Hiroshima Math. J.
30 (2000), 21–27

Two-point boundary value problem for first order implicit differential equations

Huizhao Liu and Daqing Jiang

(Received November 5, 1998)

Abstract. In this paper, we investigate a two-point boundary value problem for first order implicit differential equations. By using the monotone iterative technique, we obtain the extremal solutions for this problem, from the lower and upper solutions.

1. Introduction and main results

Initial and periodic boundary value problems for the following first order differential equation

$$u'(t) = f(t, u(t)), \qquad t \in J := [0, T],$$

where $f \in C[J \times R, R]$, have long been studied by means of the monotone iterative technique, see [1, 2]. Recently, they have been investigated again by employing the method of generalized quasilinearization, a monotone iterative technique, see [3–5]. However, more general two-point boundary value problems for a first order differential equation have not been examined yet.

In this paper, we use the monotone iterative technique to investigate a general two-point boundary value problem of the form

$$\begin{cases} u'(t) = f(t, u(t), u'(t)), & t \in J := [0, T], \\ u(0) = \lambda u(T) + \mu \end{cases} \tag{1.1}$$

where the equation is implicit, $\lambda \geq 0$, μ are given real numbers, and the nonlinear function $f(t, u, v)$ is assumed to be a Carathéodory function.

We say that $f : J \times R^2 \to R$ is a Carathéodory function, if it possesses the following three properties:

(i) For given $(u, v) \in R^2$, the function $t \to f(t, u, v)$ is measurable on J.

(ii) For almost all $t \in J$, the function $(u, v) \to f(t, u, v)$ is continuous on R^2.

2000 *Mathematics Subject Classification.* 34B15

Key words and phrases. implicit differential equations, two-point boundary value problem, extremal solution.

(iii) For given $N > 0$, there exists a $g_N \in L^p[J, R], 1 < p < +\infty$, such that

$$|f(t, u, v)| \leq g_N(t) \qquad \text{for a.e. } t \in J \qquad \text{and} \qquad |u|, |v| \leq N. \tag{1.2}$$

Clearly, the two-point boundary value problem (1.1) becomes an initial value problem if $\lambda = 0$ and a periodic boundary value problem if $\lambda = 1, \mu = 0$.

To develop a monotone method, we need the conception of upper and lower solutions.

We say that $u \in W^{1,p}[J, R], 1 < p < +\infty$, is an upper solution of the problem (1.1) if

$$\begin{cases} u'(t) \geq f(t, u(t), u'(t)) & \text{for a.e. } t \in J \\ u(0) \geq \lambda u(T) + \mu, \end{cases}$$

and a lower solution of (1.1) if the reserved inequalities hold. Here

$$W^{1,p}[J, R] := \{u \in C[J, R] : u(t) \text{ absolutely continuous on } J \text{ and } u'(t) \in L^p[J, R]\}.$$

A function $u \in W^{1,p}[J, R]$ is called a solution of the two-point boundary value problem (1.1) if it is an upper and lower solution of (1.1).

Concerning the function $f(t, u, v)$, we make the following hypotheses:

(H1) For given $\alpha, \beta \in C[J, R], \alpha(t) \leq \beta(t)$ on J, there exists a $K \in (0, 1)$ such that

$$|f(t, u, v) - f(t, u, \bar{v})| \leq K|v - \bar{v}| \qquad \text{for a.e. } t \in J,$$

whenever $\alpha(t) \leq u \leq \beta(t), v, \bar{v} \in R$.

(H2) for given $\alpha, \beta \in C[J, R], \alpha(t) \leq \beta(t)$ on J, there exists an $M \in L^p[J, R^+], 1 < p < +\infty, R^+ := (0. + \infty)$, such that

$$f(t, u, v) - f(t, \bar{u}, v) \geq -M(t)(u - \bar{u}) \qquad \text{for a.e. } t \in J,$$

whenever $\alpha(t) \leq \bar{u} \leq u \leq \beta(t), v \in R$.

The main result of this paper is as follows.

THEOREM 1. *Let α_0, β_0 be lower and upper solutions of the problem* (1.1), *respectively, and $\alpha_0(t) \leq \beta_0(t)$ on J. Suppose that $f(t, u, v)$ is a Carathéodory function satisfying* (H1) *and* (H2). *Then there exist two monotone sequences $\{\alpha_n\}, \{\beta_n\}$, nondecreasing and nonincreasing, respectively, which converge uniformly to the extremal solutions of the two-point boundary value problem* (1.1) *in the segment*

$$[\alpha_0, \beta_0] := \{u \in C[J, R] : \alpha_0(t) \leq u(t) \leq \beta_0(t) \text{ on } J\}.$$

The above theorem extends and improves some results in [1, 2].

2. Proof of Theorem 1

In the present section, we shall describe a monotone method which yields two monotone sequences that converge uniformly to extremal solutions of the problem (1.1).

To prove the validity of the monotone method, we need the following statement.

LEMMA 2. *Let* $y \in W^{1,p}[J,R], 1 \le p \le +\infty$ *and satisfy*

$$\begin{cases} y'(t) + K|y'(t)| + M(t)y(t) \ge 0 & \text{for a.e. } t \in J, \\ y(0) \ge 0 \end{cases}$$

where $K \in [0,1)$ *and* $M \in L^p[J,R^+]$. *Then* $y(t) \ge 0$ *on* J.

PROOF. If the lemma is not true, then there exists a point $b \in (0,T]$ such that $y(b) < 0$, and hence, a point $a \in [0,b)$ such that $y(t) < 0$ in $(a,b]$ and $y(a) = 0$, since $y(0) \ge 0$. As a result, we have

$$y'(t) + K|y'(t)| \ge -M(t)y(t) > 0 \qquad \text{for a.e. } t \in (a,b].$$

This implies that $y'(t) > 0$ for a.e. $t \in (a,b]$ and then leads to a contradiction $0 > y(b) > y(a) = 0$. The lemma is thus proved. □

We now assume that α_0, β_0 are lower and upper solutions of problem (1.1), respectively, $\alpha_0(t) \le \beta_0(t)$ on J and $f(t,u,v)$ is a Carathéodory function satisfying (H1) and (H2) and consider the following initial value problem

$$\begin{cases} u'(t) + M(t)u(t) = F(t,\eta(t),u'(t)), & t \in J, \\ u(0) = \lambda\eta(T) + \mu, \end{cases} \tag{2.1}$$

where $F(t,u,v) := f(t,u,v) + M(t)u$ and $\eta \in [\alpha_0, \beta_0]$.

Concerning the initial value problem (2.1), the following statement holds.

LEMMA 3. *For each fixed* $\eta \in [\alpha_0, \beta_0]$, *the initial value problem* (2.1) *has a unique solution* $u \in W^{1,p}[J,R]$.

PROOF. We define a mapping $A : L^p[J,R] \to L^p[J,R]$ by

$$(Aw)(t) := -M(t)\left\{\int_0^t w(s)ds + \lambda\eta(T) + \mu\right\} + F(t,\eta(t),w(t)),$$

where $L^p[J,R]$ denotes particular the Banach space endowed with norm

$$\|w\| := \left\{\int_0^T \exp\left(-\sigma^p \int_0^t M^p(r)dr\right)|w(t)|^p dt\right\}^{1/p},$$

$$\sigma > T^{(p-1)/p}(1-K)^{-1}.$$

From the definition of A, it follows that for any $w_1, w_2 \in L^p[J, R]$

$$\|Aw_1 - Aw_2\| \leq \left\| M(t) \int_0^t |w_1(s) - w_2(s)| ds \right\| + K\|w_1 - w_2\|. \tag{2.2}$$

Here we have used the hypothesis (H1). Note that

$$\begin{aligned}
&\left\| M(t) \int_0^t |w_1(s) - w_2(s)| ds \right\|^p \\
&\quad = \int_0^T \exp\left(-\sigma^p \int_0^t M^p(r) dr\right) M^p(t) \left(\int_0^t |w_1(s) - w_2(s)| ds\right)^p dt \\
&\quad \leq \int_0^T \exp\left(-\sigma^p \int_0^t M^p(r) dr\right) M^p(t) t^{p-1} \left(\int_0^t |w_1(s) - w_2(s)|^p ds\right) dt \\
&\quad \leq T^{p-1} \int_0^T |w_1(s) - w_2(s)|^p \left\{\int_s^T M^p(t) \exp\left(-\sigma^p \int_0^t M^p(r) dr\right) dt\right\} ds \\
&\quad \leq T^{p-1} \sigma^{-p} \int_0^T |w_1(s) - w_2(s)|^p \exp\left(-\sigma^p \int_0^s M^p(r) dr\right) ds \\
&\quad \leq T^{p-1} \sigma^{-p} \|w_1 - w_2\|^p.
\end{aligned} \tag{2.3}$$

Here we have used the Hölder inequality. The inequality (2.2) together with (2.3) yields for any $w_1, w_2 \in L^p[J, R]$

$$\|Aw_1 - Aw_2\| \leq (T^{(p-1)/p} \sigma^{-1} + K) \|w_1 - w_2\|,$$

which shows that A is a contraction mapping. The Banach contraction principle tells us that A has a unique fixed point in $L^p[J, R]$.

Let w be the unique fixed point of A. Then

$$w(t) = -M(t) \left\{\int_0^t w(s) ds + \lambda \eta(T) + \mu\right\} + F(t, \eta(t), w(t)) \qquad \text{for a.e. } t \in J. \tag{2.4}$$

Put

$$u(t) := \int_0^t w(s) ds + \lambda \eta(T) + \mu, \qquad t \in J. \tag{2.5}$$

It is easy to see that the function $u(t)$, defined by (2.4) and (2.5), is a unique solution of (2.1). This proves Lemma 3. □

Now let us define a mapping $Z : [\alpha_0, \beta_0] \to C[J, R]$ by setting $(Z\eta)(t) := u(t)$, where $u(t)$ is the unique solution of (2.1) with $\eta \in [\alpha_0, \beta_0]$. It follows by Lemma 3 that the mapping Z is well defined.

Concerning the mapping Z, we can prove the following statement.

LEMMA 4. *The mapping Z possesses the following two properties:*
(i) $\eta_1, \eta_2 \in [\alpha_0, \beta_0]$ *and* $\eta_1(t) \le \eta_2(t)$ *for all* $t \in J$ *imply that* $(Z\eta_1)(t) \le (Z\eta_2)(t)$ *on* J.
(ii) $\alpha_0(t) \le (Z\alpha_0)(t)$ *and* $(Z\beta_0)(t) \le \beta_0(t)$ *on* J.

PROOF. We now prove the property (i). Let

$$u_i(t) := (Z\eta_i)(t), \quad i = 1, 2, \qquad \text{and} \qquad y(t) = u_2(t) - u_1(t).$$

Then we have

$$\begin{cases} y'(t) + M(t)y(t) = f(t, \eta_2(t), u_2'(t)) - F(t, \eta_1(t), u_1'(t)) \\ \qquad\qquad\qquad\qquad \ge -K|y'(t)| \qquad \text{for a.e. } t \in J \\ y(0) \ge 0 \end{cases}$$

Here we have used hypotheses (H1) and (H2). From Lemma 2, we conclude that $y(t) = u_2(t) - u_1(t) \ge 0$ on J. This shows that property (i) is true.

In very much the same way, we can prove property (ii). The proof is thus complete. □

We can now define the sequences $\alpha_{n+1} := Z\alpha_n, \beta_{n+1} := Z\beta_n, n = 0, 1, 2, \dots,$ and conclude from Lemma 4 that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0 \qquad \text{on } J.$$

From (2.4) and (2.5), we know that

$$\begin{cases} \alpha_{n+1}'(t) + M(t)\alpha_{n+1}(t) = f(t, \alpha_n(t), \alpha_{n+1}'(t)) + M(t)\alpha_n(t) \quad \text{for a.e. } t \in J, \\ \alpha_{n+1}(t) = \int_0^t \alpha_{n+1}'(s)ds + \lambda\alpha_n(T) + \mu. \end{cases} \tag{2.6}$$

Whence it follows that for all $n \ge 0$

$$\begin{aligned} |\alpha_{n+1}'(t)| &\le M(t)(|\alpha_{n+1}(t)| + |\alpha_n(t)|) + |f(t, \alpha_n(t), 0)| \\ &\quad + |f(t, \alpha_n(t), \alpha_{n+1}'(t) - f(t, \alpha_n(t), 0)| \\ &\le 2NM(t) + g_N(t) + K|\alpha_{n+1}'(t)| \qquad \text{for a.e. } t \in J, \end{aligned}$$

i.e.

$$|\alpha_{n+1}'(t)| \le \frac{1}{1-K}\{2NM(t) + g_N(t)\} =: G(t) \qquad \text{for a.e. } t \in J,$$

where $N := \max\{|\alpha_0(t)| + |\beta_0(t)| : t \in J\}$ and $g_N(t)$ is determined by (1.2). This shows that the sequence $\{\alpha_n'\}$ is a bounded set of $L^p[J, R]$.

As a result, we have for any $n \geq 1, t, s \in J, t > s$,

$$|\alpha_n(t) - \alpha_n(s)| \leq \int_s^t |\alpha_n'(r)| dr \leq \int_s^t G(r) dr$$

$$\leq |t - s|^{(p-1)/p} \left(\int_0^T G^p(r) dr \right)^{1/p},$$

which implies that $\{\alpha_n(t)\}$ is equicontinuous on J. It follows by Arzela-Ascoli theorem that there is a subsequence of $\{\alpha_n(t)\}$ which is uniformly convergent and hence so is the sequence $\{\alpha_n(t)\}$ itself.

To prove the convergence of $\{\alpha_n'(t)\}$, we need the following well-known facts. (see, e.g., [6] and [7, P 31].)

(i) $L^p[J, R], 1 < p < +\infty$ is a uniformly convex, reflexive Banach space.

(ii) A reflexive Banach space is sequentially weakly complete.

(iii) A bounded set in a reflexive Banach space is weakly sequentially complete.

(iv) In a uniformly convex Banach space, $x_n \rightharpoonup x$ weakly and $\|x_n\| \to \|x\|$ imply that $x_n \to x$ strongly.

From the facts above-mentioned, we can select a subsequence of $\{\alpha_n'(t)\}$, which strongly converges to some $w^* \in L^p[J, R]$. Let α^* be the uniform limit of $\{\alpha_n(t)\}$. Inserting the (strongly) convergent subsequence of $\{\alpha_n'(t)\}$ and the corresponding subsequence of $\{\alpha_n(t)\}$ into (2.6) and then taking the limit, we obtain

$$w^* + M(t)\alpha^*(t) = f(t, \alpha^*(t), w^*(t)) + M(t)\alpha^*(t) \qquad \text{for a.e. } t \in J,$$

$$\alpha^*(t) = \int_0^t w^*(s) ds + \lambda \alpha^*(T) + \mu.$$

This shows that $\alpha^*(t)$ is a solution of the problem (1.1)

It must be point out that because all (strongly) convergent subsequences of $\{\alpha_n'(t)\}$ have the same limit $w^*(t) = \dfrac{d\alpha^*(t)}{dt}$ for a.e. $t \in J$, the selection is unnecessary and $w^*(t)$ is certainly the (strong) limit of $\{\alpha_n'(t)\}$.

In very much the same way, we can prove that $\{\beta_n(t)\}$ is uniformly convergent, $\{\beta_n'(t)\}$ (strongly) convergent, and $\beta^*(t)$, the uniform limit of $\{\beta_n(t)\}$, is a solution of the problem (1.1) as well.

Finally, it follows, employing standard argument (see [1]), that α^* and β^* are respectively minimal and maximal solutions of the problem (1.1) in the segment $[\alpha_0, \beta_0]$. This completes the proof of Theorem 1.

Acknowledgement

The authors thank the referee for his valuable suggestion

References

[1] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations,* Pitman, Boston (1985).

[2] V. Lakshmikantham and S. Leela, *Existence and Monotone method for periodic solutions of first order differential equations,* J. Math. Analysis Appl. **91** (1983), 237–243.

[3] V. Lakshmikantham, N. Shahazad and J. J. Nieto, *Methods of generalized quasilinearization for periodic boundary value problems,* Nonlinear Analysis **27** (1996), 143–151.

[4] V. Lakshmikantham, *Further improvement of generalized quasilinearzation method,* Nonlinear Analysis **27** (1996), 223–227.

[5] V. Lakshmikantham, S. Leela and F. A. McRae, Improved generalized quasilinearzation method, Nonlinear Analysis **24** (1995), 1627–1637.

[6] R. A. Adams, *Sobelev Space,* Academidc Press, New York (1975).

[7] M. S. Berger, *Nonlinearity and Functional Analysis,* Academic Press, New York (1977).

Huizhao Liu
Department of Applied Mathematics
Hebei University of Technology
Tianjun 300130
People's Republic of China

Daqing Jiang
Department of Mathematics
Northeast Normal University
Changchun 130024
People's Republic of China