HIROSHIMA MATH. J.
29 (1999), 299–312

A note on the localization of J-groups

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(Received June 4, 1998)
(Revised September 15, 1998)

ABSTRACT. Let $\widetilde{JO}(X) = \widetilde{KO}(X)/TO(X)$ be the J-group of a connected finite CW complex X. Using Atiyah-Tall [5], we obtain two computable formulae of $TO(X)_{(p)}$, the localization of $TO(X)$ at a prime p. Then we show how to use those two formulae of $TO(X)_{(p)}$ to find the J-orders of elements of $\widetilde{KO}(X)$, at least the 2 and 3 primary factors of the canonical generators of $\widetilde{JO}(\mathbf{C}P^m)$. Here $\mathbf{C}P^m$ is the complex projective space.

1. Introduction

Let $\widetilde{JO}(X) = \widetilde{KO}(X)/TO(X)$ be the J-group of a connected finite CW complex X, where $\widetilde{KO}(X)$ is the additive subgroup of the KO-ring $KO(X)$ of elements of virtual dimension zero and $TO(X) = \{E - F \in \widetilde{KO}(X) : S(E \oplus n)$ is fibre homotopy equivalent to $S(F \oplus n)$ for some $n \in \mathbf{N}\}$. Let ψ^k be the Adams operations. Then Adams [1] and Quillen [13] showed that $TO(X) = WO(X) = VO(X)$. Here

$$WO(X) = \bigcap_f \widetilde{KSO}(X)_f \tag{1}$$

where the intersection runs over all functions $f : \mathbf{N} \to \mathbf{N}$ and $\widetilde{KSO}(X)_f = \langle k^{f(k)}(\psi^k - 1)(u) : u \in \widetilde{KSO}(X) \text{ and } k \in \mathbf{N}\rangle$, and

$$VO(X) = \left\{ x \in \widetilde{KSO}(X) : \text{there exists } u \in \widetilde{KSO}(X) \text{ such that } \theta_k(x) = \frac{\psi^k(1+u)}{1+u} \in 1 + \widetilde{KSO}(X) \otimes \mathbf{Q}_k \text{ for all } k \in \mathbf{N} \right\} \tag{2}$$

where θ_k are the Bott exponential classes, and $\mathbf{Q}_k = \{n/k^m : n, m \in \mathbf{Z}\}$.

For a prime p, let $\widetilde{JO}(X)_{(p)}$ denote the localization of $\widetilde{JO}(X)$ at p. Since $\widetilde{JO}(X)$ is a finite abelian group, $\widetilde{JO}(X)_{(p)}$ is isomorphic to the p-summand of

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1991 *Mathematics Subject Classification*. Primary 55Q50, 55R50.

Key words and phrases. Fibre homotopy equivalence, Hopf line bundle, orientable real vector bundle, Adams operations, Bott classes.

$\widetilde{JO}(X)$. Moreover, since the localization is an exact functor on the category of finitely generated abelian groups, $\widetilde{JO}(X)_{(p)} \cong \widetilde{KO}(X)_{(p)}/TO(X)_{(p)}$. Using Atiyah-Tall [5] we obtain two computable formulae of $TO(X)_{(p)}$. The significance of those two localized formulae of $TO(X)$ is shown to find the J-orders of elements of $\widetilde{KO}(\mathbf{C}P^m)$.

In §2 using the fact that $\widetilde{KSO}(X)$ is an orientable γ-ring and the p-adic completion $\widetilde{KSO}(X)_p$ is an orientable p-adic γ-ring, we define a natural exponential map $\theta_k^{or} : KSO(2)(X) \to KSO(X)_p$ for each positive integer k. If k is odd, θ_k^{or} is the extension of $\theta_k^{or} : VectSO(2)(X) \to KSO(X)$ defined in Dieck [6]. From the main theorem of [5], we obtain the commutative diagram in Theorem 2.3.

Our main result is the following two formulae of $TO(X)_{(p)}$, which can be obtained directly from Theorem 2.3.

$$TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)}). \qquad \text{(Formula I)}$$

$$TO(X)_{(p)} = \left\{ x \in \widetilde{KSO}(X)_{(p)} : \theta_{k_p}^{or}(x) = \frac{\psi^{k_p}(1+u)}{1+u} \in 1 + \widetilde{KSO}(X)_p \right.$$

$$\left. \text{for some } u \in \widetilde{KSO}(X)_p \right\}. \qquad \text{(Formula II)}$$

Formula I (resp. Formula II) of $TO(X)_{(p)}$ may be thought of as the localization of $WO(X)$ (resp. $VO(X)$) at p.

Let $y = r\xi_m(\mathbf{C}) - 2$ where $\xi_m(\mathbf{C})$ is the complex Hopf line bundle over $\mathbf{C}P^m$. In §3 we apply Formulae I and II of $TO(X)_{(p)}$ to find $b_m(P_m(y; m_1, \dots, m_t))$, the J-order of $P_m(y; m_1, \dots, m_t) = m_1 y + m_2 y^2 + \cdots + m_t y^t \in \widetilde{KO}(\mathbf{C}P^m)$. Önder [10] has given the formula $TO(X)_{(2)} = (\psi^3 - 1)(\widetilde{KO}(X)_{(2)})$ by using Dieck [6] Ch. 11, and applied this formula to give computation of the 2-primary factor of $b_m(y)$. We obtain sharper results in giving a simple formula for the 2 and 3 primary factors of the J-orders of the canonical generators of $\widetilde{JO}(\mathbf{C}P^m)$. Finally in §4 we show how Formulae I and II of $TO(X)_{(p)}$ can be used to compute the group $\widetilde{JO}(X)$ for our illustrative example $X = \mathbf{C}P^4$.

2. Two computable formulae of $TO(X)_{(p)}$

Let G be a finitely generated abelian group. For a prime p let $G_{(p)} = \{g/m : g \in G \text{ and } m \in \mathbf{Z} \text{ with } (p, m) = 1\}$ denote the localization of G at p, then $G_{(p)}$ is canonically isomorphic to $\mathbf{Z}_{(p)} \otimes G$. Also, let $G_p = \varprojlim_n G/p^n G$ denote the p-adic completion of G. Then G_p is canonically isomorphic to $\mathbf{Z}_p \otimes G$.

For a rational number $q, v_p(q)$ denotes the exponent of p in the prime factorization of q.

LEMMA 2.1. (i) *Let G be a finite abelian group. Then the following groups are canonically isomorphic:*

$$G_{(p)} \cong G_p \cong G(p)$$

where $G(p) = \{g \in G : \text{the order of } g \text{ is a power of } p\}$. Consequently, if $g \in G$ has order m, then the order of $g/1$ in $G_{(p)}$ which is equal to the order of $1 \otimes g$ in G_p is equal to $p^{v_p(m)}$.

(ii) *If G is a finitely generated abelian group, then $G_{(p)}$ is canonically embedded in G_p.*

PROOF. Clear.

Now, our aim is to show how to apply the work of Atiyah-Tall [5] to find $\widetilde{JO}(X)_{(p)}$. Let $KSO(d)(X)$ be the group obtained by symmetrization of the semi-group $VectSO(d)(X)$ of all isomorphic classes of real vector bundles over X with structural group $SO(dn)$ for $n = 1, 2, \ldots$. $KSO(d)(X)$ is monomorphically embedded in $KO(X)$ as the subgroup of classes x such that $\omega_1(x) = 0$ and $\dim(x) = dn$ for some $n \in \mathbf{N}$, i.e.,

$$KSO(d)(X) = \{E - F \in KO(X) : \dim(E - F) = dn \text{ and } E, F \text{ are orientable}\}.$$

Let $\widetilde{KSO}(d)(X) = \{E - F \in KSO(d)(X) : \dim E = \dim F\}$. It is easy to see that $KSO(d)(X) = d\mathbf{Z} \oplus \widetilde{KSO}(d)(X)$ and $\widetilde{KSO}(d)(X) = \widetilde{KSO}(1)(X)$ for each $d \geq 1$. So, for simplicity, we write $\widetilde{KSO}(X)$ instead of $\widetilde{KSO}(d)(X)$. It is well known that $\widetilde{KSO}(X)$ is an orientable γ-ring and $\widetilde{KSO}(X)_p$ is an orientable p-adic γ-ring (see [5], or [6] Ch. 3).

Let k be an odd integer and J be a set of kth roots of unity $u \neq 1$ which contains from each pair u, u^{-1} exactly one element. The operations θ_k^{or} : $VectSO(2)(X) \to KSO(X)$ are defined in [6] and given by

$$\theta_k^{or}(E) = k^m \prod_{u \in J} \lambda_{-u}(E)(1-u)^{-2m} = \prod_{u \in J} \lambda_{-u}(E)(-u^{-1})^m \tag{3}$$

where $2m = \dim E$. θ_k^{or} does not depend on the choice of J [6].

If $(k, p) = 1$ then $\theta_k^{or}(E)$ is invertible in $KSO(X)_p$. So θ_k^{or} can be extended to $KSO(2)(X)$ with values in $KSO(X)_p$. Also, by using the fact that θ_k^{or} is a natural exponential map, it can be shown that $\theta_k^{or} : \widetilde{KSO}(X) \to 1 + \widetilde{KSO}(X)_p$ where $1 + \widetilde{KSO}(X)_p$ is the multiplicative group of elements $1 + w$ with $w \in \widetilde{KSO}(X)_p$. The operations $\rho_k^{or} : \widetilde{KSO}(X)_p \to 1 + \widetilde{KSO}(X)_p$ are given by

$$\rho_k^{or}(x) = \prod_{u \in J} \gamma_{u/u-1}(x). \tag{4}$$

Now we shall show how to define θ_{2k}^{or} and ρ_{2k}^{or} for $k \geq 1$. If $p \neq 2$ then $1/2 \in \mathbf{Z}_p$. So we may define $\theta_{2k}^{or} : \widetilde{KSO}(X) \to 1 + \widetilde{KSO}(X)_p$. by

$$\theta_{2k}^{or}(E-F) = \left(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \lambda_{-u}(E) \left(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \lambda_{-u}(F) \right)^{-1} \right)^{1/2}. \tag{5}$$

Similarly, we may define

$$\rho_{2k}^{or}(x) = \left(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \gamma_{u/u-1}(x) \right)^{1/2}. \tag{6}$$

Lemma 2.2 (*An analogue of Proposition* 5.3 *of* [5]). *If* $(p,k) = 1$ *then the following diagram where* $i(x) = 1 \otimes x$ *is commutative:*

$$\begin{tikzcd} \widetilde{KSO}(X) \arrow[d, "i"] \arrow[dr, "\theta_k^{or}"] & \\ \widetilde{KSO}(X)_p \arrow[r, "\rho_k^{or}"] & 1 + \widetilde{KSO}(X)_p. \end{tikzcd}$$

Remark. If $(p,k) = 1$, then $\mathbf{Q}_k \subseteq \mathbf{Z}_p$. So, using Proposition 3.15.2 of [6] and Examples 5.14 and 5.15 of [1]-II, we see that θ_k^{or} agrees with Bott operation θ_k which is denoted by ρ^k in [1]-II.

Now, we give our main theorem.

Theorem 2.3. *Let* p *be a prime number and* k_p *be a generator of* $(\mathbf{Z}/p^2\mathbf{Z})^*$, *the group of units in* $\mathbf{Z}/p^2\mathbf{Z}$. *Then the following diagram is commutative:*

$$\begin{tikzcd} & 0 \arrow[d] & & \\ & \widetilde{KSO}(X)_{(p),\Gamma} \arrow[r, "\tilde{q}"] \arrow[d, "i_\Gamma"] & \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)} \arrow[r] \arrow[d, "\tilde{\theta}_{k_p}^{or}"] & 0 \\ 0 \arrow[r] & \widetilde{KSO}(X)_{p,\Gamma} \arrow[r, "\rho_{k_p,\Gamma}^{or}"] & 1 + \widetilde{KSO}(X)_{p,\Gamma} \arrow[r] & 0 \end{tikzcd}$$

Here the index Γ *indicates that we factor out the image of* $(\psi^{k_p} - 1)$ *and* $\tilde{q}$ *is the quotient map.*

Proof. First, we show that rows and columns are well-defined and exact.

(a) Using Lemma 2.1, the fact that localization and completion are exact functors on the category of finitely generated abelian groups and the naturality of Adams' operations, we have the following identifications:

$$\begin{aligned}\widetilde{KSO}(X)_{(p)}/(\psi^{k_p}-1)(\widetilde{KSO}(X)_{(p)}) &= \widetilde{KSO}(X)_{(p)}/((\psi^{k_p}-1)(\widetilde{KSO}(X)))_{(p)}\\ &= (\widetilde{KSO}(X)/(\psi^{k_p}-1)(\widetilde{KSO}(X)))_{(p)}\\ &\subseteq \widetilde{KSO}(X)_p/((\psi^{k_p}-1)(\widetilde{KSO}(X)))_p\\ &= \widetilde{KSO}(X)_p/(\psi^{k_p}-1)(\widetilde{KSO}(X)_p).\end{aligned}$$

Hence, $i:\widetilde{KSO}(X)_{(p)}\to\widetilde{KSO}(X)_p$ defined by $i(x/m)=(m)^{-1}\otimes x$ induces a monomorphism $i_\Gamma:\widetilde{KSO}(X)_{(p),\Gamma}\to\widetilde{KSO}(X)_{p,\Gamma}$.

(b) By Theorem 4.5 of Atiyah-Tall [5], $\rho^{or}_{k_p}$ induces an isomorphism

$$\rho^{or}_{k_p,\Gamma}:\widetilde{KSO}(X)_{p,\Gamma}\to 1+\widetilde{KSO}(X)_{p,\Gamma}.$$

(c) To show that $(\psi^{k_p}-1)(KSO(X)_{(p)})\subseteq TO(x)_{(p)}$. Let

$$\frac{E-F}{m}\in\widetilde{KSO}(X)_{(p)}.$$

Then

$$(\psi^{k_p}-1)\left(\frac{E-F}{m}\right)=\left(\frac{\psi^{k_p}E-E}{m}\right)-\left(\frac{\psi^{k_p}F-F}{m}\right).$$

By Quillen [13], there is a fiberwise map of degree a power of k_p between $\psi^{k_p}E$ and E. So, by Dold's Theorem mod k in [1]-I we have

$$k_p^e(\psi^{k_p}E-E)\in TO(X)$$

for some integer e. Since $(p,k_p)=1$, we have

$$\left(\frac{\psi^{k_p}E-E}{m}\right)=\frac{k_p^e(\psi^{k_p}E-E)}{k_p^e m}\in TO(X)_{(p)}.$$

Similarly,

$$\frac{\psi^{k_p}F-F}{m}\in TO(X)_{(p)}$$

and hence

$$(\psi^{k_p}-1)\left(\frac{E-F}{m}\right)\in TO(X)_{(p)}.$$

Thus, we have an epimorphism $\tilde q:\widetilde{KSO}(X)_{(p),\Gamma}\to\widetilde{KSO}(X)_{(p)}/TO(X)_{(p)}$.

(d) It is easy to see that $\theta_{k_p}^{or} : \widetilde{KSO}(X)_{(p)} \to 1 + \widetilde{KSO}(X)_p$ given by $\theta_{k_p}^{or}(x/m) = (\theta_{k_p}^{or}(x))^{1/m}$ is an exponential map. Let

$$\frac{E - F}{m} \in TO(X)_{(p)}.$$

Then $nS(E)$ is stably fibre homotopy equivalent to $nS(F)$ for some n with $(p, n) = 1$. So by [1]-(II) Corollary 5.8,

$$\theta_{k_p}^{or}(E - F)^n = \frac{\psi^{k_p}(1 + u)}{1 + u} \in 1 + \widetilde{KSO}(X)_p$$

for some $u \in \widetilde{KSO}(X)$. Since $(p, n) = 1, (1 + u)^{1/nm} = 1 + w \in 1 + \widetilde{KSO}(X)_p$ for some $w \in \widetilde{KSO}(X)_p$. Hence

$$\theta_{k_p}^{or}\left(\frac{E - F}{m}\right) = \theta_{k_p}^{or}(E - F)^{1/m} = (\theta_{k_p}^{or}(E - F)^n)^{1/nm}$$

$$= \frac{\psi^{k_p}(1 + u)^{1/nm}}{(1 + u)^{1/nm}} = \frac{\psi^{k_p}(1 + w)}{1 + w}.$$

Thus $\theta_{k_p}^{or}$ induces a homomorphism

$$\tilde{\theta}_{k_p}^{or} : \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)} \to 1 + \widetilde{KSO}(X)_{p,\Gamma}.$$

Finally, we show the commutativity of our diagram.

Let $x/m \in \widetilde{KSO}(X)_{(p)}$. Then $\tilde{\theta}_{k_p}^{or} \circ \tilde{q}(x/m + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})) = \tilde{\theta}_{k_p}^{or}(x/m + TO(X)_{(p)}) = \theta_{k_p}^{or}(x)^{1/m} + (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p)$. On the other hand, $(\rho_{k_p,\Gamma}^{or} \circ i_\Gamma)(x/m + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})) = \rho_{k_p,\Gamma}^{or}(i(x/m) + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_p)) = \rho_{k_p}^{or}(i(x/m)) + (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p$. Now, the result follows from Lemma 2.2. This completes the proof of Theorem 2.3.

Corollary 2.4 (*Formula I of $TO(X)_{(p)}$*).

$$TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)}).$$

Proof. Since $\tilde{\theta}_{k_p}^{or} \circ \tilde{q} = \rho_{k_p,\Gamma}^{or} \circ i_\Gamma, \tilde{q}$ is injective and hence an isomorphism. So, $TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})$.

Corollary 2.5 (*Formula II of $TO(X)_{(p)}$*).

$$TO(X)_{(p)} = \left\{x \in \widetilde{KSO}(X)_{(p)} : \theta_{k_p}^{or}(x) = \frac{\psi^{k_p}(1 + u)}{1 + u} \in 1 + \widetilde{KSO}(X)_p \right.$$

$$\left. \text{for some } u \in \widetilde{KSO}(X)_p \right\}.$$

PROOF. Clearly, the right hand side of the above equality is a well-defined subgroup of $\widetilde{KSO}(X)_{(p)}$. The fact that i_Γ is injective implies that

$$i(\widetilde{KSO}(X)_{(p)}) \cap (\psi^{k_p} - 1)(\widetilde{KSO}(X)_p) = i((\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})). \tag{7}$$

The fact that $\rho^{or}_{k_p,\Gamma}$ is an isomorphism implies that

$$\rho^{or}_{k_p}(\psi^{k_p} - 1)(\widetilde{KSO}(X)_p) = (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p). \tag{8}$$

Now let $x \in TO(X)_{(p)}$. Then by Formula I of $TO(X)_{(p)}$, $x \in (\psi^{k_p} - 1)\widetilde{KSO}(X)_{(p)}$. Hence from (7) and (8)

$$\theta^{or}_{k_p}(x) = \rho^{or}_{k_p}(i(x)) = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X)_p$$

for some $u \in \widetilde{KSO}(X)_p$.

If X is a finite CW complex, then $\widetilde{JO}(X)$ is a finite abelian group. So, by Lemma 2.1, the p-primary factor of the order of $x + TO(X) \in \widetilde{JO}(X)$ is the order of $x + TO(X)_{(p)} \in \widetilde{JO}(X)_{(p)}$, the smallest power of p, p^m such that $p^m x \in TO(X)_{(p)}$.

3. J-orders of elements of $\widetilde{KO}(\mathbf{C}P^m)$

We will show how to use Formulae I and II of $TO(X)_{(p)}$ to find the J-orders of elements of $\widetilde{KO}(\mathbf{C}P^m)$. As we have shown in [9], we only need to consider the case when m is even, that is $m = 2t$ for some $t \in \mathbf{N}$. Let $P_m(y; m_1, \dots, m_t) = m_1 y + m_2 y^2 + \dots + m_t y^t \in \widetilde{KO}(\mathbf{C}P^m) = \mathbf{Z}[y] \pmod{y^{t+1}}$. In order to find the J-order $b_m(P_m(y; m_1, \dots, m_t))$ of $P_m(y; m_1, \dots, m_t)$ the following two lemmas will be useful.

LEMMA 3.1. *Let k_p be a generator of $(\mathbf{Z}/p^2\mathbf{Z})^*$. If $n \in \mathbf{N}$, then*
(i) $\nu_2(3^{2n} - 1) = 3 + \nu_2(n)$.
(ii) *For an odd prime p,*

$$\nu_p(k_p^{2n} - 1) = \begin{cases} 0 & \text{if } 2n \not\equiv 0 \bmod (p-1) \\ 1 + \nu_p(n) & \text{if } 2n \equiv 0 \bmod (p-1). \end{cases}$$

PROOF. (i) is well-known.

(ii) Let $\nu_p(k_p^{2n} - 1) = s$. Then $k_p^{2n} \equiv 1 \bmod p^s$. If $s \geq 1$, then $(\mathbf{Z}/p^s\mathbf{Z})^*$ is cyclic of order $p^{s-1}(p-1)$ with generator k_p ([7], Theorem 2, p. 43). So, $2n = p^{s-1}(p-1)d$ for some $d \in \mathbf{N}$ with $(d, p) = 1$ ([7], Lemma 3, p. 42). Hence, $s = 1 + \nu_p(n)$.

LEMMA 3.2. *Let k_p be a generator of $(\mathbf{Z}/p^2\mathbf{Z})^*$ and $r, s \in \mathbf{N}$ with $r \geq s$. Then*

(i) $\nu_2(\prod_{i=s}^{r}(3^{2i}-1)) = 3(r-s+1) + \sum_{i=s}^{r}\nu_2(i)$.
(ii) *For an odd prime* $p, \nu_p(\prod_{i=s}^{r}(k_p^{2i}-1))$

$$= \begin{cases} \left[\dfrac{2r}{p-1}\right] + \sum_{i=1}^{[2r/(p-1)]}\nu_p(i) - \left[\dfrac{2(s-1)}{p-1}\right] - \sum_{i=1}^{[2(s-1)/(p-1)]}\nu_p(i) & \text{if } p \le 2r+1 \\ 0 & \text{if } p > 2r+1. \end{cases}$$

PROOF. (i) $\nu_2(\prod_{i=s}^{r}(3^{2i}-1)) = \sum_{i=s}^{r}\nu_2(3^{2i}-1) = \sum_{i=s}^{r}(3+\nu_2(i)) = 3(r-s+1) + \sum_{i=s}^{r}\nu_2(i)$.

(ii) If $p > 2r+1$, then $\nu_p(k_p^{2i}-1) = 0$ for each $i = s, \dots, r$. Hence

$$\nu_p\left(\prod_{i=s}^{r}(k_p^{2i}-1)\right) = 0.$$

If $p \le 2r+1$, then $p-1 = 2d$ for some $d \in \{1, \dots, r\}$.

$$\nu_p\left(\prod_{i=s}^{r}(k_p^{2i}-1)\right) = \sum_{i=s}^{r}\nu_p(k_p^{2i}-1) = \sum_{\substack{i=s \\ 2i\equiv 0 \bmod (p-1)}}^{r}(1+\nu_p(i))$$

$$= \sum_{i=1}^{[2r/(p-1)]}(1+\nu_p(2di)) - \sum_{i=1}^{[2(s-1)/(p-1)]}(1+\nu_p(2di)).$$

But $\nu_p(2di) = \nu_p(i)$. So

$$\nu_p\left(\prod_{i=s}^{r}(k_p^{2i}-1)\right) = \sum_{i=1}^{[2r/(p-1)]}(1+\nu_p(i)) - \sum_{i=1}^{[2(s-1)/(p-1)]}(1+\nu_p(i))$$

$$= \left[\frac{2r}{p-1}\right] + \sum_{i=1}^{[2r/(p-1)]}\nu_p(i) - \left[\frac{2(s-1)}{p-1}\right] - \sum_{i=1}^{[2(s-1)/(p-1)]}\nu_p(i).$$

This completes the proof.

Now, let k_p be an odd generator of $(\mathbf{Z}/p^2\mathbf{Z})^*$, say $k_p = 2q+1$ (take $k_2 = 3$).

REMARK. We take k_p to be odd only to reduce the work. Our argument works equally well for the case when k_p is even.

According to Formula I of $TO(X)_{(p)}$, $\nu_p(b_m(P_m(y; m_1, \dots, m_t)))$ is the smallest non-negative integer v such that

$$p^v P_m(y; m_1, \dots, m_t) = (1-\psi^{k_p})(u) \tag{9}$$

in $\widetilde{KO}(\mathbf{C}P^m)_{(p)}$ for some $u \in \widetilde{KO}(\mathbf{C}P^m)_{(p)}$.

From [2] Theorem 2.2, and [8] Lemma 3.6,

$$\psi^{k_p}(y) = y\left(\sum_{j=0}^{q} \frac{k_p}{2j+1}\binom{q+j}{2j} y^j\right)^2 = y\left(\sum_{j=0}^{q} b_j y^j\right)^2 \tag{10}$$

where

$$b_j = \frac{k_p}{2j+1}\binom{q+j}{2j}, \qquad j = 0, \dots, q.$$

So, for $r = 2, \dots, t$

$$\psi^{k_p}(y^r) = (\psi^{k_p}(y))^r = \sum_{\substack{j=0 \\ j \le t-r}}^{2rq} C_{j,r} y^{r+j}$$

where

$$C_{j,r} = \sum_{\substack{i_1+\cdots+i_{2r}=j \\ i_1,\dots,i_{2r} \in \{0,\dots,q\}}} b_{i_1} b_{i_2} \dots b_{i_{2r}}.$$

Let $u \in \widetilde{KO}(\mathbf{C}P^m)_{(p)}$. Then $u = a_1 y + \cdots + a_t y^t$ for some $a_i \in \mathbf{Z}_{(p)}$. Using (10), it is easy to see that the coefficient of y^r in $(1 - \psi^{k_p})(u)$ is

$$-\sum_{i=j_r}^{r-1} C_{r-i,i} a_i + (1 - k_p^{2r}) a_r$$

where $j_r = \left[\dfrac{r-1}{k_p}\right] + 1$.

So, from (9), we need to find the smallest v which solves the following system of equations in $\mathbf{Z}_{(p)}$:

$$-\sum_{i=j_r}^{r-1} C_{r-i,i} a_i + (1 - k_p^{2r}) a_r = p^v m_r$$

where $r = 1, \dots, t$.

The above system has the following solutions:

$$a_r = \frac{p^v M_{k_p,r}(m_1, \dots, m_t)}{(1 - k_p^2) \dots (1 - k_p^{2r})}$$

where $M_{k_p,1}(m_1, \dots, m_t) = m_1$ and for $r = 2, \dots, t$

$$M_{k_p,r}(m_1, \dots, m_t) = \sum_{i=j_r}^{r-1} C_{r-i,i}(1 - k_p^{2(i+1)}) \dots (1 - k_p^{2(r-1)}) M_{k_p,i}(m_1, \dots, m_t)$$
$$+ m_r (1 - k_p^2) \dots (1 - k_p^{2(r-1)}).$$

Now, $a_r \in \mathbf{Z}_{(p)}$ implies that $v_p(a_r) \geq 0$. So

$$v \geq \max_{r=1,\dots,t} \left\{ v_p\left(\prod_{i=1}^{r}(1-k_p^{2i})\right) - v_p(M_{k_p,r}(m_1,\dots,m_t)), 0 : M_{k_p,r}(m_1,\dots,m_t) \neq 0 \right\}$$

is a necessary and sufficient condition on v so that (9) is satisfied. Hence, we have:

THEOREM 3.3.

$$\begin{aligned} &v_p(b_m(P_m(y;m_1,\dots,m_t))) \\ &\quad = \max_{r=1,\dots,t} \left\{ v_p\left(\prod_{i=1}^{r}(1-k_p^{2i})\right) - v_p(M_{k_p,r}(m_1,\dots,m_t)), \right. \\ &\qquad\qquad \left. 0 : M_{k_p,r}(m_1,\dots,m_t) \neq 0 \right\}. \end{aligned}$$

Now, let us use Formula II.

Let $\theta_{k_p}(P_m(y;m_1,\dots,m_t)) = 1 + \alpha_1(m_1,\dots,m_t)y + \cdots + \alpha_t(m_1,\dots,m_t)y^t$ for some $\alpha_i(m_1,\dots,m_t) \in \mathbf{Z}_p$ (see [9], Theorem 2.2). $v_p(b_m(P_m(y;m_1,\dots,m_t)))$ is the smallest non-negative integer v such that

$$\theta_{k_p}(P_m(y;m_1,\dots,m_t))^{p^v} = \frac{\psi^{k_p}(1+u)}{(1+u)} \text{ in } 1 + \widetilde{KO}(\mathbf{C}P^m)_p \tag{11}$$

for some $u \in \widetilde{KO}(\mathbf{C}P^m)_p$. Let $u = b_1 y + \cdots + b_t y^t$ for some $b_i \in \mathbf{Z}_p$. With the above symbols, the coefficient of y^r in $\psi^{k_p}(u)$ is

$$\sum_{i=j_r}^{r-1} C_{r-i,i} b_i + b_r k_p^{2r}.$$

To avoid excessive notation, we write $\theta_{k_p}(P_m(y;m_1,\dots,m_t))^{p^v} = 1 + \alpha_1 y + \cdots + \alpha_t y^t$ where α_i involves quantities containing p in some way.

From (11), we have $1 + \psi^{k_p}(u) = 1 + d_1 y + \cdots + d_t y^t$ where

$$d_n = \sum_{\substack{i+s=n \\ b_0=\alpha_0=1}} b_i \alpha_s.$$

Thus

$$\sum_{i=j_r}^{r-1} C_{r-i,i} b_i + b_r k_p^{2r} = b_r + \sum_{\substack{s>0 \\ i+s=r}} b_i \alpha_s$$

which implies that

$$b_r = \frac{\sum_{\substack{s>0 \\ i+s=r}} b_i\alpha_s - \sum_{i=j}^{r-1} C_{r-i,i}b_i}{(k_p^{2r}-1)} = \frac{L_{k_p,r}(m_1,\dots,m_t)}{(k_p^2-1)\dots(k_p^{2r}-1)}$$

where $L_{k_p,1}(m_1,\dots,m_t) = \alpha_1$ and for $r = 2,\dots,t$,

$$\begin{aligned} L_{k_p,r}(m_1,\dots,m_t) = {} & \sum_{i=1}^{r-1} L_{k_p,r-i}(m_1,\dots,m_t)\alpha_i(k_p^{2(r-i+1)}-1)\dots(k_p^{2(r-1)}-1) \\ & - \sum_{i=j_r}^{r-1} C_{r-i,i}L_{k_p,i}(m_1,\dots,m_t)(k_p^{2(i+1)}-1)\dots(k_p^{2(r-1)}-1) \\ & + \alpha_r(k_p^2-1)\dots(k_p^{2(r-1)}-1). \end{aligned}$$

Now, $b_i \in \mathbf{Z}_p$ for $i = 1,\dots,t$ implies that

$$v_p(L_{k_p,r}(m_1,\dots,m_t)) \geq v_p((k_p^2-1)\dots(k_p^{2r}-1)).$$

So, we have:

THEOREM 3.4. *$v_p(b_m(P_m(y;m_1,\dots,m_t)))$ is the smallest v such that $v_p(L_{k_p,r}(m_1,\dots,m_t)) \geq v_p((k_p^2-1)\dots(k_p^{2r}-1))$ for each $r = 1,\dots,t$.*

Using Lemma 3.2 and any one of the above two theorems, we directly obtain:

COROLLARY 3.5. *If $p > 2t+1$, then $v_p(b_m(P_m(y;m_1,\dots,m_t))) = 0$. Consequently,*

$$\widetilde{JO}(\mathbf{C}P^m) \cong \bigoplus_{\text{all primes } p \leq m+1} \widetilde{JO}(\mathbf{C}P^m)_{(p)}.$$

From Theorem 3.3, to find $b_m(P_m(y;m_1,\dots,m_t))$ we only need to find $v_p(M_{k_p,r}(m_1,\dots,m_t))$ for $r = 1,\dots,t$. Therefore, it may be a good problem if one tries to obtain a general formula for $v_p(M_{k_p,r}(m_1,\dots,m_t))$ in term of $r, k_p, m_1,\dots,m_t$. Next, we compute $v_p(M_{k_p,r}(0,\dots,m_r = 1,0,\dots,0))$ for $p = 2,3$ and then we obtain simple formulae for the 2 and 3 primary factors of the J-orders of the canonical generators of $\widetilde{JO}(\mathbf{C}P^m)$. These simple formulae have been already conjectured in [9].

For $n = 1,\dots,t$, the J-order of $y^n + TO(\mathbf{C}P^m)$ is $b_m(P_m(y;0,\dots,m_n = 1,0,\dots,0))$. Let $M_{k_p,r} = M_{k_p,r}(0,\dots,m_n = 1,0,\dots,0)/(1-k_p^2)\dots(1-k_p^{2(n-1)})$. Then $M_{k_p,r} = 0$ for $r < n, M_{k_p,n} = 1$ and for $r = n+1,\dots,t$,

$$M_{k_p,r} = \sum_{\substack{i=jr\\ i\geq n}}^{r-1} C_{r-i,i}(1-k_p^{2(i+1)})\dots(1-k_p^{2(r-1)})M_{k_p,i}$$

where $j_r = \left[\dfrac{r-1}{k_p}\right] + 1$.

Hence, from Theorem 3.3, we have

$$v_p(b_m(y^n)) = \max_{r=n,\dots,t}\left\{v_p\left(\prod_{i=n}^{r}(1-k_p^{2i})\right) - v_p(M_{k_p,r}), 0 : M_{k_p,r} \neq 0\right\}.$$

PROPOSITION 3.6. *If $p = 2$ or 3, then*

$$v_p(M_{k_p,r}) = \sum_{s=1}^{[2(r-1)/(p-1)]} v_p(s) - \sum_{s=1}^{[2(n-1)/(p-1)]} v_p(s)$$

for each $r = n,\dots,t$.

PROOF. We prove this proposition for $p = 2$ (the case $p = 3$ is similar). Recall that $k_2 = 3$. So we need to show that $v_2(M_{3,r}) = r - n + \sum_{s=n}^{r-1} v_2(s)$ for $r = n+1,\dots,t$, where

$$M_{3,r} = \sum_{\substack{i=j_r\\ i\geq n}}^{r-1} 3^{3i-r}\binom{2i}{r-i}(1-3^{2(i+1)})\dots(1-3^{2(r-1)})M_{3,i}$$

by induction on r. If $r = n+1$ then $v_2(M_{3,r}) = v_2\binom{2n}{1} = 1 + v_2(n)$. So let $n+1 < r \leq t$. We claim that $v_2(3^{3i-r}\binom{2i}{r-i}(1-3^{2(i+1)})\dots(1-3^{2(r-1)})M_{3,i}) > v_2(3^{2r-3}\binom{2(r-1)}{1}M_{3,r-1})$ for each $\max\{j_r, n\} \leq i < r-1$. Suppose that $\max\{j_r, n\} \leq i < r-1$. Then by induction hypothesis and Lemma 3.2,

$$v_2\left(3^{3i-r}\binom{2i}{r-i}(1-3^{2(i+1)})\dots(1-3^{2(r-1)})M_{3,i}\right)$$
$$= v_2\binom{2i}{r-i} + 3(r-i-1) + i - n - v_2(i) + \sum_{s=n}^{r-1} v_2(s).$$

On the other hand,

$$v_2\left(3^{2r-3}\binom{2(r-1)}{1}M_{3,r-1}\right) = r - n + \sum_{s=n}^{r-1} v_2(s).$$

So, we need to show that $v_2\binom{2i}{r-i} + 2(r-i-1) > v_2(2i)$. But this follows directly from the fact that $v_2\binom{2i}{r-i} = v_2(2i) - v_2(r-i)$ if $v_2(2i) \geq r-i-1$. This completes the proof of our claim. Hence, $\nu_2(M_{3,r}) = \nu_2\left(3^{2r-3}\binom{2(r-1)}{1}M_{3,r-1}\right) = r - n + \sum_{s=n}^{r-1}\nu_2(s)$. This completes the proof.

Unfortunately, the above proof can not be used for $p \neq 2, 3$.

THEOREM 3.7. *If $p = 2$ or 3 and $1 \le n \le t$. Then*

$$\nu_p(b_m(y^n)) = \max\left\{s - \left[\frac{2(n-1)}{p-1}\right] + \nu_p(s) : \left[\frac{2n}{p-1}\right] \le s \le \left[\frac{2t}{p-1}\right]\right\}.$$

PROOF. Let $p = 2$, then $\nu_2(b_m(y^n)) = \max\{\nu_2(\prod_{i=n}^{r}(1-3^{2i})) - \nu_2(M_{3,r}) : r = n, \dots, t\} = \max\{2r - 2n + 2 + \nu_2(2r) : r = n, \dots, t\} = \max\{s - 2(n-1) + \nu_2(s) : 2n \le s \le 2t\}$. The case $p = 3$ is similar.

REMARK. If Proposition 3.6 holds for some values of p other than 2 or 3, then Theorem 3.7 also holds for those values of p.

4. An illustrative example $\widetilde{JO}(CP^4)$

If $\widetilde{KO}(X) = \langle y_1, \dots, y_n\rangle$, then $\widetilde{JO}(X)_{(p)} = \langle \alpha_{1,p} = y_1 + TO(X)_{(p)}, \dots, \alpha_{n,p} = y_n + TO(X)_{(p)}\rangle$. So to compute $\widetilde{JO}(X)_{(p)}$, we need to find all relations between $\alpha_{1,p}, \dots, \alpha_{n,p}$, i.e., we need to find "sufficient" solutions for the equation:

$$c_1\alpha_{1,p} + \cdots + c_n\alpha_{n,p} = 0 \quad \text{in } \widetilde{JO}(X)_{(p)}, \qquad c_1, \dots, c_n \in \mathbf{Z}. \tag{12}$$

This implies that $c_1y_1 + \cdots + c_ny_n \in TO(X)_{(p)}$. Now using formulae I and II of $TO(X)_{(p)}$, one may try to find "sufficient" solutions for (12).

$\widetilde{KO}(\mathbf{C}P^4) = \{a_1y + a_2y^2 : a_1, a_2 \in \mathbf{Z}, y^3 = 0\}$. So, $\widetilde{JO}(\mathbf{C}P^4)_{(p)} = \langle \alpha_{1,p} = y + TO(\mathbf{C}P^4)_{(p)}, \alpha_{2,p} = y^2 + TO(\mathbf{C}P^4)_{(p)}\rangle = \langle\alpha_{1,p}\rangle + \langle\alpha_{2,p}\rangle$. To find relations between $\alpha_{1,p}$ and $\alpha_{2,p}$, we need to solve $c_1\alpha_{1,p} + c_2\alpha_{2,p} = 0$ in $\widetilde{JO}(\mathbf{C}P^4)_{(p)}$.

$\widetilde{JO}(\mathbf{C}P^4)_{(2)} = \langle\alpha_{1,2} = y + TO(\mathbf{C}P^4)_{(2)}\rangle + \langle\alpha_{2,2} = y^2 + TO(\mathbf{C}P^4)_{(2)}\rangle$. $\langle\alpha_{1,2}\rangle$ is cyclic of order 64 and $\langle\alpha_{2,2}\rangle$ is cyclic of order 16. Also, $2\alpha_{2,2} = 40\alpha_{1,2}$. Hence $\widetilde{JO}(\mathbf{C}P^4)_{(2)} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/64\mathbf{Z}$. Similarly, $\widetilde{JO}(\mathbf{C}P^4)_{(3)} \cong \mathbf{Z}/9\mathbf{Z}$ and $\widetilde{JO}(\mathbf{C}P^4)_{(5)} \cong \mathbf{Z}/5\mathbf{Z}$. Thus, by Corollary 3.5, we obtain a well-known result:

THEOREM 4.1. $\widetilde{JO}(\mathbf{C}P^4) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/64\mathbf{Z} \oplus \mathbf{Z}/9\mathbf{Z} \oplus \mathbf{Z}/5\mathbf{Z}$.

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