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Liftings of Pettis integrable functions

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Abstract. It is proved, that if the lifting of a bounded Pettis integrable function is appropriately measurable, then it is also Pettis integrable.

Introduction

Throughout this paper (Ω, Σ, μ) is a finite complete measure space, ρ is a lifting on $L_\infty(\mu)$, X denotes an arbitrary Banach space and X_c^{**} is the set of all $x^{**} \in X^{**}$ which are weak*-cluster points of bounded countable subsets of X.

If $f : \Omega \to X$ is a weakly measurable and scalarly bounded (or $f : \Omega \to X^{**}$ is weak*-measurable and weak*-bounded), then $\rho_1(f) : \Omega \to X^{**}$ is the unique function satisfying for each $x^* \in X^*$ and each $\omega \in \Omega$ the equality $\langle \rho_1(f)(\omega), x^* \rangle = \rho(x^* f)(\omega)$ (cf [2], VI. 4). If $f : \Omega \to X^*$ is a weak*-measurable and weak*-bounded, then $\rho_0(f) : \Omega \to X^*$ is the unique function satisfying for each $x \in X$ and each $\omega \in \Omega$ the equality $\langle \rho_0(f)(\omega), x \rangle = \rho(xf)(\omega)$.

It is known (cf [6]), that $\rho_1(f) : \Omega \to X^{**}$ and $\rho_0(f) : \Omega \to X^*$ are weak*-Borel measurable and the measures $\xi_0 := \mu\rho_0(f)^{-1}$ and $\xi_1 := \mu\rho_1(f)^{-1}$ are Radon measures on the completions Ξ_f^0 and Ξ_f^1 of the σ-algebras of weak*-Borel subsets of X^* and X^{**} respectively. If $E \in \Sigma$ and $f : \Omega \to X$ is Pettis integrable, then $\nu_f : \Sigma \to X$ is given by $\nu_f(E) := \int_E f \, d\mu$. The space of all X-valued μ-Pettis integrable functions is denoted by $P(\mu, X)$ (weakly equivalent functions are identified).

It is an open problem, whether the functions $\rho_0(f)$ and $\rho_1(f)$ are always Pettis integrable if f is bounded and Pettis integrable. Talagrand presented a few sufficient conditions in [6]. In particular, the RS-property is sufficient for the Pettis integrability of $\rho_0(f)$ for a bounded f but as he noticed in 7-3-16 of [6], the Pettis integrability of $\rho_0(f)$ need not imply the RS-property of f. In 1996 Rybakov published a paper [5] concerning the Pettis integrability of $\rho_1(f)$

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in case of separable $\nu_f(\Sigma)$, but the proof was totally false (see Math. Rev. 98h # 20007).

It is implicitly proved in Theorem 7-3-7 of Talagrand [6] that each $\rho_0(f)$ taking its values in a convex Pettis set (see [6] for the definition) is Pettis integrable. In Theorem 6-2-1 of [6] the Pettis integrability of an arbitrary weakly measurable $\rho_0(f)$ was proved under the assumption of the validity of Axiom L and the perfectness of μ. We generalize the first one and in the second case we replace the perfectness of the measure and Axiom L by a measurability condition. We prove also that liftings of McShane integrable functions are always Pettis integrable. The integrability of lifted functions becomes important while examining the completeness problem of some special subspaces of $P(\mu, X)$. It has been for instance proven in [4] that the space of scalarly bounded X^*-valued Pettis integrable functions is a Banach space (with the norm given by $\|f\|_{P_\infty} := \sup_{\|x^{**}\|\le 1} \|x^{**}f\|_\infty$), if each lifting $\rho_0(f)$ of a bounded Pettis integrable function is Pettis integrable.

1. Pettis integrability of liftings of Pettis integrable functions

LEMMA 1. *If $f : \Omega \to X^*$ is a scalarly bounded Pettis integrable function, then*

$$\langle x^{**}, \nu_f(E)\rangle = \int_E \langle x^{**}, \rho_0(f)\rangle d\mu$$

for each $x^{**} \in X_c^{**}$ and each $E \in \Sigma$.

PROOF. Let (x_α) be a countable net $\sigma(X^{**}, X^*)$ – convergent to a functional $x^{**} \in X_c^{**}$. We have for each $\omega \in \Omega$

$$\lim_\alpha \langle x_\alpha, f(\omega)\rangle = \langle x^{**}, f(\omega)\rangle \quad \text{and} \quad \lim_\alpha \langle x_\alpha, \rho_0(f)(\omega)\rangle = \langle x^{**}, \rho_0(f)(\omega)\rangle.$$

Since $\langle x_\alpha, f\rangle = \langle x_\alpha, \rho_0(f)\rangle$ μ-a.e. and there are only countably many different functionals x_α, we have $\langle x^{**}, f\rangle = \langle x^{**}, \rho_0(f)\rangle$ μ-a.e. The required equality of the integrals is now a direct consequence of the Pettis integrability of f. □

THEOREM 2. *Let $f : \Omega \to X^*$ be a scalarly bounded Pettis integrable function. If each $x^{**} \in X^{**}$ is ξ_0-measurable, then $\rho_0(f) \in P(\mu, X^*)$. The assumptions are in particular satisfied if X^* has the weak RNP.*

PROOF. In order to prove the Pettis integrability of $\rho_0(f)$, we have only to show that for an arbitrary $z \in \nu_f(\Sigma)^\perp$ (= the annihilator of $\nu_f(\Sigma)$) the equality $\langle z, \rho_0(f)\rangle = 0$ holds true μ-a.e. (according to Theorem 6.2 from [3]). Suppose that there exists $z \in \nu_f(\Sigma)^\perp$ such that $\mu\{\omega \in \Omega : \langle z, \rho_0(f)(\omega)\rangle > 0\} > 0$ and $\|z\| \le 1$ and let K_f^0 be the the weak*-closure of $\rho_0(f)(\Omega)$ in X^*. Then, since ξ_0 is a Radon measure and $\xi_0(K_f^0) = \mu(\Omega)$, there exist a weak*-compact set

$L \subseteq \{x^* \in K_f^0 : \langle z, x^* \rangle > 0\}$ and a positive real number a such that

$$\xi_0(L) > 0, \qquad z \text{ is continuous on } L \text{ and } \langle z, x^* \rangle > a \text{ for each } x^* \in L.$$

Take now an arbitrary net $(x_\alpha)_{\alpha \in A}$ in B_X that is $\sigma(X^{**}, X^*)$-convergent to z. It follows then, that there exists a net of convex combinations of the elements of $(x_\alpha)_{\alpha \in A}$ that is convergent to z in the Mackey topology $\tau(X^{**}, X^*)$. In particular the convergence to z is uniform on $\nu_f(\Sigma)$. To avoid unnecessary complications, we assume at once that the initial net $(x_\alpha)_{\alpha \in A}$ is Mackey convergent to z. Then, for each $n \in \mathbf{N}$ there exists $\alpha_n \in A$ such that

$$|\langle x_\alpha, \nu_f(E) \rangle| \leq 1/n \qquad \text{for all } E \in \Sigma \text{ and all } \alpha \geq \alpha_n.$$

Let $A_n := \{\alpha \in A : \alpha \geq \alpha_n\}$. Since for each $n \in \mathbf{N}$ and each $x^* \in L$ the net $(\langle x_\alpha, x^* \rangle)_{\alpha \in A_n}$ is convergent to $\langle z, x^* \rangle$, we can find for each collection of points $x_1^*, \ldots, x_n^* \in L$ an index $\alpha_{x_1^*, \ldots, x_n^*} \in A_n$ such that

$$|\langle z, x_i^* \rangle - \langle x_{\alpha_{x_1^*, \ldots, x_n^*}}, x_i^* \rangle| < 1/n \qquad \text{for each } i \leq n.$$

Equivalently,

$$L^n \subseteq \bigcup_{\alpha \in A_n} \{x^* : |\langle z, x^* \rangle - \langle x_\alpha, x^* \rangle| < 1/n\}^n.$$

Now, as a consequence of the compactness of L and the continuity of $z|L$, there exists a finite set $B_n \subset A_n$ such that the inclusion

$$L^n \subseteq \bigcup_{\alpha \in B_n} \{x^* : |\langle z, x^* \rangle - \langle x_\alpha, x^* \rangle| < 1/n\}^n \tag{*}$$

holds true. It follows that $z|L$ is a pointwise cluster point of the countable set $\{x_\alpha | L : x_\alpha \in \bigcup_{n=1}^\infty B_n\}$. Consequently, there exists $x_0^{**} \in X^{**}$ that is a weak*-cluster point of the set $\{x_\alpha : \alpha \in \bigcup_{n=1}^\infty B_n\}$ and $x_0^{**}|L = z|L$. It follows from the construction of x_0^{**} that $x_0^{**} \in \nu_f(\Sigma)^\perp$ and so

$$\int_{\rho_0(f)^{-1}(L)} \langle x_0^{**}, \rho_0(f) \rangle d\mu \geq a\mu(\rho_0(f)^{-1}(L)) > 0 = \langle x_0^{**}, \nu_f(\rho_0(f)^{-1}(L)) \rangle.$$

Since, $x_0^{**} \in X_c^{**}$ and f is Pettis integrable, we get a contradiction with Lemma 1. $\square$

The next result is a direct consequence of the above theorem.

THEOREM 3. *Let $f : \Omega \to X$ be a scalarly bounded Pettis integrable function. If each $x^{***} \in X^{***}$ is ξ_1-measurable, then $\rho_1(f) \in P(\mu, X^{**})$.*

COROLLARY 4. *If $f \in P(\mu, X^*)$ is scalarly bounded, $K \subset X^*$ is a convex Pettis set and $\rho_0(f) : \Omega \to K$, then $\rho_0(f) \in P(\mu, X^*)$. If $f \in P(\mu, X)$ is* *scalarly bounded,* $K \subset X^{**}$ *is a convex Pettis set and* $\rho_1(f) : \Omega \to K$, *then* $\rho_1(f) \in P(\mu, X^{**})$.

2. Liftings of McShane integrable functions

We begin with the following result that I couldn't find anywhere (the undefined notions can be founded in [6]).

LEMMA 5. *Let* ρ *be a consistent lifting and let* $\mathscr{Z} \subset \mathscr{L}_\infty(\mu)$ *be a family of* ρ*-invariant functions. If each countable subset of* $\mathscr{Z}$ *is stable, then* $\mathscr{Z}$ *itself is also stable.*

PROOF. Assume that each countable family $\mathscr{Z}' \subset \mathscr{Z}$ is stable but $\mathscr{Z}$ is not stable. Then there exist $\alpha < \beta$ and a critical set $A \in \Sigma_\mu^+$. Without loss of generality, we may assume that $\rho(A) = A$. Thus, we have for each $k, l \in \mathbf{N}$

$$\mu_{k+l}^*\left(\bigcup_{f \in \mathscr{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}\right) = [\mu(A)]^{k+l}. \tag{**}$$

One can easily see that

$$\{f < \alpha\} \subseteq \rho(\{f < \alpha\}) \qquad \text{and} \qquad \{f > \beta\} \subseteq \rho(\{f > \beta\})$$

for each $f \in \mathscr{Z}$. Moreover, the consistency of ρ implies the relation

$$\begin{aligned}
&\rho_{k+l}(\{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}) \\
&\quad = [\rho(\{f < \alpha\})]^k \times [\rho(\{f > \beta\})]^l \cap [\rho(A)]^{k+l} \\
&\quad \supseteq \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}.
\end{aligned}$$

Consequently, it follows from [2] that the set $\bigcup_{f \in \mathscr{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}$ is measurable and (**) may be replaced by

$$\mu_{k+l}\left(\bigcup_{f \in \mathscr{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}\right) = [\mu(A)]^{k+l}.$$

A standard considerations show that one can find such a countable set $\mathscr{Z}' \subset \mathscr{Z}$ that

$$\mu_{k+l}\left(\bigcup_{f \in \mathscr{Z}'} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}\right) = [\mu(A)]^{k+l}.$$

This contradicts the stability of $\mathscr{Z}'$ and completes the proof. □

THEOREM 6. *Let μ be a quasi-Radon measure (see* [1]*). If $f : \Omega \to X^*$ is weak*-bounded and McShane integrable then for each consistent lifting ρ the function $\rho_0(f)$ is Pettis integrable. Similarly, if $f : \Omega \to X$ is scalarly bounded and McShane integrable then $\rho_1(f) \in P(\mu, X^{**})$ for each consistent lifting ρ.*

PROOF. According to Theorem 3C of [1] each countable set $\{x_n f : \|x_n\| \leq 1, n = 1, 2, \dots\}$ is stable. Since ρ is consistent, each set $\{x_n \rho_0(f) : \|x_n\| \leq 1, n = 1, 2 \dots\}$ is also stable. Applying Lemma 5 we get the stability of the whole set $\{x\rho_0(f) : \|x\| \leq 1\}$. According to [6] the function $\rho_0(f)$ is properly measurable and hence it is Pettis integrable. □

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