}(m_1) \times N^{n_2}(m_2)) \leq 2\nu_2(n_1+1) + 2\nu_2(n_2+1) + 6.$$

This result is best possible when $\nu_2(n_1+1)$ and $\nu_2(n_2+1)$ are divisible by 4 (see [6]). For small values of m_1 and m_2 the best upper bounds have been obtained by T. Kobayashi in [16].

We establish similar results for products of spheres and complex projective spaces, Dold manifolds $D(u,v)$ and products of Dold manifolds.

THEOREM 1.4. *Let M be the product $\prod_{i=1}^{r} S^{m_i} \times \prod_{l=1}^{s} \mathbf{C}P^{n_l}$. If all the spheres are even dimensional then $Span(M) = 0$. If one of the m_i at least is odd, then*

$$Span(M) \leq m + 2n - k \leq m + 2\sum_{l=1}^{s} \nu_2(n_l + 1)$$

where $m = m_1 + m_2 + \cdots + m_r$.

For the proof of this theorem only the second factor of M, involving complex projective spaces, will be taken into account (see section 6). So, the upper bound given in theorem 1.4 is a good bound only if $\sum_{i=1}^{r} Span(\mathbf{S}^{m_i})$ is small with respect to $n_1 + n_2 + \cdots + n_s$, or if r is small with respect to s. For example, we believe that

$$Span(\mathbf{S}^{2u-1} \times \mathbf{C}P^{v}) = \rho(2u) + 2\nu_2(v+1) - 1,$$

where $\rho(2n)$ is the Hurwitz-Radon-Eckmann number (see for example [18]). Invoking Clifford algebra constructions, it is possible to show that

$$Span(\mathbf{S}^{2u-1} \times \mathbf{C}P^{v}) \geq \rho(2u) + 2\nu_2(v+1) - 2.$$

COROLLARY 1.1. *Let $N = \prod_{i=1}^{r} D(u_i, v_i)$. If all the integers u_i, $i = 1, 2, \ldots, r$, are even then $Span(N) = 0$. If one of the integers u_i at least is odd, then*

$$Span(N) \leq \sum_{i=1}^{r} (u_i + 2\nu_2(v_i + 1)).$$

In particular:

$$Span(D(2u+1, v)) \leq 2u + 1 + 2\nu_2(v+1).$$

For $r = 2$ and $\rho(2u_i + 2)$ small with respect to v_i, (i.e. $\max(u_1, u_2) \leq v_1 + v_2$) the corollary improves a result of Sohn in [21].

The paper is organized as follows: In section 2, we shall see that Theorem 1.1 is a straightforward consequence of a criterion about geometric dimension mentioned in [12] and [14]. We give a proof of this criterion in section 3. From section 4 to 6 we prove Theorems 1.2 to 1.4.

2. The geometric dimension and the vector field problem

Let X be a finite CW-complex and let x be an element of $\widetilde{KO}(X)$. The geometric dimension of x, denoted $gdim(x)$, is the smallest integer k such that $x+\underline{k}$ is represented by a k-dimensional real vector bundle. Here, $\underline{k}$ denotes the trivial k-dimensional real vector bundle over X. If M is a smooth, closed and connected m-manifold, we call *geometric dimension of* M and we denote it by $gdim(M)$, the geometric dimension of the stable class τ_0 of the tangent bundle of M

$$\tau_0 = \tau_M - \underline{m}.$$

It is a well known result that

$$Span(M) \leq m - gdim(M). \tag{2.1}$$

Consequently, if we can give a lower bound for $gdim(M)$, we obtain an upper bound of $Span(M)$. The following result established in [12] and [14] is a useful criterion to give lower bounds for $gdim(M)$.

THEOREM 2.1. *If $x \in \widetilde{KO}(M)$ is the image of a stable complex class, (i.e. $x = ry$ with $y \in \widetilde{KU}(M)$ and $r: KU(M) \to KO(M)$ the canonical map), and if $gdim(x) \leq k$, the following relation is satisfied in $\widetilde{KU}(M)$*

$$2^{n-1}\gamma_{1/2}(y) \equiv 0 \pmod{2^{n-j-1}},$$

where $2n > dim(M)$, $j = \left[\frac{k}{2}\right]$ and γ_t is as in Theorem 1.1.

We will give a proof of this theorem in section 3. Now we can show that Theorem 1.1 is a straightforward consequence of Theorem 2.1. Let M be the product $M_1 \times M_2$, where M_i is a smooth, closed, connected and stably complex m_i-manifold for $i = 1, 2$. If $\tau_0(i) = \tau_{M_i} - \underline{m_i}$ denotes the stable class of the tangent bundle over M_i, we have the following relations:

$$\tau_0(i) = ry_i, \quad \text{with } y_i \in KU(M_i), \quad i = 1, 2.$$
$$\begin{aligned} \tau_0 &= \tau_{M_1 \times M_2} - \underline{m_1 + m_2} = p_1^*(\tau_0(1)) + p_2^*(\tau_0(2)) \\ &= p_1^*(ry_1) + p_2^*(ry_2) = r(p_1^*(y_1) + p_2^*(y_2)), \end{aligned}$$

where $p_i: M_1 \times M_2 \to M_i$ is the canonical projection.

Hence, the stable class τ_0 of the tangent bundle over $M_1 \times M_2$ comes from a complex stable class. If $Span(M_1 \times M_2) \geq m_1 + m_2 - k$, by the inequality (2.1) we have $gdim(\tau_0) \leq k$. Then, according to Theorem 2.1, in $KU(M_1 \times M_2)$ the following relation holds:

$$2^{n-1}\gamma_{1/2}(p_1^*(y_1) + p_2^*(y_2)) \equiv 0 \pmod{2^{n-j-1}}. \tag{2.2}$$

By the Künneth theorem in KU-theory [3] the homomorphism

$$KU^0(M_1) \otimes KU^0(M_2) \to KU^0(M_1 \times M_2)$$
$$x \otimes y \mapsto p_1^*(x) \cdot p_2^*(y)$$

maps $KU^0\ (M_1) \otimes KU^0(M_2)$ onto a direct summand.

We have $2^{n-1}\gamma_{1/2}(p_1^*(y_1) + p_2^*(y_2)) = 2^{n-1}p_1^*(\gamma_{1/2}(y_1)) \cdot p_2^*(\gamma_{1/2}(y_2))$. The latter element corresponds via the Künneth isomorphism to $2^{n-1}\gamma_{1/2}(y_1) \otimes \gamma_{1/2}(y_2)$ and Theorem 1.1 follows from (2.2). □

Let $f : M \to BSO(2n)$, $2n > dim(M)$, be the classifying map of $x \in \widetilde{KO}(M)$. Since $x = ry$, we can lift the map f to $BU(n)$. We shall denote the classifying map of y by g. If we assume that $gdim(x) = k$, we can lift f to $BSO(k)$ and further to $B(n,k)$, the latter space being the pull-back space of the diagram

$$\begin{tikzcd} & BSO(k) \arrow[d] \\ BU(n) \arrow[r] & BSO(2n). \end{tikzcd}$$

We have the following commutative diagram

(2.3)
$$\begin{tikzcd} & B(n,k) \arrow[r, "f_k"] \arrow[d, "p"] & BSO(k) \arrow[d, "q"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "g"] & BU(n) \arrow[r, "r_n"] & BSO(2n). \end{tikzcd}$$

With the same hypothesis as in Theorem 2.1 we can give a second criterion concerning the geometric dimension of real stably complex vector bundles.

THEOREM 2.2. *If $gdim(x) \leq k$, the following relations are satisfied in $H^*(B(n,k); \mathbf{Z})$,*

$$g^*(c_i) \equiv 0 \ (\mathrm{mod}\, 2), \qquad \left[\frac{k}{2}\right] + 1 \leq i \leq n-1,$$

where c_i is the i-th universal Chern class.

PROOF. In [12] and [15], we have determined the additive structure of $H^*(B(n,k); \mathbf{Z})$. There are abelian group isomorphisms:

$$H^*(B(n,k); \mathbf{Z}) \cong \begin{cases} \mathbf{Z}[c_1, \dots, c_t] \otimes \Delta(a_t, b_{t+1}, \dots, b_{n-1}) & \text{if } k = 2t \\ \mathbf{Z}[c_1, \dots, c_t] \otimes \Delta(b_{t+1}, \dots, b_{n-1}) & \text{if } k = 2t+1 \end{cases}$$

where $\Delta(x_1,\dots,x_m)$ is the free abelian group generated by the elements

$$x_{i_1}x_{i_2}\dots x_{i_s} \quad \text{with } 1 \le i_1 < i_2 < \dots < i_s \le m,$$

c_i is the image of the i-th universal Chern class under the map p^* and the elements b_i satisfy the relations

$$c_i = 2b_i, \quad i = \left[\frac{k}{2}\right] + 1, \dots, n-1.$$

Then, by the commutativity of the diagram (2.3), we have

$$g^*(c_i) = \tilde{f}^*(p^*(c_i)) = \tilde{f}^*(2b_i) = 2\tilde{f}^*(b_i)$$

for $i = \left[\frac{k}{2}\right] + 1, \dots, n-1.$ □

We shall also need the two following results:

PROPOSITION 2.1. *Let* τ_0 *and* $\tau_0(i)$ *be the stable classes of the tangent bundles of* $M_1 \times M_2$ *and* M_i *respectively* $(i = 1,2)$. *Then*:

(a) $$gdim(\tau_0) \ge \max(gdim(\tau_0(1)), gdim(\tau_0(2))),$$

(b) $$gdim(\tau_0) \le gdim(\tau_0(1)) + gdim(\tau_0(2)).$$

PROOF. (a) If $gdim(\tau_0) = k$, the stable class τ_0 may be written as $\tau_0 = \xi - \underline{k}$ where ξ is a real k-dimensional vector bundle. Then we have

$$\tau_0(1) = i_1^*(p_1^*(\tau_0(1)) + p_2^*(\tau_0(2))) = i_1^*(\tau_0) = i_1^*(\xi) - \underline{k}$$

and so $gdim(\tau_0(1)) \le k = gdim(\tau_0)$. In the same way we show $gdim(\tau_0(2)) \le gdim(\tau_0)$.

(b) If $gdim(\tau_0(i)) = k_i$, the stable class $\tau_0(i)$ may be written as $\tau_0(i) = \xi_i - \underline{k_i}$, where ξ_i is a real k_i-dimensional vector bundle, for $i = 1,2$ and so

$$\tau_0 = (p_1)^*(\tau_0(1)) + (p_2)^*(\tau_0(2)) = (p_1)^*(\xi_1) \oplus (p_2)^*(\xi_2) - \underline{k_1 + k_2},$$

hence $gdim(\tau_0) \le k_1 + k_2 = gdim(\tau_0(1)) + gdim(\tau_0(2))$. □

PROPOSITION 2.2. *If* M_1 *and* M_2 *are as above, then*

$$Span(M_1 \times M_2) \ge Span(M_1) + Span(M_2).$$

PROOF. If there are k_i linearly independent tangent vector fields over M_i, for $i = 1,2$, then there are at least $k_1 + k_2$ over $M_1 \times M_2$. □

3. Spinor representations and generators of $KU(B(n,k))$

Let $Spin^c(2n)$ be the group $(Spin(2n) \times U(1))/(\mathbf{Z}/2)$. Here $\mathbf{Z}/2$ is the subgroup generated by $(\varepsilon, -1)$, where ε denotes the generator of the kernel of $\pi : Spin(2n) \to SO(2n)$, the 2-fold covering map of $SO(2n)$. The composition of the projection $Spin(2n) \times U(1) \to Spin(2n)$ and π sends the subgroup $\mathbf{Z}/2$ to the identity matrix of $SO(2n)$, and induces a map

$$\tilde{\pi} : Spin^c(2n) \to SO(2n).$$

We can also see the group $Spin^c(2n)$ as $\pi^{-1}(SO(2n) \times SO(2))$, where $SO(2n) \times SO(2)$ is identified with a subgroup of $SO(2n+2)$ and $\pi : Spin(2n+2) \to SO(2n+2)$ is as above.

The canonical inclusion $U(n) \subset SO(2n)$ lifts to $Spin^c(2n)$. Then, the map $BU(n) \xrightarrow{r_n} BSO(2n)$, which is induced by this inclusion on the classifying spaces, lifts to $BSpin^c(2n)$ (see [4]), i.e. we have maps

$$BU(n) \xrightarrow{\tilde{f}_{2n}} BSpin^c(2n) \xrightarrow{B_{\tilde{\pi}}} BSO(2n), \quad \text{with} \quad B_{\tilde{\pi}} \circ \tilde{f}_{2n} = r_n. \tag{3.1}$$

The pull-back diagram of Lie groups

$$\begin{tikzcd} Spin^c(2k) \arrow[r] \arrow[d] & SO(2k) \arrow[d] \\ Spin^c(2n) \arrow[r] & SO(2n) \end{tikzcd}$$

gives rise to a pull-back diagram on the classifying space level and together with (3.1) we obtain the pull-back diagram

$$\begin{tikzcd} B(n,2k) \arrow[r, "\overline{f_{2k}}"] \arrow[d, "p"'] & BSpin^c(2k) \arrow[r] \arrow[d, "\psi"] & BSO(2k) \arrow[d] \\ BU(n) \arrow[r, "\tilde{f}_{2n}"] & BSpin^c(2n) \arrow[r] & BSO(2n) \end{tikzcd} \tag{3.2}$$

In the following we concentrate on the left hand square. The diagram induces a commutative diagram in KU-theory.

It is a well known result that the ring $KU(BG)$ is isomorphic to the completed representation ring $\widehat{RU}(G)$, when G is a compact, connected Lie group (see [5]). This is our motive to use below some information about the representation rings of $Spin^c(2n)$, $Spin^c(2k)$ and $U(n)$ to define generators of $KU(B(n,2k))$ (see also [12]). In a first step we consider the projection $Spin(2n) \times U(1) \xrightarrow{\varphi} Spin^c(2n)$. It induces an injection of representation rings

$$\varphi^* : RU(Spin^c(2n)) \to RU(Spin(2n)) \otimes RU(U(1)).$$

Let ρ be the representation defined as the identity of $U(1)$, and let Δ_{2n}^{+}, Δ_{2n}^{-} be the canonical irreducibles spinor representations of $Spin(2n)$. The representations $\Delta_{2n}^{\pm} \otimes \rho$ of $Spin(2n) \times U(1)$ give rise to the representations $\tilde{\Delta}_{2n}^{\pm}$ of $Spin^c(2n)$ (the elements $(\varepsilon, -1)$ acts trivially). The latter induce two elements in $KU(BSpin^c(2n))$ that we still denote $\tilde{\Delta}_{2n}^{+}$ and $\tilde{\Delta}_{2n}^{-}$. There is a relation between these two elements and some generators of $KU(B(n, 2k))$ defined in [12] and [13].

PROPOSITION 3.1. (a) *In $KU(B(n, 2k))$, there are elements α_k and β_{k+1} such that the following relations are satisfied*

i)
$$\tilde{f}_{2k}^{*}(\tilde{\Delta}_{2k}^{-}) = \sum_{r=0}^{k-1} 2^{k-r-1}\gamma^r + \alpha_k$$

ii)
$$\tilde{f}_{2k}^{*}(\tilde{\Delta}_{2k}^{+}) = \sum_{r=0}^{k-1} 2^{k-r-1}\gamma^r + \gamma^k - \alpha_k + \beta_{k+1}$$

iii)
$$2^{n-k}\beta_{k+1} = \sum_{r=k+1}^{n-1} 2^{n-r}\gamma^r.$$

(b) *In $KU(B(n, 2k+1)$, there is an element β'_{k+1} satisfying*

iii)
$$2^{n-k}\beta'_{k+1} = \sum_{r=k+1}^{n-1} 2^{n-r}\gamma^r.$$

Here the elements γ^r are the images of the universal generators of $KU(BU(n))$ under the map $KU(BU(n)) \to KU(B(n,j))$, $j = 2k$, $2k+1$.

PROOF. Let T, T', T'' be maximal tori of the Lie groups $SO(2n)$, $Spin(2n)$, $Spin^c(2n)$ respectively. Via the canonical inclusion $U(n) \subset SO(2n)$, T is also a maximal torus of $U(n)$. Following [7], we know that

$$RU(T') \cong RU(T)[u]/(u^2 = \alpha_1 \cdot \alpha_2 \ldots \alpha_n)$$

where the α_j are the 1-dimensional canonical irreducible representations of T and u is an irreducible representation of T' mapping ε to $-1 \in U(1)$. With this description of $RU(T')$ and identifying $RU(Spin(2n))$ with its image in $RU(T')$, we can write

$$\Delta_{2n}^{+} + \Delta_{2n}^{-} = u^{-1}\prod_{r=1}^{n}(\alpha_r + 1)$$

and

$$\tilde{\Delta}_{2n}^{+} + \tilde{\Delta}_{2n}^{-} = (\Delta_{2n}^{+} + \Delta_{2n}^{-}) \otimes \rho = u^{-1} \prod_{r=1}^{n} (\alpha_r + 1) \otimes \rho = \prod_{r=1}^{n} (\alpha_r + 1) u^{-1} \otimes \rho$$

in $RU(Spin(2n)) \otimes RU(U(1)) \subset RU(T') \otimes RU(U(1))$.

Both elements $\tilde{\Delta}_{2n}^{+} + \tilde{\Delta}_{2n}^{-}$ and $u^{-1} \otimes \rho$ belong to $RU(T'') \subset RU(T') \otimes RU(U(1))$ and the image of the element $\tilde{\Delta}_{2n}^{+} + \tilde{\Delta}_{2n}^{-}$ in $RU(U(n))$ shall be determined, if we know the image of $u^{-1} \otimes \rho$. Invoking the explicit description of the map $U(n) \to Spin^c(2n)$ given in [4], we see that the image of $u^{-1} \otimes \rho$ in $RU(T)$ is the trivial representation and hence

$$\tilde{f}_{2n}^{*}(\tilde{\Delta}_{2n}^{+} + \tilde{\Delta}_{2n}^{-}) = \prod_{r=1}^{n} (\alpha_r + 1) = \prod_{r=1}^{n} (\alpha_r - 1 + 2) = \sum_{r=0}^{n} 2^{n-r} \gamma^r.$$

The image of $\Delta_{2n}^{+} + \Delta_{2n}^{-}$ in $RU(Spin(2k))$ is equal to $2^{n-k}(\Delta_{2k}^{+} + \Delta_{2k}^{-})$. By homotopy commutativity of the diagram (3.2), the element $\tilde{f}_{2k}^{*}(\Delta_{2k}^{+} + \Delta_{2k}^{-})$ of $KU(B(n, 2k))$ satisfies the following relation

$$2^{n-k} \tilde{f}_{2k}^{*}(\tilde{\Delta}_{2k}^{+} + \tilde{\Delta}_{2k}^{-}) = \sum_{r=0}^{n-1} 2^{n-r} \gamma^r,$$

where γ^r denotes the image of the r-th universal class under the map ρ^*. Consequently, the element

$$\beta_{k+1} = \tilde{f}_{2k}^{*}(\tilde{\Delta}_{2k}^{+} + \tilde{\Delta}_{2k}^{-}) - \sum_{r=0}^{k} 2^{k-r} \gamma^r \tag{3.3}$$

satisfies

$$2^{n-k} \beta_{k+1} = \sum_{r=k+1}^{n-1} 2^{n-r} \gamma^r.$$

So we have proved part iii) of Proposition 3.1.

We know that the Euler class in KU-theory of the sphere fibration

$$S^{2k-1} \to BSpin^c(2k-1) \to BSpin^c(2k)$$

is the element $\tilde{\Delta}_{2k}^{+} - \tilde{\Delta}_{2k}^{-}$ (see [4]). We denote by ε_k the image of this class in $KU(B(n, 2k))$, (i.e. the Euler class of the induced fibration under the map $\tilde{f}_{2k}$) and we can write:

$$\tilde{f}_{2k}^{*}(2\tilde{\Delta}_{2k}^{-}) = \sum_{r=0}^{k} 2^{k-r} \gamma^r + \beta_{k+1} - \varepsilon_k = \sum_{r=0}^{k-1} 2^{k-r} \gamma^r + \gamma^k + \beta_{k+1} - \varepsilon_k.$$

We set

$$\alpha_k = \tilde{f}_{2k}^*(\widetilde{\Delta_{2k}}) - \sum_{r=0}^{k-1} 2^{k-r-1}\gamma^r$$

satifying relation i) of Proposition 3.1; furthermore $\varepsilon_k = \gamma^k + \beta_{k+1} - 2\alpha_k$.

Relation ii) is a straightforward consequence of relations i) and (3.3).

To prove part (b) of the proposition we consider the canonical map $B(n, 2k+1) \xrightarrow{p_0} B(n, 2k+2)$. In KU-theory the homomorphism p_0^* maps the Euler class ε_{k+1} to zero. We set $\beta'_{k+1} = p_0^*(\alpha_{k+1})$ and calculate

$$\begin{aligned} 2^{n-k}\beta'_{k+1} &= 2^{n-k-1}p_0^*(2\alpha_{k+1}) = 2^{n-k-1}p_0^*(\gamma^{k+1} + \beta_{k+2}) \\ &= p_0^*(2^{n-k-1}\gamma^{k+1} + 2^{n-k-1}\beta_{k+2}) \end{aligned}$$

Relation iii) for the case $B(n, 2k+2)$ implies (b). □

The generator β_{k+1} may be defined in another way, with the help of Thom and Bott isomorphisms (see [12]).

Now we can see Theorem 2.1 as a consequence of the above Proposition. Let $f : X \to BO(2n)$ be a classifying map of $x = ry$ in $\widetilde{KO}(X)$, where r and y are as in section 2. The map f lifts to $BU(n)$ and we denote g the classifying map of y. If $gdim(x) = k$, f lifts to $BSO(k)$ and there is a map $\tilde{f} : X \to B(n, k)$ such that the following diagram is commutative

$$\begin{tikzcd} & B(n,k) \arrow[r, "f_k"] \arrow[d, "p"] & BSO(k) \arrow[d, "q"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "g"] & BU(n) \arrow[r, "r_n"] & BSO(2n). \end{tikzcd}$$

If k is even, we apply $\tilde{f}^*$ to the relation iii) of Proposition 3.1. We obtain in $KU(X)$, with $j = \frac{k}{2} = \left[\frac{k}{2}\right]$, and identifying $\gamma^r \in KU(BU(n))$ with its image in $KU(B(n, k))$:

$$\begin{aligned} 2^{n-1}\gamma_{1/2}(y) &= \sum_{r=0}^{n-1} 2^{n-r-1}\gamma^r(y) \\ &= \sum_{r=0}^{n-1} 2^{n-r-1}\tilde{f}^*(\gamma^r) \\ &= \tilde{f}^*\left(\sum_{r=0}^{j} 2^{n-r-1}\gamma^r + \sum_{r=j+1}^{n-1} 2^{n-r-1}\gamma^r\right) \\ &= 2^{n-j-1}\tilde{f}^*\left(\sum_{r=0}^{j} 2^{j-r}\gamma^r + \beta_{j+1}\right) \\ &\equiv 0 \pmod{2^{n-j-1}}. \end{aligned}$$

If k is odd, say $k = 2j + 1$, we proceed as before invoking (b) of Proposition 3.1. $\square$

4. Proof of Theorem 1.2

By a well known theorem of H. Hopf, the span of the complex projective spaces $\mathbf{C}P^n$ and their products is zero, since the Euler characteristic of these manifolds is non-zero. But, to study the lens space case, it will be convenient to invoke the following facts on $\mathbf{C}P^n$ (see for example [18]). The complex K-theory of the complex projective space $\mathbf{C}P^n$ is given by

$$KU^q(\mathbf{C}P^n) \cong \begin{cases} \mathbf{Z}[\mu]/(\mu^{n+1}) & \text{if } q = 0 \\ 0 & \text{if } q = 1, \end{cases}$$

where μ denotes the stable class of the canonical complex line bundle over $\mathbf{C}P^n$. Since the KU-theory of $\mathbf{C}P^n$ is torsion free, $\gamma_{1/2}(x)$ makes sense in $KU(\mathbf{C}P^n) \otimes \mathbf{Q}$. We have $\gamma_{1/2}(\mu) = 1 + \frac{1}{2}\mu$ and $\gamma_{1/2}((n+1)\mu) = \left(1 + \frac{1}{2}\mu\right)^{n+1}$. The stable class of the tangent bundle $\tau_{\mathbf{C}P^n} - 2n$ over $\mathbf{C}P^n$ may be identified with $r((n+1)\mu)$ (see [22]). It follows that the stable class of the tangent bundle of $\mathbf{C}P^{n_1} \times \mathbf{C}P^{n_2}$ corresponds to the element $(n_1 + 1)\mu_1 \otimes (n_2 + 1)\mu_2$ of $KU^0(\mathbf{C}P^{n_1}) \otimes KU^0(\mathbf{C}P^{n_2})$ and we calculate:

$$\begin{aligned} & 2^{n-1}\gamma_{1/2}((n_1+1)\mu_1) \otimes \gamma_{1/2}((n_2+1)\mu_2) = 2^{n-1}\gamma_{1/2}(\mu_1)^{n_1+1} \otimes \gamma_{1/2}(\mu_2)^{n_2+1} \\ (4.1) \qquad & = \sum_{s=0}^{n_1} \sum_{t=0}^{n_2} 2^{n-s-t-1} \binom{n_1+1}{s} \binom{n_2+1}{t} \mu_1^s \otimes \mu_2^t \end{aligned}$$

We now turn to the lens spaces. The space $L^n(2^m)$ is the quotient space $\mathbf{S}^{2n+1}/(\mathbf{Z}/2^m)$ where the action on the sphere $\mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$ of the group $\mathbf{Z}/2^m$ generated by $\zeta = exp(i\pi/2^{m-1})$ is given by:

$$\zeta^k z = (\zeta^k z_0, \zeta^k z_1, \ldots, \zeta^k z_n).$$

It is well known that the KU-theory and the integral cohomology of $L^n(2^m)$ are given by:

$$KU^q(L^n(2^m)) \cong \begin{cases} \mathbf{Z} & \text{if } q = 1 \\ \mathbf{Z}[\sigma]/\langle \sigma^{n+1}, (\sigma+1)^{2^m} \rangle & \text{if } q = 0. \end{cases}$$

$$H^q(L^n(2^m); \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & \text{if } q = 0, 2n+1 \\ \mathbf{Z}/2^m & \text{if } q \text{ even}, 0 < q \le 2n \\ 0 & \text{otherwise.} \end{cases}$$

Here $\sigma = \pi^*(\mu)$, where $\pi : L^n(2^m) \to \mathbf{C}P^n$ is the canonical map. The group $H^{2r}(L^n(2^m); \mathbf{Z}) \cong \mathbf{Z}/2^m$ is generated by z^r where z is the first Chern class of σ. For a complete description, the reader is referred to [18].

Recall that the stable class $\tau_{L^n(2^m)} - \underline{2n+1}$ of the tangent bundle of $L^n(2^m)$ may be identified with $r((n+1)\sigma)$ (see [22]), and that the stable class of the tangent bundle of $L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})$, is the element $\tau_0 = r(p_1^*((n_1+1)\sigma_1) + p_2^*((n_2+1)\sigma_2))$. The latter element is the pull back of the stable tangent bundle of $\mathbf{C}P^{n_1} \times \mathbf{C}P^{n_2}$ with respect to the projection

$$L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2}) \to \mathbf{C}P^{n_1} \times \mathbf{C}P^{n_2}.$$

Now we want to find a lower bound for $gdim(\tau_0)$. We proceed in two steps. First we apply the cohomology criterion of theorem 2.2. This criterion gives us a first bound for $gdim(\tau_0)$ (see Prop. 4.2). Next we use this bound and Theorem 2.1 to prove Theorem 1.2. We start with some technical lemmas.

LEMMA 4.1. *Let $g : L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2}) \to BU(n)$ be the classifying map of $p_1^*((n_1+1)\sigma_1) + p_2^*((n_2+1)\sigma_2)$. Then for $l = 1, 2, \dots, n$, we have*

$$g^*(c_l) = \sum_{i=\max(0,l-n_2)}^{\min(l,n_1)} \binom{n_1+1}{i}\binom{n_2+1}{l-i} z_1^i \otimes z_2^{l-i},$$

where g^* is the map induced by g in integral cohomology, c_l is the l-th universal Chern class, and $z_i = c_1(\sigma_i) \in H^2(L^{n_i}(2^{m_i}); \mathbf{Z}) \cong \mathbf{Z}/2^{m_i}$, for $i = 1, 2$, and $n \geq n_1 + n_2 + 2$.

PROOF.

$$\begin{aligned}
g^*(c_l) &= c_l(p_1^*((n_1+1)\sigma_1) + p_2^*((n_2+1)\sigma_2)) \\
&= \sum_{i=0}^{l} c_i(p_1^*((n_1+1)\sigma_1))\, c_{l-i}(p_2^*((n_2+1)\sigma_2)) \\
&= \sum_{i=0}^{l} p_1^*(c_i((n_1+1)\sigma_1))\, p_2^*(c_{l-i}((n_2+1)\sigma_2)) \\
&= \sum_{i=0}^{l} \binom{n_1+1}{i} p_1^*(c_1(\sigma_1)^i) \binom{n_2+1}{l-i} p_2^*(c_1(\sigma_2)^{l-i}) \\
&= \sum_{i=0}^{l} \binom{n_1+1}{i}\binom{n_2+1}{l-i} z_1^i \otimes z_2^{l-i}
\end{aligned}$$

We know that $z_1^i = 0$ for $i \geq n_1 + 1$ and that $z_2^{l-i} = 0$ for $l - i \geq n_2 + 1$. This achieves the proof. $\square$

LEMMA 4.2. *Let $n+1 = 2^s(2u+1)$ and $s \geq 1$ be integers. The following congruences are satisfied,*

$$\binom{n+1}{i} \equiv \begin{cases} 1 \pmod 2 & \text{if } i = n+1-2^s \\ 0 \pmod 2 & \text{if } n+2-2^s \leq i \leq n. \end{cases}$$

Notice, if n is even, then $\binom{n+1}{n}$ is odd.

PROOF. Recall that $v_2\left(\binom{n}{k}\right) = \alpha(k) + \alpha(n-k) - \alpha(n)$ where $\alpha(n)$ is the number of 1 in the dyadic expansion of n. Then, we have

$$\begin{aligned} v_2\left(\binom{n+1}{n+1-2^s}\right) = v_2\left(\binom{n+1}{2^s}\right) &= \alpha(2^s) + \alpha(n+1-2^s) - \alpha(n+1) \\ &= 1 + \alpha(2^{s+1}u) - \alpha(2^s(2u+1)) \\ &= 1 + \alpha(u) - \alpha(2u+1) = 1 + \alpha(u) - \alpha(u) - 1 = 0. \end{aligned}$$

Moreover, as $\binom{n+1}{i} = \binom{n+1}{n+1-i}$, we can reduce the case $i \geq n+2-2^s$ to the case $i \leq 2^s - 1$.

Let us give the dyadic expansion of $n+1$ and i,

$$\begin{aligned} n+1 = 2^s(2u+1) = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_t}, \quad &\text{with } s_1 > s_2 > \cdots > s_t = s, \\ i = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_r}, \quad &\text{with } s-1 \geq q_1 > q_2 > \cdots > q_r. \end{aligned}$$

It is easy to see that

$$n+1-i = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_{t-1}} + \sum_{j=q_r}^{s-1} 2^j - \sum_{v=1}^{r-1} 2^{q_v}.$$

We observe that $\alpha(n+1) = t$ and $\alpha(i) = r$, then we can write

$$\alpha(n+1-i) = t + s - r - q_r = \alpha(n+1) - \alpha(i) + s - q_r > \alpha(n+1) - \alpha(i). \qquad \square$$

LEMMA 4.3. *Consider the integers $n_i + 1 = 2^{s_i}(2u_i+1)$ with $u_i \geq 1$ $(i = 1,2)$, and $l = n_1 + n_2 + 2 - 2^{s_1} - 2^{s_2}$. We have $g^*(c_l) \not\equiv 0 \pmod 2$.*

PROOF. According to Lemma 4.1, we have

$$g^*(c_l) = \sum_{i=\max(0,l-n_2)}^{\min(l,n_1)} \binom{n_1+1}{i}\binom{n_2+1}{l-i} z_1^i \otimes z_2^{l-i}.$$

Using Lemma 4.2, we see that $\binom{n_1+1}{i}$ is even if

$$n_1 + 1 - 2^{s_1} < i \leq \min(l, n_1) \leq n_1,$$

we also see that $\binom{n_2+1}{l-i}$ is even if

$$l - n_2 \leq \max(0, l - n_2) \leq i < n_1 + 1 - 2^{s_1},$$

since in this last case $n_2 + 1 - 2^{s_2} < n_1 + n_2 + 2 - 2^{s_1} - 2^{s_2} - i = l - i \leq n_2$.

Finally $\binom{n_1+1}{i}\binom{n_2+2}{l-i}$ is odd if $i = n_1 + 1 - 2^{s_1}$, since $l - i = n_2 + 1 - 2^{s_1}$. So,

we have established

$$g^*(c_l) \equiv \binom{n_1+1}{2^{s_1}}\binom{n_2+1}{2^{s_2}} z_1^{n_1+1-2^{s_1}} \otimes z_2^{n_2+1-2^{s_2}} \not\equiv 0 \pmod 2. \qquad \square$$

LEMMA 4.4. *Consider the integer* $n + 1 = 2^s(2u + 1)$. *We have*

$$gdim(\tau_{L^n(2^m)} - \underline{2n+1}) \geq 2n + 2 - 2^{s+1}.$$

PROOF. We know that $\tau_{L^n(2^m)} - \underline{2n+1} = r((n+1)\sigma)$. Moreover, if $g : L^n(2^m) \to BU$ denotes the classifying map of the stable bundle $(n+1)\sigma$,

$$g^*(c_l) = c_l((n+1)\sigma) = \binom{n+1}{l} c_1(\sigma)^l.$$

Assume that $gdim(\tau_{L^n(2^m)} - \underline{2n+1}) = 2n + 1 - 2^{s+1}$. Then according to Theorem 2.2

$$g^*(c_l) \equiv 0 \pmod 2 \quad \text{for } l \geq n + 1 - 2^s,$$

which is inconsistent with Lemma 4.2. $\square$

LEMMA 4.5 *If* $n = 2^s - 1$ *and* $m \geq [log_2 n] + 1$, *then*

$$gdim(\tau_{L^n(2^m)} - \underline{2n+1}) \geq \left[\frac{n}{2}\right].$$

PROOF. According to [24] we have $gdim(\tau_{L^n(2^m)} - \underline{2n+1}) \geq r_2(n, m)$ where

$$r_2(n, m) = \max\left\{0 \leq r \leq \left[\frac{n}{2}\right] \,\middle|\, \nu_2\left(\binom{n+1}{r}\right) < m + n - 2r\right\}.$$

In our case $\nu_2\left(\binom{n+1}{r}\right) = \nu_2\left(\binom{2^s}{r}\right) = s - \nu_2(r)$.

In particular if $r = \left[\frac{n}{2}\right] = 2^{s-1} - 1$,

$$s - \nu_2(r) = s = [log_2 n] + 1 \le m < m + 1 = m + n - 2r. \qquad \square$$

PROPOSITION 4.1. *Let* $n_i + 1 = 2^{s_i}(2u_i + 1)$ *be an integer with* $u_i \ge 1$ $(i = 1, 2)$. *Then*

$$gdim(\tau_0) \ge 2n_1 + 2n_2 + 4 - 2^{s_1+1} - 2^{s_2+1}.$$

PROOF. Assume that $gdim(\tau_0) = 2n_1 + 2n_2 + 3 - 2^{s_1+1} - 2^{s_2+1}$. Then, according to Theorem 2.2, we should have $g^*(c_l) \equiv 0 \pmod 2$ for all

$l = n_1 + n_2 + 2 - 2^{s_1} - 2^{s_2}, \ldots, n_1 + n_2 + 1$, which is inconsistent with the result of Lemma 4.3. $\square$

PROPOSITION 4.2 a) *Consider the integers* $n_1 + 1 = 2^{s_1}(2u_1 + 1)$ *with* $u_1 \ge 1$, $n_2 + 1 = 2^{s_2}$ *and* $m_2 \ge [log_2 n_2] + 1$. *Then we have*

$$gdim(\tau_0) \ge \max\left(2n_1 + 2 - 2^{s_1+1}, \left[\frac{n_2}{2}\right]\right).$$

b) *Consider the integers* $n_i + 1 = 2^{s_i}$ *and* $m_i \ge [log_2 n_i] + 1$, $(i = 1, 2)$. *Then we have*

$$gdim(\tau_0) \ge \max\left(\left[\frac{n_1}{2}\right], \left[\frac{n_2}{2}\right]\right).$$

PROOF. By Proposition 2.1

$$gdim(\tau_0) \ge \max(gdim(\tau_0(1)), gdim(\tau_0(2))),$$

where $\tau_0(i) = \tau_{L^{n_i}(2^{m_i})} - \underline{2n_i + 1}$.

Moreover, according to Lemmas 4.4 and 4.5 we can assert that, under the hypothesis of a),

$$gdim(\tau_0(1)) \ge 2n_1 + 2 - 2^{s_1+1} \quad \text{and} \quad gdim(\tau_0(2)) \ge \left[\frac{n_2}{2}\right]$$

and under the hypothesis of b),

$$gdim(\tau_0(1)) \ge \left[\frac{n_1}{2}\right] \quad \text{and} \quad gdim(\tau_0(2)) \ge \left[\frac{n_2}{2}\right]. \qquad \square$$

Now, we apply the criterion of Theorem 1.1 to the stable classes $y_i = (n_i + 1)\sigma_i$, $i = 1, 2$. If $Span(L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})) = 2(n_1 + n_2 + 1) - k$, the following relation is satisfied in $KU(L^{n_1}(2^{m_1})) \otimes KU(L^{n_2}(2^{m_2}))$:

$$2^{n-1}\gamma_{1/2}((n_1 + 1)\sigma_1) \otimes (n_2 + 1)\sigma_2) \equiv 0 \pmod{2^{n-j-1}}$$

with $n \ge n_1 + n_2 + 2$ and $j = \left[\frac{k}{2}\right]$.

The left hand side of this congruence is the image of the left hand side of (4.1) under the canonical projection $L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2}) \to \mathbf{C}P^{n_1} \times \mathbf{C}P^{n_2}$ and (4.1) implies

$$(4.2) \qquad \sum_{i=0}^{n_1} \sum_{l=0}^{n_2} 2^{n-i-l-1} \binom{n_1+1}{i} \binom{n_2+1}{l} \sigma_1^i \otimes \sigma_2^l \equiv 0 \pmod{2^{n-j-1}}.$$

We shall consider the projection

$$\pi_1 \otimes \pi_2 : \mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2] \to KU(L^{n_1}(2^{m_1})) \otimes KU(L^{n_2}(2^{m_2})).$$

The relation (4.2) lifts to $\mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2]$ modulo $ker(\pi_1 \otimes \pi_2)$, that is to say modulo the ideal of $\mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2]$ generated by

$$\sigma_1^{n_1+1} \otimes 1, 1 \otimes \sigma_2^{n_2+1}, ((1+\sigma_1)^{2^{m_1}} - 1) \otimes 1 \text{ and } 1 \otimes ((1+\sigma_2)^{2^{m_2}} - 1).$$

We obtain in $\mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2]$:

$$\begin{aligned}(4.3) \qquad & \sum_{i=0}^{n_1} \sum_{l=0}^{n_2} 2^{n-i-l-1} \binom{n_1+1}{i} \binom{n_2+1}{l} \sigma_1^i \otimes \sigma_2^l \\ & = 2^{n-j-1} \sum_{i=0}^{n_1} \sum_{l=0}^{n_2} a_{il} \sigma_1^i \otimes \sigma_2^l + ((1+\sigma_1)^{2^{m_1}} - 1) p_1(\sigma_1, \sigma_2) \\ & + ((1+\sigma_2)^{2^{m_2}} - 1) p_2(\sigma_1, \sigma_2)\end{aligned}$$

where $p_1(\sigma_1, \sigma_2)$, $p_2(\sigma_1, \sigma_2)$ are certain polynomials and the coefficients a_{il} are integers.

We need the following result to conclude.

LEMMA 4.6. *If $m \geq [log_2(n)]$, then*

$$(x+1)^{2^m} - 1 = 2^{m-[log_2 n]} p(x) + x^{n+1} q(x)$$

where $p(x)$, $q(x)$ are polynomials in the indeterminate x and $deg(p(x)) \leq n$.

PROOF. We have

$$(x+1)^{2^m} - 1 = \sum_{i=1}^{2^m} \binom{2^m}{i} x^i$$

and since

$$\nu_2\left(\binom{2^m}{i}\right) = m - \nu_2(i) \geq m - [log_2 n],$$

$i = 1, 2, \ldots, n$, the lemma follows. $\square$

We shall now assume that $m_i \geq [log_2 n_i] + 1 (i = 1, 2)$, and we set $n = n_1 + n_2 + 2$. Using Propositions 4.1 and 4.2, we obtain for $j = \left[\frac{k}{2}\right]$ and $n - j - 1$:

1) If $n_i + 1 = 2^{s_i}(2u_i + 1)$, $u_i \geq 1$ $(i = 1, 2)$, we have

$$j \geq n_1 + n_2 + 2 - 2^{s_1} - 2^{s_2}, \quad n - j - 1 \leq 2^{s_1} + 2^{s_2} - 1.$$

2) If $n_1 + 1 = 2^{s_1}(2u_1 + 1)$, $u_1 \geq 1$, $n_2 + 1 = 2^{s_2}$, we have

$$j \geq \max\left(n_1 + 1 - 2^{s_1}, \left[\frac{n_2}{4}\right]\right), \quad n - j - 1 \leq \min\left(n_2 + 2^{s_1}, n_1 + \left[\frac{3(n_2 + 2)}{4}\right]\right).$$

3) If $n_i + 1 = 2^{s_i}$, $(i = 1, 2)$, we have

$$j \geq \max\left(\left[\frac{n_1}{4}\right], \left[\frac{n_2}{4}\right]\right), \quad n - j - 1 \leq \min\left(n_2 + \left[\frac{3(n_1 + 2)}{4}\right], n_1 + \left[\frac{3(n_2 + 2)}{4}\right]\right).$$

Under the above hypothesis the relation (4.3) becomes in $\mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2]$:

$$\sum_{i=0}^{n_1} \sum_{l=0}^{n_2} 2^{n-i-l-1} \binom{n_1 + 1}{i} \binom{n_2 + 1}{l} \sigma_1^i \otimes \sigma_2^l \equiv 0 \ (\mathrm{mod}\, 2^{n-j-1}). \tag{4.4}$$

As the generators $\sigma_1^i \otimes \sigma_2^l$ are free in $\mathbf{Z}[\sigma_1] \otimes \mathbf{Z}[\sigma_2]$, (4.4) induces the congruence relations:

$$2^{n-i-l-1} \binom{n_1 + 1}{i} \binom{n_2 + 1}{l} \equiv 0 \ (\mathrm{mod}\, 2^{n-j-1}), \tag{4.5}$$

for $0 \leq i \leq n_1$ and $0 \leq l \leq n_2$. In particular, if $i = n_1$ and $l = n_2$ in (4.5), one gets

$$2^{n-i-l-1}(n_1 + 1)(n_2 + 1) \equiv 0 \ (\mathrm{mod}\, 2^{n-j-1}).$$

In other words, we have:

$$n - n_1 - n_2 - 1 + v_2(n_1 + 1) + v_2(n_2 + 1) \geq n - j - 1$$

so

$$j \geq n_1 + n_2 - v_2(n_1 + 1) - v_2(n_2 + 1).$$

If one of the following three conditions is satisfied

1) $n_i + 1 = 2^{s_i}(2u_i + 1)$, $u_i \geq 1$ and $m_i \geq [log_2 n_i] + 2^{s_1} + 2^{s_2} (i = 1, 2)$

2) $n_1 + 1 = 2^{s_1}(2u_1 + 1)$, $u_1 \geq 1$, $n_2 + 1 = 2^{s_2}$ and $m_i \geq [log_2 n_i] + \min\left(n_2 + 2^{s_1}, n_1 + \left[\frac{3(n_2 + 2)}{4}\right]\right)$ $(i = 1, 2)$

3) $n_i + 1 = 2^{s_i}$ and $m_i \geq [log_2 n_i] + \min\left(n_2 + \left[\frac{3(n_1+2)}{4}\right], n_1 + \left[\frac{3(n_2+2)}{4}\right]\right)$ $(i = 1, 2)$, then

$$Span(L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})) \leq 2\nu_2(n_1 + 1) + 2\nu_2(n_2 + 1) + 2.$$

Using Proposition 2.2 and Theorem 1.1 of [6], we observe that

$$\begin{aligned} Span(L^{n_1}(2^{m_1}) \times L^{n_2}(2^{m_2})) &\geq Span(L^{n_1}(2^{m_1})) + Span(L^{n_2}(2^{m_2})) \\ &= 2\nu_2(n_1 + 1) + 2\nu_2(n_2 + 1) + 2. \end{aligned}$$

This achieves the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

Let $\mathbf{H}$ be the field of quaternions and let m be a positive integer. Let Q_m be the group of order 2^{m+1}, generated by x and y such that $x^{2^{m-1}} = y^2$ and $xyx = y$. We can see Q_m as a subgroup of $\mathbf{S}^3 \subset \mathbf{H}$, taking $x = exp(i\pi/2^{m-1})$ and $y = j$. Here quaternions are represented by $z_1 + jz_2$ with z_1, $z_2 \in \mathbf{C}$. *We call Q_m—spherical space form, or quaternionic spherical space form,* the quotient manifold $N^n(m) = \mathbf{S}^{4n+3}/Q_m$, where the action of the group Q_m on $\mathbf{S}^{4n+3} \subset \mathbf{H}^{n+1}$ is given by:

$$q \cdot (x_0, x_1, \ldots, x_n) = (qx_0, qx_1, \ldots, qx_n).$$

We recall that to any group representation of Q_m corresponds a vector bundle over $N^n(m)$. We denote by α_0, α_1 and δ_1 the stable classes of the bundles corresponding to the complex representations a_0, a_1 and ζ defined by:

$$a_0(x) = 1, \quad a_0(y) = -1$$
$$a_1(x) = -1, \quad a_1(y) = -1$$
$$\zeta(z_1 + jz_2) = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}.$$

Notice that the representation ζ is nothing else than the representation induced by the canonical representation of $\mathbf{S}^3 \subset \mathbf{H}$ in $U(2)$. The latter representation defines a canonical 2-dimensional complex vector bundle ρ over the quaternionic projective space $\mathbf{H}P^n = \mathbf{S}^{4n+3}/\mathbf{S}^3$. Its stable class $z = \rho - 2$ is mapped on to δ_1 by the homomorphism induced by the projection

$$\mathbf{S}^{4n+3}/Q_m = N^n(m) \xrightarrow{\pi} \mathbf{S}^{4n+3}/\mathbf{S}^3 = \mathbf{H}P^n \subset \mathbf{H}P^\infty,$$

$$\delta_1 = \pi^*(z) \in \widetilde{KU}(N^n(m)) \tag{5.0}$$

According to [22] we can identify the stable class of $\tau_{N^n(m)}$ in $\widetilde{KO}(N^n(m))$ with $r((n+1)\delta_1)$.

Consider the elements $\beta(s)$ in $KU(N^n(m))$ inductively defined by the formulas

$$\begin{cases} \beta(0) = \delta_1 \\ \beta(s) = \beta(s-1)^2 + 4\beta(s-1) & \text{for } s \geq 1. \end{cases}$$

For all integer $s \geq 1$, let $a'(s)$ and $b'(s)$ be the integers such that $0 \leq b'(s) < 2^s$ and

$$2^s a'(s) + b'(s) = \begin{cases} 2n+1 & \text{if } n \text{ odd} \\ 2n & \text{if } n \text{ even.} \end{cases}$$

and for all integer $i = 2^s + d$ such that $0 \leq d < 2^s$ and $0 \leq s < m$, let

$$a(i) = \begin{cases} a'(s+1)+1 & \text{if } 2d \leq b'(s+1) \\ a'(s+1) & \text{if } 2d > b'(s+1) \end{cases}$$

$$u(i) = \begin{cases} 2^{m-1+a'(1)} & \text{if } i = 1 \\ 2^{m-s-2+a'(s)} & \text{if } i = 2^s > 1 \\ 2^{m-s-3+a(i)} & \text{if } i = 2^s + d \geq 3, \quad 0 < d < 2^s. \end{cases}$$

Now we can give the additive structure of $KU^*(N^n(m))$. The result is due to K. Fujii and M. Sugawara in [10] and we will adopt their notation in what follows. As abelian groups there are isomorphisms:

$$KU^1(N^n(m)) \cong \mathbf{Z}$$

and

$$\widetilde{KU}^0(N^n(m)) \cong \mathbf{Z}/2^{n+1} \cdot \langle \alpha_0 \rangle \oplus \mathbf{Z}/2^{n+1} \cdot \langle \overline{\alpha_1} \rangle \oplus \sum_{i=1}^{M} \mathbf{Z}/u(i) \cdot \langle \delta_i \rangle, \tag{5.1}$$

where $M = \min(2^{m-1}, n)$. Here $\mathbf{Z}/t \cdot \langle x \rangle$ denotes the cyclic group of order t generated by x.

The generators $\overline{\alpha_1}$ and δ_i are defined by

$$\overline{\alpha_1} = \alpha_1 - 2\sum_{s=1}^{m-3} \beta(s) \prod_{t=s+1}^{m-3} (2 + \beta(t)).$$

$$\delta_i = \beta(s) + \sum_{j=1}^{s} 2^{(2^j-1)(a'(s)+1)} \beta(s-j), \quad i = 2^s, \ 1 \leq s \leq m-1, \tag{5.2}$$

$$\delta_i = \delta_1^{d-1}\beta(1)\prod_{j=0}^{s-1}(2+\beta(j)) - 2^{a(i)-1}\delta_1^d\beta(s) + \sum_{j=2}^{s+1} 2^{(2^j-1)a(i)-1}\delta_1^d\beta(s+1-j), \tag{5.3}$$
$$i = 2^s + d, 1 \leq s \leq m-1, 0 < d < 2^s.$$

We need some complementary technical results.

LEMMA 5.1. *For all* $1 \leq s \leq m-1$, *we have*

$$\beta(s) = \delta_1^{2^s} + q_s(\delta_1)$$

where $q_s(\delta_1)$ *is a polynomial of degree* $2^s - 1$ *with even integer coefficients.*

PROOF. It is easy to see that the assertion is valid for $s = 0$ and $s = 1$. Moreover, if it is true for $s \geq 1$, the recurrence relation

$\beta(s+1) = \beta(s)^2 + 4\beta(s)$ implies that it is true for $s+1$. □

LEMMA 5.2. *For all* $i = 2^s + d \leq n$ *with* $0 \leq d \leq 2^s$ *and* $0 \leq s \leq m-1$, *the integer* $a(i)$ *satisfies the condition* $a(i) \geq 2$.

PROOF. Recall that

$$2^{s+1}a'(s+1) + b'(s+1) = \begin{cases} 2n+1 & \text{if } n \text{ odd} \\ 2n & \text{if } n \text{ even.} \end{cases}$$

For the two cases we have $a'(s+1) \geq 1$, since $b'(s+1) < 2^{s+1} \leq 2n$. Then, if $2d \leq b'(s+1)$, by definition $a(i) = a'(s+1) + 1 \geq 2$. If $2d > b'(s+1)$, we also have $a(i) = a'(s+1) \geq 2$, since $a'(s+1) = 1$ would imply

$$\begin{aligned} 2n &\leq 2^{s+1}a'(s+1) + b'(s+1) \\ &< 2^{s+1} + 2d \\ &\leq 2n \end{aligned}$$

which is impossible. □

LEMMA 5.3. *Let* $u(n)$ *be as above. Then*

$$\nu_2(u(n)) \geq m - [log_2 n] - 1.$$

PROOF. For $n = 2^s > 1$, $u(n)$ is given by $2^{m-s-2+a'(s)} = 2^{m-log_2 n}$,

and for $n = 2^s + d \geq 3$ with $0 < d < 2^s$, $u(n)$ is given by $2^{m-s-3+a(n)} = 2^{m-[log_2 n]-1}$. □

LEMMA 5.4. *For all* $1 \leq i \leq M$, *there is an odd integer* A_i *and a polynomial* $p_i(\delta_1)$ *of degree* $i-1$ *with even integer coefficients such that*

$$\delta_i = A_i\delta_1^i + p_i(\delta_1).$$

PROOF. By definition, the result is true for $i = 1$. If $i = 2^s$, we replace in (5.2) the elements $\beta(s)$ and $\beta(s-j)$ by the expression given in Lemma 5.1. Then δ_i becomes

$$\delta_i = \delta_1^{2^s} + q_s(\delta_1) + \sum_{j=1}^{s} 2^{(2^j-1)(a'(s)+1)}(\delta_1^{2^{s-j}} + q_{s-j}(\delta_1)).$$

If $i = 2^s + d$ with $0 < d < 2^s$, we do the same with the relation (5.3) and obtain

$$\delta_i = \delta_1^{d-1}(\delta_1^2 + 4\delta_1)\prod_{j=0}^{s-1}(2 + \delta_1^{2^j} + q_j(\delta_1)) - 2^{a(i)-1}\delta_1^d(\delta_1^{2^s} + q_s(\delta_1))$$
$$+ \sum_{j=2}^{s+1} 2^{(2^j-1)a(i)-1}\delta_1^d(\delta_1^{2^{s+1-j}} + q_{s+1-j}(\delta_1))$$

and hence $\delta_i = (1 - 2^{a(i)-1})\delta_1^{d+2^s} + p(\delta_1)$, where $p(\delta_1)$ is a polynomial in δ_1 of degree $< i$ with even integer coefficients. We conclude with Lemma 5.2. □

It follows from (5.1) and Lemma 5.4 by induction on i that the elements $\delta_1, \delta_2, \dots, \delta_i$ and $\delta_1, \delta_1^2, \dots, \delta_1^i$ generate the same subgroup of $\widetilde{KU}^0(N^n(m))$; (all groups under consideration have order a power of 2).

Invoking (5.1) again and assuming that $2^{m-1} \geq n$, we set

$$\widetilde{KU}^0(N^n(m)) \cong G \oplus \mathbf{Z}/u(n) \cdot \langle\delta_n\rangle,$$

where G is the subgroup generated by α_0, $\overline{\alpha_1}$, δ_1, $\delta_1^2, \dots, \delta_1^{n-1}$ and we get for the projection $\rho\colon \widetilde{KU}^0(N^n(m)) \to \mathbf{Z}/u(n) \cdot \langle\delta_n\rangle$:

$$\rho(\delta_1^i) = \begin{cases} 0 & \text{if } i = 1, \dots, n-1 \\ A \cdot \delta_n & \text{if } i = n, \text{where } A \text{ is an odd integer.} \end{cases}$$

Now consider the stable class τ_0 of the tangent bundle of $N^n(m)$. According to [18] and [22] and by (5.0) we have

$$\tau_0 = r(n+1)\delta_1 = r\pi^*((n+1)z), \quad z \in KU(\mathbf{H}P^n)$$

The γ-operations on the element $z \in \widetilde{KU}(\mathbf{H}P^n)$ are given by $\gamma_t(z) = 1 + zt(1-t)$ (see [18]). It follows that

$$\gamma_{1/2}((n+1)z) = \left(1 + \frac{z}{4}\right)^{n+1} \in KU(\mathbf{H}P^n) \otimes \mathbf{Q} = \mathbf{Q}[z]/(z^{n+1})$$

and further in $KU(\mathbf{H}P^{n_1}) \otimes KU(\mathbf{H}P^{n_2})$

(5.4) $$2^{n-1}\gamma_{1/2}((n_1+1)z_1) \otimes \gamma_{1/2}((n_2+1)z_2)$$
$$= \sum_{i=0}^{n_1}\sum_{l=0}^{n_2}\binom{n_1+1}{i}\binom{n_2+1}{l}2^{n-2i-2l-1}z_1^i \otimes z_2^l.$$

We now apply Theorem 1.1 to the stable classes $y_i = (n_i + 1)\delta_{1,i}$, $i = 1, 2$. If $Span(M) = 4n_1 + 4n_2 + 6 - k$, then the following relation is valid in $KU(N^{n_1}(m_1)) \otimes KU(N^{n_2}(m_2))$,

$$2^{n-1}\gamma_{1/2}((n_1+1)\delta_{1,1}) \otimes \gamma_{1/2}((n_2+1)\delta_{1,2}) \equiv 0 \ (\mathrm{mod}\, 2^{n-j-1}), \tag{5.5}$$

here $n \geq 2n_1 + 2n_2 + 4$ and $j = \left[\dfrac{k}{2}\right]$.

By (5.0), the left hand side of this congruence is the image of the left hand side of (5.4) under the map

$$N^{n_1}(m_1) \times N^{n_2}(m_2) \to \mathbf{H}P^{n_1} \times \mathbf{H}P^{n_2},$$

and (5.5) implies

$$\sum_{i=0}^{n_1}\sum_{l=0}^{n_2}\binom{n_1+1}{i}\binom{n_2+1}{l}2^{n-2i-2l-1}\delta_{1,1}^i \otimes \delta_{1,2}^l \equiv 0 \ (2^{n-j-1}). \tag{5.6}$$

Under the projection

$$KU^0(N^{n_1}(m_1)) \otimes KU^0(N^{n_2}(m_2)) \to (G_1 \oplus \mathbf{Z}/u(n_1) \cdot \langle\delta_{n_1}\rangle) \otimes (G_2 \oplus \mathbf{Z}/u(n_2) \cdot \langle\delta_{n_2}\rangle)$$
$$\to \mathbf{Z}/u(n_1) \cdot \langle\delta_{n_1}\rangle \otimes \mathbf{Z}/u(n_2) \cdot \langle\delta_{n_2}\rangle \cong \mathbf{Z}/\min(u(n_1), u(n_2))$$

the relation (5.6) reduces in the latter group to

$$A \cdot (n_1+1)(n_2+1)2^3 \equiv 0 \ (\mathrm{mod}\, 2^{2n_1+2n_2+3-j}) \tag{5.7}$$

provided $2^{m_i-1} \geq n_i$ $(i = 1, 2)$.

The integer $u(n_i)$ is a power of 2 and by Lemma 5.3 we have $\nu_2(u(n_i)) \geq m_i - [log_2 n_i] - 1$. So, if the hypothesis of Theorem 1.3 is satisfied, i.e. if $m_i > [log_2 n_i] + \nu_2(n_1+1) + \nu_2(n_2+1) + 4$, $(i = 1, 2)$, then $min(\nu_2(u(n_1)), \nu_2(u(n_2))) > \nu_2(n_1+1) + \nu_2(n_2+1) + 3$ and the congruence (5.7) is satisfied in $\mathbf{Z}/\min(u(n_1), u(n_2))$ if and only if

$$j \geq 2n_1 + 2n_2 - \nu_2(n_1+1) - \nu_2(n_2+1).$$

This implies

$$\begin{aligned} Span(M) &= 4n_1 + 4n_2 + 6 - k \leq 4n_1 + 4n_2 + 6 - 2j \\ &\leq 2\nu_2(n_1+1) + 2\nu_2(n_2+1) + 6. \end{aligned}$$

and achieves the proof of Theorem 1.4. □

We notice that this result is best possible when $\nu_2(n_1+1)$ and $\nu_2(n_2+1)$ are zero modulo 4 since by Proposition 2.2 and Theorem 1.1 of [6] we have

$$\begin{aligned} Span(N^{n_1}(m_1) \times N^{n_2}(m_2)) &\geq Span(N^{n_1}(m_1)) + Span(N^{n_2}(m_2)) \\ &= 2\nu_2(n_1+1) + 2\nu_2(n_2+1) + 6. \end{aligned}$$

6. Proof of Theorem 1.4

Let M be the product $\prod_{i=1}^{r}\mathbf{S}^{m_i}\times\prod_{l=1}^{s}\mathbf{C}P^{n_l}$ and set $m=m_1+m_2+\cdots+m_r$, $n=n_1+n_2+\cdots+n_s$. If all the spheres are of even dimension, then the Euler chacarteristic of M is non-zero and $Span(M)=0$. In the following we shall suppose that one of the spheres at least is odd dimensional.

The tangent bundle of M is isomorphic to $\bigoplus_{i=1}^{r}p_i^*(\tau_{S^{m_i}})\oplus\bigoplus_{l=1}^{s}q_l^*(\tau_{\mathbf{C}P^{n_l}})$, where $p_i : M\to\mathbf{S}^{m_i}$ and $q_l : M\to\mathbf{C}P^{n_l}$ are the canonical projections. The tangent bundles of the spheres are stably trivial and the complex tangent bundle of $\mathbf{C}P^{n_l}$ is stably isomorphic to $(n_l+1)\mu_l$, where μ_l denotes the stable class of the canonical line bundle over $\mathbf{C}P^{n_l}$ (see [22]). For τ_0, the complex stable class of the tangent bundle on M, it follows that

$$\tau_0=\sum_{l=1}^{s}q_l^*((n_l+1)\mu_l).$$

As in the beginning of section 4, we have $\gamma_{1/2}(\mu_l)=1+\frac{1}{2}\mu_l$ and so $\gamma_{1/2}((n_l+1)\mu_l)=\left(1+\frac{1}{2}\mu_l\right)^{n_l+1}$. An obvious generalization of Theorem 1.1 to products of more than two factors implies: If $Span(M)=m+2n-k$ then the following relation is satisfied in $\bigotimes_{l=1}^{s}KU(\mathbf{C}P^{n_l})\subset KU(S)\otimes\bigotimes_{l=1}^{s}KU(\mathbf{C}P^{n_l})$, $(S=\mathbf{S}^{m_1}\times\mathbf{S}^{m_2}\times\cdots\times\mathbf{S}^{m_r})$,

$$2^{N-1}\bigotimes_{l=1}^{s}\left(1+\frac{1}{2}\mu_l\right)^{n_l+1}\equiv 0\ (\operatorname{mod} 2^{N-j-1}),$$

where $2N>m+2n$ and $j=\left[\frac{k}{2}\right]$. Expanding the latter relation, we get

$$\sum_{u=0}^{n}\sum_{u_1+u_2+\cdots+u_s=u}2^{N-u-1}\prod_{l=1}^{s}\binom{n_l+1}{u_l}\mu_1^{u_1}\otimes\cdots\otimes\mu_s^{u_s}\equiv 0\ (\operatorname{mod} 2^{N-j-1}).$$

Since $\mu_1^{u_1}\otimes\cdots\otimes\mu_s^{u_s}$ are free generators, we obtain, concentrating on the coefficient of $\mu_1^{n_1}\otimes\cdots\otimes\mu_s^{n_s}$,

$$2^{N-n-1}\prod_{l=1}^{s}\binom{n_l+1}{n_l}=2^{N-n-1}\prod_{l=1}^{s}(n_l+1)\equiv 0\ (\operatorname{mod} 2^{N-j-1}).$$

This implies

$$N-n-1+\sum_{l=1}^{s}\nu_2(n_l+1)\geq N-j-1$$

and further

$$2n - 2\sum_{l=1}^{s} \nu_2(n_l + 1) \leq 2j \leq k.$$

Finally,

$$Span(M) \leq m + 2n - k \leq m + 2\sum_{l=1}^{s} \nu_2(n_l + 1),$$

which finishes the proof of Theorem 1.4. □

The Dold manifold $D(u, v)$ is the quotient of the product manifold

$$\mathbf{S}^u \times \mathbf{C}P^v$$

by the $\mathbf{Z}/2$-action

$$\mathbf{S}^u \times \mathbf{C}P^v \to \mathbf{S}^u \times \mathbf{C}P^v, \quad (x, z) = (-x, \bar{z}).$$

Hence $\mathbf{S}^u \times \mathbf{C}P^v$ is a 2-fold covering of $D(u, v)$, and generally,

$$\prod_{i=1}^{r} \mathbf{S}^{u_i} \times \mathbf{C}P^{v_i}$$

is a covering manifold of the product manifold

$$\prod_{i=1}^{r} D(u_i, v_i).$$

Corollary 1.1 is therefore a direct consequence of Theorem 1.4. (If $\tilde{M} \to M$ is a covering, then obviously $Span(M) \leq Span(\tilde{M})$.)

References

[1] J. F. Adams, Vector fields on spheres, Ann. of Math. **75** (1962), 603–632.
[2] M. F. Atiyah, Immersions and embeddings of manifolds, Topology **1** (1962), 125–132.
[3] M. F. Atiyah, K-Theory, Benjamin Inc. (1967).
[4] M. F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology **3** (Supplement 1) (1964), 3–38.
[5] M. F. Atiyah, F. Hirzebruch, Vector bundles and homogeneous spaces. Am. Math. Soc. Proc. Symp. Pure Math. **3** (1961), 7–38.
[6] J. C. Becker, The span of spherical space forms, Amer. J. of Math. **94** (1972), 991–1026.
[7] R. Bott, Lectures on K(X), Mimeographed notes, Harvard University, Cambridge, Mass., (1962).
[8] M. Fujii, K_U-groups of Dold manifolds, Osaka J. Math. **3** (1966), 49–61.

[9] M. Fujii, Ring structures of K_U-cohomologies of Dold manifolds, Osaka J. Math. **6** (1969), 107–115.

[10] F. Fujii, M. Sugawara, The additive structure of $K(S^{4n+3}/Q_t)$, Hiroshima Math. J. **13** (1983), 507–521.

[11] A. Hurwitz, Über die Komposition der quadratischen Formen, Math. Ann. **88** (1923), 1–25.

[12] B. Junod, Sur l'espace classifiant des fibrés vectoriels réels de dimension k stablement complexes, C. R. Acad. Sci. Paris, t. 296, série I, (1983), 215–218.

[13] B. Junod, Quelques résultats de non-immersion des formes sphériques quaternioniques, C. R. Acad. Sci. Paris, t 313, série I, (1991), 103–106.

[14] B. Junod, Sur l'espace classifiant des fibrés vectoriels réels stablement complexes, Thesis (1987).

[15] B. Junod, Some non-immersion results for lens spaces $L^n(m)$, Math. J. Okayama Univ. Vol. **37**, (1995), 137–151.

[16] T. Kobayashi, Non-embeddability and non-immersibility of products of quaternionic spherical space forms, Mem. Fac. Sci. Kochi Uni. (Math) **13** (1992), 1–14.

[17] J. Korbaš, P. Zvengrowski, The vector field problem: a survey with emphasis on specific manifolds, Expo. Math. **12** (1994), 3–30.

[18] N. Mahammed, R. Piccinini, U. Suter, Some applications of topological K-theory, North Holland (1980).

[19] H. Ōshima, On stable homotopy types of stunted spaces, Pub. R.I.M.S. Kyoto Univ. **11** (1976), 497–521.

[20] J. Radon, Lineare Scharen orthogonaler Matrizen, Abh. Math. Sem, Univ. Hamburg **1** (1922), 1–14.

[21] M.-Y. Sohn, Span of product Dold manifolds, Kyungpook Math. J. **31** (1991), 19–24.

[22] R. H. Szczarba, On tangent bundles of fibre spaces and quotient spaces, Amer. J. of Math. **86** (1964), 685–697.

[23] J. J. Ucci, Immersions and embeddings of Dold manifolds, Topology **4** (1965), 283–293.

[24] M. Yasuo, γ-dimension and products of lens spaces, Mem. Fac. Sci. Kyushu Univ. **31** (1977), 113–126.

Institut de Mathématiques
Université de Neuchâtel
Emile-Argand 11
2000 Neuchâtel
Suisse