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Enumerating embeddings of n-manifolds into complex projective n-space

Dedicated to Professor Fuichi Uchida on his 60th birthday

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Abstract. Let $f : M \to N$ be an embedding between differentiable manifolds and set $\pi_1(N^M, \mathrm{Emb}(M,N), f) = [M \subset N]_f$, where $\mathrm{Emb}(M,N)$ denotes the space of embeddings of M to N. Then it is known that there is a $\pi_1(N^M, f)$-action on $[M \subset N]_f$ such that $[M \subset N]_f/\pi_1(N^M, f)$ is equivalent to the set $[M \subset N]_{[f]}$ of isotopy classes of embeddings homotopic to f. In this paper, we will study the set $[M^n \subset CP^n]_f$ for an n-manifold M^n. Further we will determine the sets $[RP^n \subset CP^n]_{[f]}$ and $[CP^n \subset CP^{2n}]_{[f]}$.

1. Introduction and statement of results

Throughout this paper, n-manifolds mean n-dimensional connected differentiable manifolds without boundary and embeddings stand for differentiable embeddings of compact manifolds to manifolds. For any map $f : M \to N$, we denote by $[M \subset N]_{[f]}$ the set of isotopy classes of embeddings homotopic to f. A. Haefliger's existence theorem [3] implies that for any compact n-manifold M^n and any map $f : M^n \to CP^n$ $(n > 2)$, there exists an embedding homotopic to f. Henceforth we would like to determine the set $[M^n \subset CP^n]_{[f]}$.

Set $\pi_1(N^M, \mathrm{Emb}(M,N), f) = [M \subset N]_f$, where $\mathrm{Emb}(M,N)$ denotes the space of embeddings of M to N. Then it is known (cf. [2], [7], [8], [12]) that there is a $\pi_1(N^M, f)$-action on $[M \subset N]_f$ such that

$$(1.1) \qquad [M \subset N]_f/\pi_1(N^M, f) = [M \subset N]_{[f]}.$$

In this paper, we will study the set $[M^n \subset CP^n]_f$ for an n-manifold M^n and a map $f : M^n \to CP^n$. Furthermore we will determine the isotopy sets of embeddings $[RP^n \subset CP^n]_{[f]}$ and $[CP^n \subset CP^{2n}]_{[f]}$.

The integral cohomology of CP^n is given by

$$H^*(CP^n; Z) = Z[z]/(z^{n+1})(\deg z = 2).$$

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THEOREM 1.1. *Let M^n be a compact n-manifold* $(n > 3)$ *and* $f : M^n \to CP^n$ *a map. If n is even and M^n is orientable, assume that* $f^*\rho_2 z = 0$ *or* $H_1(M^n; Z)$ *does not have* Z_2 *as its direct summand. Then there exist the following exact sequences:*

$$0 \to H^n(M^n; Z)/f^*(z)H^{n-2}(M^n; Z) \to [M^n \subset CP^n]_f$$
$$\to H^{n-1}(M^n; Z) \to 0, \qquad \text{if } n \equiv 1(2), \quad w_1(M^n) = 0,$$
$$0 \to H^n(M^n; Z_2)/f^*\rho_2(z)H^{n-2}(M^n; Z_2) \to [M^n \subset CP^n]_f$$
$$\to Z \oplus \ker Sq^1 \to 0, \qquad \text{if } n \equiv 0(2), \quad w_1(M^n) \neq 0,$$
$$0 \to H^n(M^n; Z_2)/f^*\rho_2(z)H^{n-2}(M^n; Z_2) \to [M^n \subset CP^n]_f$$
$$\to H^{n-1}(M^n; Z_2) \to 0, \qquad \text{otherwise},$$

where ρ_2 *is the reduction* mod 2 *and* $Sq^1 : H^{n-1}(M^n; Z_2) \to H^n(M^n; Z_2)$.

COROLLARY 1.2. *Let M^n be a compact n-manifold. If* $f : M^n \to CP^n$ *induces an epimorphism* $f_\# : \pi_2(M^n) \to \pi_2(CP^n) = Z$, *then*

$$[M^n \subset CP^n]_f = \begin{cases} H^{n-1}(M^n; Z) & \text{if } n \equiv 1(2), w_1(M^n) = 0, \\ Z \oplus \ker Sq^1 & \text{if } n \equiv 0(2), w_1(M^n) \neq 0, \\ H^{n-1}(M^n; Z_2) & \text{otherwise.} \end{cases}$$

COROLLARY 1.3. *If M^n is simply connected, then for any* $f : M^n \to CP^n$,

$$[M^n \subset CP^n]_f = [M^n \subset CP^n]_{[f]}$$
$$= \begin{cases} H^n(M^n; Z)/f^*(z)H^{n-2}(M^n; Z) & \text{for } n \text{ odd}, \\ H^n(M^n; Z_2)/f^*\rho_2(z)H^{n-2}(M^n; Z_2) & \text{for } n \text{ even}. \end{cases}$$

In particular, for $n \geq 2$,

$$[CP^n \subset CP^{2n}]_{[f]} = (Z/(\deg f^* : H^2(CP^{2n}; Z) \to H^2(CP^n; Z))Z) \otimes Z_2.$$

COROLLARY 1.4. *If* $n > 3$, *then for any* $f : RP^n \to CP^n$ *there exist countably many distinct isotopy classes of embeddings homotopic to f.*

REMARK. B.-H Li and P. Zhang [9] have investigated the set $[M^n \subset N^{2n}]_f$ in a different way. Some results of [9] and this paper overlap, e.g., Corollary 1.3. Combination of the results of [9] and this paper enriches the study of $[M^n \subset CP^n]_f$ and hence $[M^n \subset CP^n]_{[f]}$.

2. Larmore's approach to $[M \subset N]_f$

We recall Larmore's method [7], [8] of computing the set $\pi_1(N^M, \mathrm{Emb}(M, N), f) = [M \subset N]_f$ for an embedding $f : M \to N$.

For a manifold V without boundary, let $RV = (V^2 - \varDelta V) \cup_\phi SV \times [0, \varepsilon)$, where $\phi : SV \times (0, \varepsilon) \to V^2 - \varDelta V$ is a map defined by $\phi(v, t) = (\exp(tv), \exp(-tv))$. Here we use a Riemannian metric on V and SV stands for the total space of the sphere bundle associated with the tangent bundle of V. A free Z_2-action on RV is induced from the antipodal map of SV and the interchanging of elements of V^2. The spaces R^*V and V^* are defined as quotient spaces

$$R^*V = RV/Z_2 \qquad \text{and} \qquad V^* = (V^2 - \varDelta V)/Z_2.$$

Then R^*V is a $2n$-manifold ($n = \dim V$) with boundary $PV(\approx SV/Z_2)$ and $R^*V - PV = V^*$. The pair of spaces $(R^*(V \times R^\infty), P(V \times R^\infty))$ denotes the inductive limit of $(R^*(V \times R^k), P(V \times R^k))$ and $R^*i_V : (R^*V, PV) \subset (R^*(V \times R^\infty), P(V \times R^\infty))$ denotes the natural inclusion.

For a space X, we define a space ΓX by

$$\Gamma X = (X^2 \times S^\infty)/Z_2,$$

where the involution on $X^2 \times S^\infty$ is given by $(x, y, v) \to (y, x, -v)$. The natural inclusion $\varDelta X \times S^\infty \subset X^2 \times S^\infty$ induces a natural inclusion $k : X \times P^\infty \subset \Gamma X$. A homotopy equivalence $\psi_V : (R^*(V \times R^\infty), P(V \times R^\infty)) \to (\Gamma V, V \times P^\infty)$ has been constructed in [8, p. 84].

Let $\zeta_V = \psi_V R^* i_V : (R^*V, PV) \to (\Gamma V, V \times P^\infty)$. For an embedding $f : M \to N$, we denote by $[(R^*M, PM), \zeta_N]_{\zeta_N R^* f}$ the set of homotopy classes of homotopy liftings of $\zeta_N R^* f : (R^*M, PM) \to (\Gamma N, N \times P^\infty)$ to (R^*N, PN).

THEOREM 2.1 (Larmore). *If* $2\dim N > 3(\dim M + 1)$, *then for an embedding* $f : M \to N$, *there is a bijection*

$$[M \subset N]_f = [(R^*M, PM), \zeta_N]_{\zeta_N R^* f}.$$

Let $\theta_N = \zeta_N | R^*N : R^*N \to \Gamma N$ and $\rho_N = \zeta_N | PN : PN \to N \times P^\infty$ be the restrictions of ζ_N to R^*N and PN, respectively, and regard them as fibrations in a standard way. Both fibrations have $(n-2)$-connected fibers ($n = \dim N$) [7] (or [8, §5]). Let $\pi_q \theta_N$ and $\pi_q \rho_N$ be sheaves of q-th homotopy groups of fibrations θ_N and ρ_N, respectively, (in this case, both are local systems), and $\pi_q \zeta_N$ a subsheaf of $\pi_q \theta_N$ such that

$$\pi_q \zeta_N = \begin{cases} \pi_q \theta_N & \text{over } \Gamma N - N \times P^\infty, \\ \pi_q \rho_N & \text{over } N \times P^\infty. \end{cases}$$

The sheaves $\pi_q \theta_N$, $\pi_q \rho_N$, and $\pi_q \zeta_N$ for $q = 2n-1, 2n$ are given in [8, Lemmas 5.3.2–5.3.4]. Let $Z[u]$ be a sheaf of coefficients, locally isomorphic to Z, associated with $u = w_1(N^2 \times S^\infty \to \Gamma N) \in H^1(\Gamma N; Z_2)$, and $Z[u]^0$ a subsheaf of $Z[u]$ defined by $Z[u]^0 = Z[u]_{\Gamma N - N \times P^\infty}$.

LEMMA 2.2 (cf. Larmore [8]). *Let* $N = CP^n$ $(n \geq 3)$. *Then*

(1) $\pi_{2n-1}\theta_N$, $\pi_{2n-1}\rho_N$ *and* $\pi_{2n-1}\zeta_N$ *are trivial sheaves of the group* Z.

(2) *The natural projection* $\pi_1 : Z + Z_2 \to Z$ *induces the following exact sequences of sheaves over* ΓN, *which are split if* n *is odd:*

$$0 \to Z_2 \times \Gamma N \to \pi_{2n}\theta_N \xrightarrow{\pi_1} Z[u] \longrightarrow 0,$$

$$0 \to Z_2 \times \Gamma N \to \pi_{2n}\zeta_N \xrightarrow{\pi_1} Z[u]^0 \longrightarrow 0.$$

Let $L_u(Z + Z_2, 2n+1)$ and $L_u(Z, 2n+1)$ be the fiber bundles over ΓN with fiber $K(Z + Z_2, 2n+1)$ and $K(Z, 2n+1)$ associated with the local systems $\pi_{2n}\theta_N$ and $Z[u]$, respectively (see e.g., [10, §3]). The map $\pi_1 : \pi_{2n}\theta_N \to Z[u]$ in Lemma 2.2 induces a bundle map

(2.1) $$\pi_1 : L_u(Z + Z_2, 2n+1) \to L_u(Z, 2n+1) \qquad \text{over } \Gamma N.$$

The 2-stage Postnikov tower for $\zeta_N = (\theta_N, \rho_N) : (R^*N, PN) \to (\Gamma N, N \times P^\infty)$ $(N = CP^n)$ is constructed in §4 as follows:

(2.2)

$$\begin{tikzcd}
PN \arrow[rr, "\subset"] \arrow[d] & & R^*N \arrow[d] \\
E_2' \arrow[rr] \arrow[d] & & E_2 \arrow[d] \\
E_1' \arrow[rr] \arrow[dr, "k_1'"] \arrow[dd, "p_1'"'] & & E_1 \arrow[dr, "k_1"] \arrow[dd, "p_1"'] \\
& K(Z_2, 2n+1) \arrow[rr] & & L_u(Z + Z_2, 2n+1) \arrow[r, "\pi_1"] & L_u(Z, 2n+1) \\
N \times P^\infty \arrow[rr, "k", "\subset"'] & & \Gamma N \arrow[r, "W"] & K(Z, 2n),
\end{tikzcd}$$

(2.3) $\rho_2 W = \varphi(1 \otimes 1) \in H^{2n}(\Gamma N; Z_2)$ in [14, §2] (see also [18, Proposition 2.6]),

(2.4) $\pi_1 k_1$, or $\pi_{1*}k_1 \in H^{2n+1}(E_1; Z[p_1^* u])$, corresponds to the relation $(z \otimes 1 - 1 \otimes z)W = 0$.

Here $H^2(\Gamma N; Z[u]^0) = H^2(\Gamma N, N \times P^\infty; Z[u]) = Z\langle z \otimes 1 - 1 \otimes z\rangle$ (see Lemma 4.1(2)).

By the standard spectral sequence argument, we have the following

LEMMA 2.3 (cf. Larmore [8, (6.1-1)]). *Let* $N = CP^n$ $(n \geq 3)$. *Then for any embedding* $f : M^n \to N$, *there exists an exact sequence*

$$H^{2n-2}(R^*M;(\theta_N R^*f)^{-1}\pi_{2n-1}\zeta_N) \xrightarrow{d_2} H^{2n}(R^*M;(\theta_N R^*f)^{-1}\pi_{2n}\zeta_N)$$
$$\longrightarrow [(R^*M,PM),\zeta_N]_{\zeta_N R^*f} \longrightarrow H^{2n-1}(R^*M;(\theta_N R^*f)^{-1}\pi_{2n-1}\zeta_N) \longrightarrow 0,$$

where d_2 is a cohomology operation associated with the Postnikov invariant of the 2-stage Postnikov tower for ζ_N.

3. Proofs

Before proving Theorem 1.1, we give the proofs of Corollaries 1.2–1.4.

Proof of Corollary 1.2. If $f_\# : \pi_2(M^n) \to \pi_2(CP^n)(= Z)$ is surjective, then so is $f_* : H_2(M^n;Z) \to H_2(CP^n;Z)$ because $\pi_2(CP^n) \cong H_2(CP^n;Z)$. Hence $H^2(M^n;Z(\text{or } Z_2))$ has a direct summand $Z\langle f^*(z)\rangle$ (or $Z_2\langle f^*\rho_2(z)\rangle$) and so the first terms in the short exact sequences of Theorem 1.1 vanish. Therefore, Corollary 1.2 follows. Note that when $w_1(M) = 0$, Corollary 1.2 coincides with [9, Corollary 1.3]. □

Proofs of Corollaries 1.3–1.4. In general, $\pi_1((CP^n)^{M^n}, f) = H^2(M \times (S^1,*);\pi_2(CP^n))(\cong H^1(M;Z))$, by the Eilenberg classification theorem [15, p. 243]. Hence $\pi_1((CP^n)^{M^n}, f) = 0$ if M^n is simply connected or $M^n = RP^n$. Thus Theorem 1.1, together with (1.1), leads to Corollaries 1.3–1.4. □

The rest of this section is devoted to the proof of Theorem 1.1. Theorem 2.1 for $f : M^n \to N = CP^n$, together with Lemmas 2.2–2.3, gives rise to an exact sequence

$$(3.1) \qquad 0 \to \operatorname{coker} d_2 \to [M \subset CP^n]_f \to H^{2n-1}(R^*M;Z) \to 0,$$

where $d_2 : H^{2n-2}(R^*M;Z) \to H^{2n}(R^*M;(\theta_N R^*f)^{-1}\pi_{2n}\zeta_N)$ is determined by the Postnikov invariant k_1 of the Postnikov tower (2.2).

The cohomology group $H^{2n-1}(R^*M;Z)(\cong H^{2n-1}(M^*;Z))$ is calculated by Haefliger [4] (cf. [11, 11.9, 11.19]) as follows:

$$(3.2) \qquad H^{2n-1}(R^*M;Z) = \begin{cases} H^{n-1}(M;Z) & \text{if } n \equiv 1(2), w_1(M) = 0, \\ Z \oplus \ker Sq^1 & \text{if } n \equiv 0(2), w_1(M) \neq 0, \\ H^{n-1}(M;Z_2) & \text{otherwise}, \end{cases}$$

where $Sq^1 : H^{n-1}(M;Z_2) \to H^n(M;Z_2)$.

Let $v = (\theta_N R^*f)^*(u) \in H^1(R^*M;Z_2)$. Since R^*M is a $2n$-manifold with boundary PM, the map π_1 in Lemma 2.2 induces isomorphisms

$$H^{2n}(R^*M;(\theta_N R^*f)^{-1}\pi_{2n}\zeta_N) \overset{\pi_1^*}{\cong} H^{2n}(R^*M;Z[v]^0) \cong H^{2n}(R^*M,PM;Z[v])$$

Hence, by (2.4) we have

$$\text{(3.3)} \qquad \operatorname{coker} d_2 \cong \operatorname{coker} \pi_{1*} d_2 : H^{2n-2}(R^*M; Z) \to H^{2n}(R^*M, PM; Z[v]),$$

where

$$\text{(3.4)} \qquad \pi_{1*} d_2(x) = (\zeta_N R^* f)^*(z \otimes 1 - 1 \otimes z) \cup x.$$

Let $\Lambda^2 V (= V^2/Z_2)$ be the 2-fold symmetric product of V, and $\Delta V = \Delta V/Z_2$. Then $\Lambda^2 V - \Delta V = V^* = R^* V - PV$. The cohomology of $(\Lambda^2 V, \Delta V)$ has been determined by Larmore [6]. We freely use his definitions and notations except for $v = w_1(V^2 - \Delta V \to V^*) \in H^1(V^*; Z_2)$ (v means m in [6]). We set $Z[v]^{\Lambda^2 V} = Z[v]$ as in [6].

There exists an excision isomorphism

$$\text{(3.5)} \qquad e : H^*(\Lambda^2 V, \Delta V; G) \cong H^*(R^* V, PV; G) \qquad \text{for } G = Z, Z[v] \text{ and } Z_2.$$

For an n-manifold M, let $H^n(M; Z) = Z\langle M\rangle$ or $= Z_2\langle \beta_2 M'\rangle$, according as M is orientable or not, and let $H^n(M; Z_2) = Z_2\langle M\rangle$. Then, by [6] and [17, Proposition 5.2], we have

LEMMA 3.1 (Larmore, Yasui). (1) *If* $n \equiv 1(2)$, $w_1(M) = 0$, *then*

$$H^{2n}(\Lambda^2 M, \Delta M; Z[v]) = Z\langle \Delta(M, M)\rangle;$$

(2) *otherwise* $\rho_2 : H^{2n}(\Lambda^2 M, \Delta M; Z[v]) \xrightarrow{\cong} H^{2n}(\Lambda^2 M, \Delta M; Z_2) = Z_2\langle \Lambda M \Lambda M\rangle$ *is an isomorphism.*

Let $i : R^*M \subset (R^*M, PM)$ and $j : PM \subset R^*M$ be the natural inclusions. The commutative diagram below indicates that the map ρ in [14], and so [18, (2.2)], is reworded as

$$\text{(3.6)} \qquad \rho = j^* \theta_M^* : H^*(\Gamma M; Z_2) \to H^*(R^*M; Z_2) \xrightarrow{\cong} H^*(M^*; Z_2).$$

$$\begin{tikzcd}
M^* \arrow[r, "j", "\simeq"'] \arrow[d, equal] \arrow[dr, "\theta'_M"] & R^*M \arrow[r, "R^* i_M"] \arrow[dr, "\theta_M"] & R^*(M \times R^\infty) \arrow[d, "\psi_M", "\simeq"'] \\
M^* & (M^2 - \Delta M) \times z_2 S^\infty \arrow[l, "p'"', "\simeq"] \arrow[r, "i'"] & \Gamma M,
\end{tikzcd}$$

where p' and i' are the natural projection and inclusion, respectively, and θ'_N is determined in the diagram. Further [18, Lemma 3.3(2)] is reworded as

$$\text{(3.7)} \qquad i^* e(\Lambda x \Lambda y) = \theta_M^*(x \otimes y + y \otimes x + xy \otimes 1 + 1 \otimes xy) \in H^*(R^*M; Z_2).$$

Sublemma. (1) *If* $n \equiv 1(2), w_1(M) = 0,$ *let* $H^{n-1}(M; Z) \equiv \sum_{1 \le i \le \alpha} Z\langle x_i \rangle$ mod *torsion. Then*

$$H^{2n-2}(R^*M; Z) \equiv \sum_{1 \le i \le \alpha} Z\langle (1/2) i^* e(\Lambda x_i \Lambda x_i)\rangle + \sum_{1 \le i < j \le \alpha} Z\langle i^* e(\Lambda x_i \Lambda x_j)\rangle$$

$$+ \{i^* e(\Lambda x \Lambda M) \mid x \in H^{n-2}(M; Z)\} \bmod \textit{torsion}.$$

(2) *Otherwise* $\rho_2 H^{2n-2}(R^*M; Z)$ *contains the subgroup*

$$\{\theta_M^*(U_M(x \otimes 1)) \mid x \in H^{n-2}(M; Z_2)\} \qquad \text{if } n \equiv 0(2), w_1(M) = 0,$$

$$\{\theta_M^*(Sq^1(x \otimes M' + M' \otimes x)) \mid x \in H^{n-2}(M; Z_2)\} \quad \text{if } w_1(M) \ne 0,$$

where $U_M \in H^n(M^2; Z_2)$ *is the* Z_2*-Thom class of* M.

Proof. The statement (1) is obtained in the same way as in the proof of [18, Theorem 4.3] for $n \equiv 0(2)$, $w_1(M) = 0$. Details are omitted. On the other hand, (2) for $n \equiv 0(2)$ follows from (3.6) and [18, Lemma 2.9(2)]; while (2) for $w_1(M) \ne 0$ is obvious. □

Let $\pi : (N^2, \Delta N) \to (\Lambda^2 N, \Delta N)$ be the natural projection. By [6], the element $\Lambda x \in H^2(\Lambda^2 V, \Delta V; Z[v])$ for $x \in H^2(V; Z)$ satisfies

$$\pi^*(\Lambda x) = x \otimes 1 - 1 \otimes x \in H^2(V^2, \Delta V; Z).$$

Lemma 3.2. *If V is simply connected, then for any $x \in H^2(V; Z)$, we have*

$$e(\Lambda x) = \zeta_V^*(x \otimes 1 - 1 \otimes x) \in H^2(R^*V, PV; Z[v]).$$

Proof. Let $\pi : V^2 - \Delta V \to V^*$ be the natural projection. Then, by a simple calculation, we have $\pi^* j^* i^* e(\Lambda x) = \pi^* j^* i^* \zeta_V^*(x \otimes 1 - 1 \otimes x)$ in $H^2(V^2 - \Delta V; Z)$. Here j^* is an isomorphism. Both i^* and π^* are injective, because we see easily that $H^1(R^*V; Z[v]) \to H^1(PV; Z[v]) (= Z_2\langle \beta_2^v 1\rangle)$ is surjective and that $H^1(V^*; Z) = 0$ by considering the cohomology spectral sequence of $V^2 - \Delta V \to V^* \to P^\infty$, respectively. This leads to the lemma. □

Hence, for an embedding $f : M^n \to CP^n$, there are relations

(3.8) $\quad e(\Lambda f^*(z)) = e(\Lambda^2 f)^*(\Lambda z) = (R^* f)^* e(\Lambda z) = (\zeta_N R^* f)^*(z \otimes 1 - 1 \otimes z).$

Lemmas 3.1–3.2, (3.3)–(3.5) and (3.8) imply

(3.9)
$\operatorname{coker} d_2$

$$\cong \begin{cases} H^{2n}(R^*M, PM; Z[v]) / e(\Lambda f^*(z)) H^{2n-2}(R^*M; Z) & \text{if } n \equiv 1(2), w_1(M) = 0, \\ H^{2n}(R^*M, PM; Z_2) / e(\Lambda f^* \rho_2(z)) \rho_2 H^{2n-2}(R^*M; Z) & \text{otherwise.} \end{cases}$$

The following lemma, together with (3.1)–(3.2) and (3.9), implies Theorem 1.1.

LEMMA 3.3. *Under the assumption of Theorem* 1.1,

$$H^{2n}(R^*M, PM; Z[v])/e(\Lambda f^*(z))H^{2n-2}(R^*M; Z)$$

$$\cong H^n(M; Z)/f^*(z)H^{n-2}(M; Z) \qquad \text{if } n \equiv 1(2), w_1(M) = 0,$$

$$H^{2n}(R^*M, PM; Z_2)/e(\Lambda f^*\rho_2(z))\rho_2 H^{2n-2}(R^*M; Z)$$

$$\cong H^n(M; Z_2)/f^*\rho_2(z)H^{n-2}(M; Z_2) \qquad \text{otherwise}.$$

PROOF. Case 1: $n \equiv 1(2), w_1(M) = 0$. Since $H^{2n}(R^*M, PM; Z[v]) = Z$ by Lemma 3.1, it is sufficient to calculate $(e\Lambda f^*(z))(H^{2n-2}(R^*M; Z)/\text{torsion})$. By [6, Theorem 14], we have the following relations

$$(\Lambda x_i \Lambda x_j)\Lambda f^*(z) = 0 \qquad \text{for } 1 \leq i \leq j \leq \alpha,$$

$$(\Lambda x \Lambda M)\Lambda f^*(z) = \pm\Delta(xf^*(z), M) \qquad \text{for } x \in H^{n-2}(M; Z) \text{ of order infinite.}$$

Hence $e(\Lambda f^*(z))H^{2n-2}(R^*M; Z) \cong f^*(z)H^{n-2}(M; Z)$.

Case 2: $w_1(M) \neq 0$. If $f^*\rho_2(z) = 0$, then the lemma is obvious. Therefore we assume that $f^*\rho_2 z \neq 0$. For $x \in H^{n-2}(M; Z_2)$, we have, by (3.7) and [6, Theorem 11],

$$\theta_M^*(Sq^1(x \otimes M' + M' \otimes x))e(\Lambda f^*\rho_2(z)) = i^*e(\Lambda Sq^1 x \Lambda M' + \Lambda x \Lambda M)e(\Lambda f^*\rho_2(z))$$
$$= e(\Lambda x f^*\rho_2(z)\Lambda M).$$

Since $f^*\rho_2(z)H^{n-2}(M; Z_2) = H^n(M; Z_2)$ by the assumption $f^*\rho_2(z) \neq 0$, we have the lemma in case $w_1(M) \neq 0$.

Case 3: $n \equiv 0(2), w_1(M) = 0$. If $f^*\rho_2(z) = 0$, then the lemma follows immediately. We may assume that $f^*\rho_2(z) \neq 0$. In this case $(Sq^1 + w_1(M)) \cdot H^{n-2}(M; Z_2) = 0$ by the assumption of Theorem 1.1. Therefore, by [18, (2.5) and Proposition 2.6], $U_M(x \otimes 1) \in H^{2n-2}(\Gamma M; Z_2)$ for $x \in H^{n-2}(M; Z_2)$ can be described as

$$U_M(x \otimes 1) = (M \otimes x + x \otimes M) + \sum (x' \otimes x'' + x'' \otimes x')$$

for some $x', x'' \in H^{n-1}(M; Z_2)$ with $x' \neq x''$. Using (3.7) and [6, Theorem 11], we have

$$\theta_M^*(U_M(x \otimes 1)e(\Lambda f^*\rho_2(z))) = e\Big(\Big(\Lambda M \Lambda x + \sum \Lambda x' \Lambda x''\Big)\Lambda f^*\rho_2(z)\Big)$$
$$= e(\Lambda M \Lambda x f^*\rho_2(z)),$$

thereby completing the proof of the case 3. □

Thus we have Theorem 1.1.

4. Construction of the Postnikov tower

In this section, N stands for CP^n. We use the results in [14, §2] on $H^*(\Gamma N; Z_2)$ freely. Let β_2^u be the Bockstein operator associated with the exact sequence $0 \to Z[u] \to Z[u] \to Z_2 \to 0$ for $u \in H^1(\Gamma N; Z_2)$.

LEMMA 4.1. *Let* $N = CP^n$. *Then*

(1) *the reduction* mod 2 *induces an isomorphism*

$$\rho_2 : H^{odd}(\Gamma N; Z[u]) = \sum_{0 \le i, 0 \le j \le n} Z_2\langle \beta_2^u(u^{2i} \otimes (z^j)^2\rangle \xrightarrow{\cong} H^{odd}(\Gamma N : Z_2),$$

(2) *the natural inclusion* $q : N^2 \subset \Gamma N$ *induecs an isomorphism*

$$q^* : H^{even}(\Gamma N; Z[u]) \xrightarrow{\cong} \sum_{0 \le i < j \le n} Z\langle z^j \otimes z^i - z^i \otimes z^j \rangle,$$

(3) *the natural inclusion induces an isomorphism* $H^2(\Gamma N, N \times P^\infty; Z[u]) \cong H^2(\Gamma N; Z[u])$,

(4) $\theta_N^* : H^{odd}(\Gamma N; Z[u]) \to H^{odd}(R^*N; Z[v])$ *is surjective.*

PROOF. The E_2-term of the cohomology spectral sequence for $N^2 \subset \Gamma N \to P^\infty$ is given by $E_2^{s,t} = H^s(P^\infty; H^t(N^2; Z)_{\tilde{\phi}})$, where $H^t(N^2; Z)_{\tilde{\phi}}$ is the local system associated with $\tilde{\phi} : \pi_1(P^\infty) = Z_2\langle a \rangle \to \mathrm{Aut}(H^t(N^2; Z))$ defined as follows: Let $\phi : \pi_1(P^\infty) \to \mathrm{Aut}(Z)$ be a non-trivial map and $T : N^2 \to N^2$ be the switching map. Then $\tilde{\phi}(a) = T^*\phi(a)_* : H^t(N^2; Z) \xrightarrow{\phi(a)_*} H^t(N^2; Z) \xrightarrow{T^*} H^t(N^2; Z)$. By [5, §3], we have

$$\begin{aligned} &H^s(P^\infty; Z^2\langle z^i \otimes z^j - z^j \otimes z^i, z^i \otimes z^j \rangle_{\tilde{\phi}}) \\ &\quad = \begin{cases} 0 & \text{if } s \ne 0, i \ne j, \\ Z\langle z^i \otimes z^j - z^j \otimes z^i \rangle & \text{if } s = 0, i \ne j; \end{cases} \end{aligned}$$

$$H^s(P^\infty; Z\langle z^i \otimes z^i \rangle_{\tilde{\phi}}) = \begin{cases} Z_2 & \text{if } s \text{ is odd,} \\ 0 & \text{if } s \text{ is even.} \end{cases}$$

Thus $H^*(\Gamma N; Z[u])$ has no odd torsion. In $H^*(\Gamma N; Z_2)$, we have $\rho_2\beta_2^u(u^{2j} \otimes (z^i)^2) = (Sq^1 + u)(u^{2j} \otimes (z^i)^2) = u^{2j+1} \otimes (z^i)^2$ and $\rho_2\beta_2^u(I^*) = 0$ by [1, Lemma 11] (see also [18, p. 563]). Hence (1) follows immediately. This implies that all differentials in the spectral sequence are trivial and so (2) follows. A simple calculation yields that $H^1(\Gamma N; Z[u]) \cong H^1(N \times P^\infty; Z[u]) = Z_2\langle \beta_2^u(1) \rangle$ and $H^2(N \times P^\infty; Z[u]) = 0$, and so (3) follows. In the same way as in (1), we see that $H^*(R^*M; Z[v])$ has no odd torsion. $H^{odd}(R^*N; Z_2) = vH^{even}(R^*N; Z_2)$ because of $H^{\mathrm{odd}}(\Gamma N; Z_2) = uH^{\mathrm{even}}(\Gamma N; Z_2)$ and the surjectivity of $\theta_N^* : H^*(\Gamma N; Z_2) \to H^*(R^*N; Z_2)$. Hence $H^{odd}(R^*N; Z[v]) = \beta_2^v H^{even}(R^*N; Z_2) = \theta_N^* \beta_2^u H^{even}(\Gamma N; Z_2)$. Thus (4) follows. $\square$

Construction of the Postnikov tower for ζ_N. Let F be the homotopy fiber of $\theta_N : R^*N \to \Gamma N$ and $\iota_F \in H^{2n-1}(F; Z) (= Z$, see §2) the fundamental class of F. Then ι_F is transgressive. We denote $\tau(\iota_F) = W \in H^{2n}(\Gamma N; Z) \cap \ker \theta_N^*$. Since θ_N^* is surjective on Z_2-cohomology [14, §2], we have $\rho_2 W \neq 0$ and therefore

$$(4.1) \qquad \rho_2(W) = \varphi(1 \otimes 1).$$

The first stage Postnikov tower for θ_N is the principal fibration $p_1 : E_1 \to \Gamma N$ with classifying map W and there is a homotopy lifting $q_1 : R^*N \to E_1$ of θ_N.

Let F' be the homotopy fiber of q_1. Then F' is also the homotopy fiber of $\iota_F : F \to K(Z, 2n-1)$. Further F' is $(2n-1)$-connected and $\pi_{2n}(F') = \pi_{2n}(F) = Z + Z_2$. The $\pi_1(E_1)$-action on $\pi_{2n}(F')$ is induced from the $\pi_1(\Gamma N)$-action on $\pi_{2n}(F)$. The fundamental class $\iota_{F'} \in H^{2n}(F'; Z + Z_2)$ of F' is transgressive, e.g., [10, Theorem 4.1]. To calculate $\operatorname{coker} d_2$ in (3.1), the equality (3.3) indicates that it is sufficient to determine $\pi_{1*}\tau(\iota_{F'}) \in H^{2n+1}(E_1; Z[p_1^* u]) \cap \ker q_1^*$. Consider the diagram (cf. [13, Lemma 4])

$$\begin{tikzcd}
R^*N \times K(Z, 2n-1) \arrow[r, "v"] \arrow[d, shift left, "\pi"] & E_1 \arrow[d, "p_1"] & \\
R^*N \arrow[u, shift left, "s"] \arrow[ur, "q_1"] \arrow[r, "\theta_N"] & \Gamma N \arrow[r, "W"] & K(Z, 2n).
\end{tikzcd}$$

LEMMA 4.2. $\ker \theta_N^* \cap H^{2n+1}(\Gamma N; Z[u]) \subset \ker p_1^*$.

PROOF. We see that $\ker \theta_N^* \cap H^{2n+1}(\Gamma N; Z_2) = Z_2\langle \varphi(u \otimes 1) \rangle$ by [14] (see [18, §2]) and $\varphi(u \otimes 1) = \rho_2 \beta_2^u \varphi(1 \otimes 1)$ by a simple calculation, while using the relation on $Sq^1(u^i \otimes x^2)$ [1, Lemma 11] (see also [18, p. 563]. Thus $\ker \theta_N^* \cap H^{2n+1}(\Gamma N; Z[u]) = Z_2\langle \beta_2^u \varphi(1 \otimes 1) \rangle$ by Lemma 4.1. On the other hand $\beta_2^u \varphi(1 \otimes 1) = \beta_2^u \rho_2(W) \in \ker p_1^*$ by (4.1). $\square$

As in [13, Property 5], Lemmas 4.1(4) and 4.2 lead to an exact sequence

$$0 \longrightarrow H^{2n+1}(E_1; Z[p_1^* u]) \xrightarrow{v^*} H^{2n+1}(R^*N \times K(Z, 2n-1); Z[v] \otimes Z)$$

$$\xrightarrow{\tau_1} H^{2n+2}(\Gamma N; Z[u]).$$

Here $H^{2n+1}(R^*N \times K(Z, 2n-1); Z[v] \otimes Z) = H^{2n+1}(R^*N; Z[v]) \oplus Z\langle \theta_N^*(z \otimes 1 - 1 \otimes z) \times \iota_{2n-1} \rangle$ by Lemma 4.1 and the fact that θ_N is $(2n-2)$-equivalence, and $\tau_1(\theta_N^*(z \otimes 1 - 1 \otimes z) \times \iota_{2n-1}) = (z \otimes 1 - 1 \otimes z) W$. Since

$\theta_N^*(W)=0$ implies $i^*q^*(W)=0$ for $i: N^2-\Delta N\subset N^2$, the element $q^*(W)$ can be described as $q^*(W)=mU_N$ for some m, where U_N denotes the integral Thom class of N. Hence $(z\otimes 1-1\otimes z)q^*(W)=0$ because $(x\otimes 1)U_N=(1\otimes x)U_N$, and so $(z\otimes 1-1\otimes z)W=0$ by Lemma 4.1(2). Further there exists a unique element $k_1\in H^{2n+1}(E_1;Z[p_1^*u])$ satisfying the two conditions $H^{2n+1}(E_1;Z[p_1^*u])\cap\ker q_1^*=Z\langle k_1\rangle$ and $v^*(k_1)=(z\otimes 1-1\otimes z)\iota_{2n-1}$.

Summing up the argument, we get the Postnikov tower for θ_N. The Postnikov tower for $\zeta_N=(\theta_N,\rho_N):(R^*N,PN)\to(\Gamma N,N\times P^\infty)$, which is used in §2, is induced from that of θ_N.

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