Hiroshima Math. J.
37 (2007), 397–408

A uniqueness result on some differential polynomials sharing 1-points

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(Received August 21, 2006)
(Revised January 9, 2007)

Abstract. We study the uniqueness of meromorphic functions when two nonlinear differential polynomials generated by two meromorphic functions share the same simple and double 1-points and improve an earlier result given by Fang-Fang [1] and a recent result of Lahiri-Mandal [10].

1. Introduction definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbf{C}$. If for some $a \in \mathbf{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a counting multiplicities. Let m be a positive integer or infinity and $a \in \mathbf{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbf{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f, g share the value a counting multiplicities.

During the last few years a great deal of work has been carried out on the uniqueness problem concerning differential polynomials generated by two meromorphic functions. (cf. [1], [2], [4], [6], [9], [10], [11], [12]). In [4, 6] Lahiri studied the uniqueness problem of meromorphic functions when two linear differential polynomials share the same 1-points. In [4] Lahiri asked the following question: What can be said if two non linear differential polynomials generated by two meromorphic functions share 1 counting multiplicities? Several authors like Fang-Fang [1], Fang-Hong [2], Lin [11], Yi-Lin [12] investigate the problem of uniqueness of meromorphic functions when two nonlinear differential polynomials share the same 1-points.

In 2001 Fang and Hong [2] proved the following result.

2000 *Mathematics Subject Classification*: 30D35.

Key words and phrases: Meromorphic functions, uniqueness, nonlinear differential polynomials, shared set.

Theorem A. *Let f and g be two transcendental entire functions and $n \geq 11$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 counting multiplicities, then $f \equiv g$.*

Also in 2002 Fang and Fang [1] improved the above result and proved the following theorem.

Theorem B. *Let f and g be two nonconstant entire functions and $n \geq 9$ be an integer. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$, then $f \equiv g$.*

The following example shows that Theorem B is not valid when f and g are two meromorphic functions.

Example 1.1.

$$f(z) = \frac{(n+2)}{(n+1)} \frac{e^z + \cdots + e^{(n+1)z}}{1 + e^z + \cdots + e^{(n+1)z}}$$

and

$$g(z) = \frac{(n+2)}{(n+1)} \frac{1 + e^z + \cdots + e^{nz}}{1 + e^z + \cdots + e^{(n+1)z}}$$

Clearly $f(z) = e^z g(z)$. Also $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 counting multiplicities but $f \not\equiv g$.

So it is a natural query that if in Theorem B f and g be two non constant meromorphic functions then under which condition $f \equiv g$?

In this regard recently Lahiri and Mandal [10] proved the following result for meromorphic functions.

Theorem C. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ and $n \geq 17$ be an integer. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Now one may ask the following question which is the motivation of the paper: Is it possible in Theorem C to reduce the lower bound of n from 17? In this paper we give an affirmative answer to the above question. We now state the following theorem which is the main result of the paper.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions and $n > \max\left\{\frac{27}{2} - 5\min(\Theta(\infty; f), \Theta(\infty; g)), \frac{4}{\Theta(\infty; f) + \Theta(\infty; g)} - 1\right\}$, be an integer, where $\Theta(\infty; f) + \Theta(\infty; g) > 0$. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$ then $f \equiv g$.*

From Theorem 1.1 we can immediately deduce the following corollary which improve Theorem C.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n+1}$, and $n(\geq 14)$ be an integer. If $E_{2)}(1;f^n(f-1)f')=E_{2)}(1;g^n(g-1)g')$ then $f\equiv g$.*

The following example shows that the condition $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n+1}$ is sharp in Corollary 1.1.

Example 1.2 [9]. Let $f=\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $g=h\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$ and $h=\frac{\alpha^2(e^z-1)}{e^z-\alpha}$ where $\alpha=\exp\left(\frac{2\pi i}{n+2}\right)$ and n is a positive integer. Since $h\not\equiv 1$ we have $f=\frac{(n+2)(1+h+h^2+\cdots+h^n)}{(n+1)(1+h+h^2+\cdots+h^{n+1})}$ and $g=h\frac{(n+2)(1+h+h^2+\cdots+h^n)}{(n+1)(1+h+h^2+\cdots+h^{n+1})}$. So it follows from Mohon'ko's Lemma {See [13]} $T(r,f)=(n+1)T(r,h)+O(1)$ and $T(r,g)=(n+1)T(r,h)+O(1)$. Further we see that $h\neq\alpha,\alpha^2$ and a root of $h=1$ is not a pole of f and g. Hence $\Theta(\infty;f)=\Theta(\infty;g)=\frac{2}{n+1}$. Also $f^{n+1}\left(\frac{f}{n+1}-\frac{1}{n+1}\right)\equiv g^{n+1}\left(\frac{g}{n+1}-\frac{1}{n+1}\right)$ and $f^n(f-1)f'\equiv g^n(g-1)g'$ but $f\not\equiv g$.

Though we use the standard notations and definitions of the value distribution theory available in [3], we explain some definitions and notations which are used in the paper.

Definition 1.1 [5]. For $a\in\mathbf{C}\cup\{\infty\}$ we denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)$ $(N(r,a;f\mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)$ and $\overline{N}(r,a;f\mid\geq m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ *and* $\overline{N}(r,a;f\mid>m)$ are defined analogously.

Definition 1.2 (cf. [16]). We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$.

Definition 1.3. Let m be a positive integer and for $a\in\mathbf{C}$, $E_{m)}(a;f)=E_{m)}(a;g)$. For an a-point z of f, g, we denote by $\mu(z,a,f)$, $\mu(z,a,g)$ their multiplicities respectively. We denote by $\overline{N}_L(r,a;f,g)$ $(\overline{N}_L(r,a;g,f))$ the intregeted reduced counting function of a-points z with $\mu(z,a,f)>\mu(z,a,g)\geq m+1$ $(\mu(z,a,g)>\mu(z,a,f)\geq m+1)$, by $\overline{N}_E^{[m+1}(r,a;f,g)$ the integrated reduced counting function of those a-points z with $\mu(z,a,f)=\mu(z,a,g)\geq m+1$, by $\overline{N}^{[m+1}(r,a;f\mid g\neq a)$ $(\overline{N}^{[m+1}(r,a;g\mid f\neq a))$ the integrated reduced counting functions of a-points z with $\mu(z,a,f)\geq m+1$ and $g(z)\neq a$ $(\mu(z,a,g)\geq m+1$ and $f(z)\neq a)$.

Definition 1.4. We denote by $\overline{N}(r,a;f\,|=k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k where $k \geq 2$ is an integer. For $k=1$ we refer Definition 1.1.

Definition 1.5 [7]. Let $a,b \in \mathbf{C}\cup\{\infty\}$. We denote by $N(r,a;f\,|\,g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.6 [7]. Let $a,b \in \mathbf{C}\cup\{\infty\}$. We denote by $N(r,a;f\,|\,g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right)$$

and

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Lemma 2.1 [10]. *If f, g be two nonconstant meromorphic functions such that $E_{1)}(1;f)=E_{1)}(1;g)$ and $h\not\equiv 0$ then*

$$N(r,1;f\,|\leq 1)\leq N(r,0;h)\leq N(r,\infty;h)+S(r,f)+S(r,g).$$

Lemma 2.2. *Let $E_{2)}(1;f)=E_{2)}(1;g)$ and $h\not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;h)\leq{}&\overline{N}(r,0;f\,|\geq 2)+\overline{N}(r,0;g\,|\geq 2)+\overline{N}(r,\infty;f\,|\geq 2)+\overline{N}(r,\infty;g\,|\geq 2)\\&+\overline{N}^{[3}(r,1;f\,|\,g\neq 1)+\overline{N}^{[3}(r,1;g\,|\,f\neq 1)+\overline{N}_L(r,1;f,g)\\&+\overline{N}_L(r,1;g,f)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) the common zeros of $f-1$ and $g-1$ whose multiplicities are different, (iii) those 1-points of f (g) which are not the 1-points of g (f), (iv) zeros of f' which are not the zeros of

$f(f-1)$, (v) zeros of g' which are not zeros of $g(g-1)$. Since all the poles of h are simple, the lemma follows from above.

Lemma 2.3 [8]. *If* $N(r,0;f^{(k)}\,|\,f\neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\,|\,f\neq 0)\le k\overline{N}(r,\infty;f)+N(r,0;f\,|<k)+k\overline{N}(r,0;f\,|\ge k)+S(r,f).$$

Lemma 2.4. *Let* $E_{2)}(1;f)=E_{2)}(1;g)$. *Then*

$$\overline{N}^{[3}(r,1;f\,|\,g\neq 1)\le \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_0(r,0;f')+S(r,f),$$

where $\overline{N}_0(r,0;f')$ *is the counting function of those zeros of* f' *which are not the zeros of* $f(f-1)$.

Proof. Using Lemma 2.3 we get

$$\begin{aligned}\overline{N}^{[3}(r,1;f\,|\,g\neq 1)&\le \overline{N}(r,1;f\,|\ge 3)\\&\le \frac{1}{2}N(r,0;f'\,|\,f=1)\\&\le \frac{1}{2}N(r,0;f'\,|\,f\neq 0)-\frac{1}{2}N_0(r,0;f')\\&\le \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_0(r,0;f')+S(r,f).\end{aligned}$$

Lemma 2.5. *Let* $E_{2)}(1;f)=E_{2)}(1;g)$. *Then*

$$\begin{aligned}&2\overline{N}_L(r,1;f,g)+2\overline{N}_L(r,1;g,f)+2\overline{N}_E^{[3}(r,1;f,g)+\overline{N}(r,1;f\,|=2)\\&\quad+2\overline{N}^{[3}(r,1;g\,|\,f\neq 1)\le N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

Proof. Since $E_{2)}(1;f)=E_{2)}(1;g)$, we note that the simple and double 1-points of f and g are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q=3$ the possible values of p are as follows (i) $p=3$ (ii) $p\ge 4$ (iii) $p=0$. Similarly when $q=4$ the possible values of p are (i) $p=3$ (ii) $p=4$ (iii) $p\ge 5$ (iv) $p=0$. If $q\ge 5$ we can similarly find the possible values of p. Now the lemma follows from above explanation. This completes the proof of the lemma.

Lemma 2.6 [14]. *Let* f *be a nonconstant meromorphic function and* $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, *where* $a_0,a_1,a_2\dots,a_n$ *are constants and* $a_n\neq 0$. *Then* $T(r,P(f))=nT(r,f)+O(1)$.

Lemma 2.7 [10]. *Let f and g be two nonconstant meromorphic functions. Then*

$$f^n(f-1)f'g^n(g-1)g' \not\equiv 1,$$

where $n \geq 5$ is an integer.

Lemma 2.8 [10]. *Let f and g be two nonconstant meromorphic functions and*

$$F_1 = f^{n+1}\left(\frac{f}{n+2} - \frac{1}{n+1}\right) \qquad \text{and} \qquad G_1 = g^{n+1}\left(\frac{g}{n+2} - \frac{1}{n+1}\right),$$

where $n \geq 4$ is an integer. Then $F_1' \equiv G_1'$ implies $F_1 \equiv G_1$.

Lemma 2.9 [10]. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1},$$

where $n \geq 2$ is an integer. Then

$$f^{n+1}(af+b) \equiv g^{n+1}(ag+b)$$

implies $f \equiv g$, where a and b are finite non-zero constants and n is an integer.

Lemma 2.10 [15]. *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N(r,0;f) + S(r,f).$$

Lemma 2.11. *Let $E_{2)}(1;f) = E_{2)}(1;g)$ and $h \not\equiv 0$. Then*

$$\begin{aligned} T(r,f) + T(r,g) \leq\ & 2\{N_2(r,0;f) + N_2(r,\infty;f) + N_2(r,0;g) + N_2(r,\infty;g)\} \\ & + \overline{N}^{[3}(r,1;f \mid g \neq 1) + \overline{N}^{[3}(r,1;g \mid f \neq 1) - 2\overline{N}_E^{[3}(r,1;f,g) \\ & - m(r,1;f) - m(r,1;g) + S(r,f) + S(r,g). \end{aligned}$$

Proof. By the second fundamental theorem of Nevenlinna we get

$$\begin{aligned} T(r,f) + T(r,g) \leq\ & \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) \\ & + \overline{N}(r,1;f) + \overline{N}(r,1;g) - N_0(r,0;f') - N_0(r,0;g') \\ & + S(r,f) + S(r,g). \end{aligned} \tag{2.1}$$

By Lemmas 2.1, 2.2 and 2.5 we get

$$
\begin{aligned}
(2.2)\qquad & \overline{N}(r,1;f)+\overline{N}(r,1;g)\\
&\le N(r,1;f\,|{=}1)+\overline{N}(r,1;f\,|{=}2)\\
&\quad+\overline{N}_L(r,1;f,g)+\overline{N}_L(r,1;g,f)+\overline{N}_E^{[3}(r,1;f,g)\\
&\quad+\overline{N}^{[3}(r,1;f\,|\,g\neq 1)+\overline{N}(r,1;g)\\
&\le \overline{N}(r,0;f\,|{\ge}2)+\overline{N}(r,\infty;f\,|{\ge}2)+\overline{N}(r,0;g\,|{\ge}2)\\
&\quad+\overline{N}(r,\infty;g\,|{\ge}2)+\overline{N}^{[3}(r,1;f\,|\,g\neq 1)\\
&\quad+\overline{N}^{[3}(r,1;g\,|\,f\neq 1)+\overline{N}_L(r,1;f,g)+\overline{N}_L(r,1;g,f)\\
&\quad+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+\overline{N}(r,1;f\,|{=}2)\\
&\quad+\overline{N}_L(r,1;f,g)+\overline{N}_L(r,1;g,f)+\overline{N}_E^{[3}(r,1;f,g)\\
&\quad+\overline{N}^{[3}(r,1;f\,|\,g\neq 1)+N(r,1;g)-2\overline{N}_E^{[3}(r,1;f,g)\\
&\quad-2\overline{N}_L(r,1;f,g)-2\overline{N}_L(r,1;g,f)-\overline{N}(r,1;f\,|{=}2)\\
&\quad-2\overline{N}^{[3}(r,1;g\,|\,f\neq 1)+S(r,f)+S(r,g)\\
&\le \overline{N}(r,0;f\,|{\ge}2)+\overline{N}(r,\infty;f\,|{\ge}2)+\overline{N}(r,0;g\,|{\ge}2)\\
&\quad+\overline{N}(r,\infty;g\,|{\ge}2)+2\overline{N}^{[3}(r,1;f\,|\,g\neq 1)+T(r,g)\\
&\quad-m(r,1;g)+O(1)-\overline{N}_E^{[3}(r,1;f,g)-\overline{N}^{[3}(r,1;g\,|\,f\neq 1)\\
&\quad+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).
\end{aligned}
$$

From (2.1) and (2.2) we get

$$
\begin{aligned}
(2.3)\qquad T(r,f)\le{}& N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\
&+2\overline{N}^{[3}(r,1;f\,|\,g\neq 1)-\overline{N}_E^{[3}(r,1;f,g)-\overline{N}^{[3}(r,1;g\,|\,f\neq 1)\\
&-m(r,1;g)+S(r,f)+S(r,g).
\end{aligned}
$$

Similarly we can obtain

$$
\begin{aligned}
(2.4)\qquad T(r,g)\le{}& N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\
&+2\overline{N}^{[3}(r,1;g\,|\,f\neq 1)-\overline{N}_E^{[3}(r,1;f,g)-\overline{N}^{[3}(r,1;f\,|\,g\neq 1)\\
&-m(r,1;f)+S(r,f)+S(r,g).
\end{aligned}
$$

Adding (2.3) and (2.4) we get the conclusion of the lemma.

Lemma 2.12. *Let f and g be two meromorphic functions and $n \geq 7$ be an integer. Also let $F = f^n(f-1)f'$, $G = g^n(g-1)g'$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$. If*

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{2.5}$$

where $a \neq 0$, b are two constants then $f \equiv g$.

Proof. Since

$$\begin{aligned} T(r,F) &= T(r, f^n(f-1)f') \\ &\leq T(r, f^n(f-1)) + T(r, f') \\ &\leq (n+1)T(r,f) + 2T(r,f) + S(r,f) \\ &= (n+3)T(r,f) + S(r,f) \end{aligned}$$

and

$$T(r,G) \leq (n+3)T(r,g) + S(r,g)$$

it follows that $S(r,F)$ can be replaced by $S(r,f)$ and $S(r,G)$ can be replaced by $S(r,g)$. Using Lemma 2.6 we note that

$$\begin{aligned} T(r,F) + m\left(r, \frac{1}{f'}\right) &= N(r,\infty; f^n(f-1)f') + m(r, f^n(f-1)f') + m\left(r, \frac{1}{f'}\right) \\ &\geq N(r,\infty; f^n(f-1)) + N(r,\infty; f') + m(r, f^n(f-1)) \\ &= (n+1)T(r,f) + N(r,\infty; f') + O(1). \end{aligned}$$

Therefore

$$\begin{aligned} T(r,F) &\geq (n+1)T(r,f) - T(r,f') + N(r,0;f') + N(r,\infty;f') + S(r,f) \\ &\geq (n+1)T(r,f) - T(r,f) + N(r,\infty;f) + N(r,0;f') + S(r,f) \\ &= nT(r,f) + N(r,\infty;f) + N(r,0;f') + S(r,f). \end{aligned} \tag{2.6}$$

Similarly we can obtain

$$T(r,G) \geq nT(r,g) + N(r,\infty;g) + N(r,0;g') + S(r,g) \tag{2.7}$$

Without loss of generality we suppose that $\{r \geq 0; T(r,f) \leq T(r,g)\}$, is of infinite linear measure. Now we consider the following cases.

Case I $b \neq 0, -1$: If $a - b - 1 \neq 0$ then from (2.5) we get

$$\overline{N}\left(r, -\frac{a-b-1}{b+1}; G\right) = \overline{N}(r,0;F)$$

By the second fundamental theorem and Lemma 2.10 we get

$$\begin{aligned} T(r,G) &\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r,-\frac{a-b-1}{b+1};G\right) + S(r,G) \\ &= \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,0;F) + S(r,g) \\ &\le \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,1;g) + N(r,0;g') \\ &\quad + \overline{N}(r,0;f) + \overline{N}(r,1;f) + N(r,0;f') + S(r,g) \\ &\le \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,1;g) + N(r,0;g') \\ &\quad + \overline{N}(r,\infty;f) + 2N(r,0;f) + \overline{N}(r,1;f) + S(r,g) \\ &\le 2T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + 4T(r,f) + S(r,g) \\ &\le 6T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + S(r,g). \end{aligned}$$

Hence by (2.7) and for $n \ge 7$ we see that $\{r \ge 0; (n-6)T(r,g) \le S(r,g)\}$, is of infinite linear measure which is impossible.

Next if $a-b-1=0$ then from (2.5) we get

$$F = \frac{(b+1)G}{bG+1}.$$

So

$$\overline{N}\left(r,-\frac{1}{b};G\right) = \overline{N}(r,\infty;F)$$

By the second fundamental theorem of Nevanlinna and Lemma 2.10 we get

$$\begin{aligned} T(r,G) &\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r,-\frac{1}{b};G\right) + S(r,G) \\ &\le \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,1;g) \\ &\quad + N(r,0;g') + \overline{N}(r,\infty;f) + S(r,g) \\ &\le 2T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + T(r,f) + S(r,g) \\ &\le 3T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + S(r,g). \end{aligned}$$

Again from (2.7) and for $n \ge 7$ we see that $\{r \ge 0; (n-3)T(r,g) \le S(r,g)\}$, is of infinite linear measure, which is a contradiction.

Case II: If $b=-1$ (2.5) becomes

$$F = \frac{a}{(a+1)-G}.$$

If $a+1 \neq 0$ then

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F)$$

we deduce a contradiction as in Case I.

If $a+1=0$ then $FG \equiv 1$ i.e.

$$f^n(f-1)f'g^n(g-1)g' \equiv 1,$$

which is impossible by Lemma 2.7.

Case III: If $b=0$ then (2.5) gives

$$F = \frac{G+a-1}{a}.$$

If $a-1 \neq 0$ then

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F)$$

We can similarly deduce a contradiction as in Case I.

If $a-1=0$ then $F \equiv G$. Let $F_1' \equiv F$ and $G_1' \equiv G$. Then by Lemma 2.8 we have $F_1 \equiv G_1$.

Therefore

$$f^{n+1}\left(\frac{f}{n+2} - \frac{1}{n+1}\right) \equiv g^{n+1}\left(\frac{g}{n+2} - \frac{1}{n+1}\right).$$

Hence using Lemma 2.9 for $a = \frac{1}{n+2}$ and $b = \frac{-1}{n+1}$ we get $f \equiv g$.

3. Proof of Theorem 1.1

Let F and G be defined as in Lemma 2.12. Suppose $H \not\equiv 0$. Then from Lemmas 2.11 and 2.4 we get

$$\begin{aligned}
(3.1)\quad & T(r,F) + T(r,G) \\
&\leq 2\{N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G)\} \\
&\quad + \frac{1}{2}\{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0:G) + \overline{N}(r,\infty;G)\} \\
&\quad + S(r,F) + S(r,G) \\
&\leq 2\{2\overline{N}(r,0;f) + N(r,1;f) + N(r,0;f') + 2\overline{N}(r,\infty;f)\} \\
&\quad + 2\{2\overline{N}(r,0;g) + N(r,1;g) + N(r,0;g') + 2\overline{N}(r,\infty;g)\}
\end{aligned}$$

$$+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,0;f')+\overline{N}(r,\infty;f)$$
$$+\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}(r,0;g')\}$$
$$+S(r,f)+S(r,g)$$
$$\leq \frac{9}{2}\overline{N}(r,0;f)+\frac{5}{2}N(r,1;f)+\frac{5}{2}N(r,0;f')+\frac{9}{2}\overline{N}(r,\infty;f)$$
$$+\frac{9}{2}\overline{N}(r,0;g)+\frac{5}{2}N(r,1;g)+\frac{5}{2}N(r,0;g')+\frac{9}{2}\overline{N}(r,\infty;g)$$
$$+S(r,f)+S(r,g).$$

Now using (2.6) and (2.7) in (3.1) and by Lemma 2.10 we obtain

$$nT(r,f)+nT(r,g)\leq 7T(r,f)+\frac{3}{2}N(r,0;f')+\frac{7}{2}\overline{N}(r,\infty;f) \tag{3.2}$$
$$+7T(r,g)+\frac{3}{2}N(r,0;g')+\frac{7}{2}\overline{N}(r,\infty;g)$$
$$+S(r,f)+S(r,g)$$
$$\leq \frac{17}{2}T(r,f)+\frac{17}{2}T(r,g)+5\overline{N}(r,\infty;f)$$
$$+5\overline{N}(r,\infty;g)+S(r,f)+S(r,g).$$

So for $0<\varepsilon<n-\frac{27}{2}+5\min\{\Theta(\infty;f);\Theta(\infty;g)\}$ we get from (3.2)

$$\left(n-\frac{27}{2}+5\Theta(\infty;f)-\varepsilon\right)T(r,f)+\left(n-\frac{27}{2}+5\Theta(\infty;g)-\varepsilon\right)T(r,g)$$
$$\leq S(r,f)+S(r,g),$$

which is a contradiction. Hence $H\equiv 0$. So

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}.$$

Also by the given condition of the theorem $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n+1}$. So by Lemma 2.12 we obtain $f\equiv g$. This completes the proof of the theorem.

Acknowledgement

The author is grateful to Prof. M. L. Fang for supplying him the electronic file of the paper [1].

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