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C-injectivity and C-projectivity

Dedicated to Patricia Baccouche

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Abstract. The definition of p-injectivity motivates us to generalize the notion of injectivity and projectivity, noted respectively C-injectivity and C-projectivity. Noetherian, semi-simple Artinian, quasi-Frobenius, regular hereditary and self-injective regular rings are considered in terms of C-injectivity and C-projectivity. Partial answers are given to Matlis' Problem and Boyle's Conjecture.

Introduction

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. A left A-module M is called semi-simple if the intersection of all maximal left submodules of M is zero [23]. A is called semi-primitive or semi-simple (resp. (a) left non-singular; (b) right non-singular) if $J = 0$ (resp. (a) $Z = 0$; (b) $Y = 0$). For any left A-module M, $Z(M) = \{y \in M \mid l(y)$ is an essential left ideal of $A\}$ is the singular submodule of M. ${}_AM$ is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). Right singular submodules are similarly defined. Thus $Z = Z({}_AA)$ and $Y = Z(A_A)$. Following Faith [12], write "A is VNR" if A is a von Neumann regular ring. It is well-known that A is VNR if and only if every left (right) A-module is flat ([2], [15]). Also, A is VNR if and only if every left (right) A-module is p-injective [34]. Note that the Harada-Auslander characterization may be weakened as follows: A is VNR if and only if every cyclic singular left A-module is flat [36, Theorem 5] (cf. G. O. Michler's comment in Math. Reviews 80i# 16021 and [21, p. 2652]).

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([12, p. 122], [28, p. 577], [29, p. 340], [34]). A is called a left p-injective ring if

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${}_AA$ is p-injective. P-injectivity is similarly defined on the right side. Note that flatness and p-injectivity are distinct concepts. P-injective modules are introduced in [34] to study VNR rings, V-rings, self-injective rings and their generalizations. P. Menal and P. Vamos proved that any ring may be embedded in an FP-injective ring (and hence in a p-injective ring) [12, p. 308]. P-injective rings and their generalizations are extensively studied since several years ([12, Theorem 6.4], [8], [9], [10], [16], [17], [24], [25], [42]).

Quoting Kasch [19], one may say that "the concepts of projective and injective modules are among the most important fundamental concepts of the theory of rings and modules."

We here consider the following analogous generalizations of injective and projective modules, using cyclic modules:

Definitions. (1) A left A-module M is called C-injective if, given any left A-module N and any cyclic left submodule C of N, every left A-homomorphism of C into M extends to one of N into M.

(We shall later see that C-injectivity is strictly between injectivity and p-injectivity.)

(2) A left A-module P is called C-projective if, given any cyclic left A-modules M, N with an epimorphism $g : M \to N$ and any left A-homomorphism $f : P \to N$, there exists a left A-homomorphism $h : P \to M$ such that $gh = f$.

§1. C-injective modules

Proposition 1.1. *Let M be a C-injective left A-module. Then any cyclic left submodule of M has an injective hull in M. Consequently, every cyclic C-injective left A-module is injective.*

Proof. Let C be a cyclic left submodule of M, ${}_AE$ an injective hull of ${}_AC$. If $g : C \to M$ and $j : C \to E$ are the inclusion maps, there exists a left A-homomorphism $h : E \to M$ such that $hj = g$. For any $u \in \ker h \cap C$, $u = g(u) = hj(u) = h(u) = o$ and since ${}_AC$ is essential in ${}_AE$, then $\ker h = 0$ which implies that h is a monomorphism. Therefore ${}_Ah(E) \approx {}_AE$ is an injective left A-module. Now $C = g(C) = hj(C) = h(C) \subseteq h(E) \subseteq M$ which proves that C has an injective hull in M. In particular, if $C = M$, then ${}_AC$ is injective. □

As usual, A is called a left (resp. right) Kasch ring if every maximal left (resp. right) ideal of A is an annihilator.

Applying [32, Corollary 9] to Proposition 1.1, we get

Corollary 1.1.1. *If A is a left Kasch ring containing a maximal left ideal which is principal and C-injective, then A is left pseudo-Frobenius.*

Since a right pseudo-Frobenius ring is left Kasch, the next corollary follows.

Corollary 1.1.2. *If A is a right Kasch ring containing a maximal right ideal and a maximal left ideal which are principal and C-injective, then A is both right and left pseudo-Frobenius.*

Recall that A is a left (resp. right) V-ring if every simple left (resp. right) A-module is injective.

Remark 1. P-injective modules effectively generalize C-injective modules (otherwise, by Proposition 1.1, all VNR rings would be self-injective V-rings!).

Lemma 1.2. *A direct sum of left A-modules is C-injective if and only if each direct summand is C-injective.*

(Using the natural injections and projections, the proof is direct and depends primarily on the definition of C-injectivity).

Theorem 1.3. *If every C-injective left A-module is injective, then A is left Noetherian.*

Proof. Let M be a direct sum of injective left A-modules $M_i (i \in I)$. Since each M_i is C-injective, then M is C-injective by Lemma 1.2. By hypothesis, ${}_AM$ is injective and A is therefore left Noetherian by [11, Theorem 20.1]. □

Remark 2. Theorem 1.3 ensures that C-injectivity effectively generalizes injectivity. (Otherwise, all rings would be Noetherian!)

Theorem 1.4. *The following conditions are equivalent:*

(1) *A is a principal left ideal ring;*

(2) *Every finitely generated left ideal of A is principal and every p-injective left A-module is injective;*

(3) *Every finitely generated left ideal of A is principal and every C-injective left A-module is injective.*

Proof. Assume (1). Since every left ideal is principal, then a left A-module is injective if and only if it is p-injective. Therefore (1) implies (2).

Since any C-injective left A-module is p-injective, then (2) implies (3).

Assume (3). By Theorem 1.3, A is left Noetherian. Then every left ideal of A, being finitely generated, is principal and thus (3) implies (1). □

Proposition 1.5. *The following conditions are equivalent:*

(1) *A is a principal left ideal ring which is quasi-Frobenius;*

(2) *A is a principal right ideal ring which is quasi-Frobenius;*

(3) *Every finitely generated one-sided ideal of A is the annihilator of an element of A and every p-injective left A-module is injective;*

(4) *Every finitely generated one-sided ideal of A is the annihilator of an element of A and every C-injective left A-module is injective.*

Proof. (1) and (2) are equivalent by [11, Proposition 25.4.6B].

Therefore (1) implies (3).

Since any C-injective left A-module is p-injective, then (3) implies (4).

Assume (4). Let F be a finitely generated left ideal of A. Then $F = l(b) = l(bA)$, $b \in A$. Now $bA = r(c) = r(Ac)$, $c \in A$ and Ac is a left annihilator. Therefore $Ac = l(r(Ac)) = l(bA) = F$ which shows that every finitely generated left ideal of A is principal. By Theorem 1.4, A is a principal left ideal ring.

Now since every principal right ideal of A is a right annihilator, by [18, Theorem 1 (i)], A is a left p-injective ring which yields A left self-injective. Since A is left Noetherian by Theorem 1.3, then (4) implies (1) by [11, Theorem 24.20]. □

Proposition 1.6. *If A is a VNR ring, then every cyclic submodule of a projective C-injective left A-module is injective.*

Proof. Apply [4, Theorem 5.4] to Proposition 1.1 and Lemma 1.2. □

§ 2. Boyle's Conjecture and Matlis' Problem

Recall that A is a left QI-ring if every quasi-injective left A-module is injective. Of course, quasi-injectivity does not imply C-injectivity (otherwise, any simple left A-module would be C-injective and hence injective by Proposition 1.1 and consequently, all rings would be V-rings!).

A result of A. Koehler [11, Proposition 20.4B] asserts that A is a left QI-ring if and only if the direct sum of any two quasi-injective left A-modules is quasi-injective. It is still unknown whether every left QI-ring is left hereditary (Boyle's Conjecture [12, p. 60]).

Proposition 2.1. *The following conditions are equivalent:*

(1) *A left A-module is C-injective if and only if it is quasi-injective;*

(2) *A is a left Noetherian left QI-ring whose C-injective left modules are injective.*

Proof. Assume (1). If ${}_AN$ is a direct sum of two quasi-injective left A-modules, then ${}_AN$ is a direct sum of two C-injective left A-modules and by Lemma 1.2, ${}_AN$ is C-injective and hence quasi-injective. By [11, Proposition 20.4B], A is a left QI-ring which is also left Noetherian. Thus (1) implies (2).

It is easily seen that (2) implies (1). □

In [22], it is proved that if A is left Noetherian, then for any completely decomposable left A-module M, every direct summand of M is completely decomposable. (${}_AM$ is completely decomposable if M is a direct sum of indecomposable injective submodules; ${}_AN$ is indecomposable injective if N is injective and the only direct summands of N are 0 and N.)

Matlis' Problem (still open): For an arbitrary ring A, if M is a completely decomposable left A-module, is every direct summand of M completely decomposable?

In [37, Theorem 3], we showed that if M is a completely decomposable left A-module such that there exists an injective submodule of M which contains the singular submodule $Z(M)$, then every direct summand of M is completely decomposable.

Singular modules play an important role in ring theory. A standard reference is K. R. Goodearl's Classic [13].

Note that the ring A considered in the next proposition needs not be left Noetherian.

Proposition 2.2. *Let A be a left non-singular ring with the following property: any direct sum of the injective hulls of cyclic singular left A-modules is injective. Then, for any C-injective left A-module M, $Z(M)$ is injective.*

Proof. Suppose that $Z(M) \neq 0$. If $u \in Z(M)$, $u \neq o$, then Au has an injective hull V contained in M by Proposition 1.1. Since $Z = 0$, then $V \subseteq Z(M)$ (cf. [33, Theorem 4]). Write S for the set of the injective hulls of all cyclic singular A-modules contained in M. Let E denote the set of all independent families $\{E_j\}$ of elements of S. Then E is an inductive set and by Zorn's Lemma, E admits a maximal member $\{E_i\}$ $(i \in I_o)$. Now $N = \bigoplus_{i \in I_o} E_i \subseteq Z(M)$ and ${}_AN$ is injective by hypothesis. Therefore $Z(M) = N \oplus Q$. If $q \in Q$, $q \neq o$, then Aq has an injective hull W contained in $Z(M)$. Since $Aq \cap N = 0$, then $W \cap N = 0$ which yields a member of E which strictly contains $\{E_i\}$ $(i \in I_o)$, contradicting its maximality in E. Therefore $Q = 0$ and $Z(M) = N$ is an injective left A-module. □

The next corollary follows immediately.

Corollary 2.2.1. *If A is left non-singular, left Noetherian then for any C-injective left A-module M, $Z(M)$ is injective.*

For any left A-module M with an injective hull K, K/M is a divisible singular left A-module. If every divisible singular left A-module is injective, then A is a left hereditary ring (cf. [38, p. 192]).

Corollary 2.2.2. *Let A be a left Noetherian ring whose divisible singular left modules are C-injective. Then A is left hereditary.*

Proof. Since every divisible singular left A-module is p-injective, then every principal left ideal of A is projective by [36, Proposition 4]. Therefore $Z = 0$. By Corollary 2.2.1, every divisible singular left A-module is injective. Then A is left hereditary (cf. [38, p. 192]). □

Boyle's Conjecture has the following partial answer.

Corollary 2.2.3. *If A is a left QI-ring whose divisible singular left modules are C-injective, then A is left hereditary.*

We are also in a position to give a partial answer to Matlis' Problem.

Proposition 2.3. *Let A be a left non-singular ring such that any direct sum of the injective hulls of cyclic singular left A-modules is injective. Suppose that every singular left A-module is C-injective. Then for any completely decomposable left A-module M, every direct summand of M is completely decomposable.*

Proof. By Proposition 2.2, if N is a singular left A-module, then ${}_AN$ being C-injective implies that $Z(N) = N$ is injective. Consequently, for any completely decomposable left A-module M, there exists an injective left submodule which contains $Z(M)$ and by [37, Theorem 3], every direct summand of M is completely decomposable. □

Since any direct sum of singular left A-modules is singular, the next corollary follows.

Corollary 2.3.1. *If every singular left A-module is injective, then for any completely decomposable left A-module M, every direct summand of M is completely decomposable.*

Rings whose singular modules are injective need not be Noetherian and the converse is not true either (even for commutative rings).

§ 3. C-projective modules

Proposition 3.1. *Let S be a cyclic C-projective left A-module. Then S is a projective left A-module.*

Proof. Let M, N be two left A-modules with an epimorphism $g : M \to N$, with a left A-homomorphism $f : S \to N$. If $S = As$, let $f(s) = n \in N$. Define a left A-homomorphism $f_o : S \to An$ by $f_o(as) = an$ for all $a \in A$. Then for all $u \in S$, $f(u) = f_o(u)$. If $j : An \to N$ is the inclusion map, then $jf_o = f$. Since g is an epimorphism, there exists $m \in M$ such that $g(m) = n$. If g_o is the restriction of g to Am, then $g_o : Am \to An$ and

$jg_o(am) = gi(am)$, $a \in A$, where $i : Am \to M$ is the inclusion map. Now g_o is an epimorphism of Am onto An, and since ${}_AS$ is C-projective, there exists a left A-homomorphism $h_o : S \to Am$ such that $g_o h_o = f_o$. Set $h = ih_o$. Then $h : S \to M$ is a left A-homomorphism such that for all $w \in S$, $gh(w) = gih_o(w) = jg_o h_o(w) = jf_o(w) = f(w)$ which proves that $gh = f$, whence ${}_AS$ is projective. □

Proposition 3.2. *Let A be a VNR ring. Then any C-projective left A-module is projective.*

Proof. Let P denote a non-zero C-projective left A-module. It is well-known that there exists a free left A-module F with an epimorphism $g : F \to P$. Let M be a non-zero cyclic left submodule of F. Since A is VNR, by [4, Theorem 5.4], ${}_AM$ is a direct summand of ${}_AF$. Then there exists a non-zero left A-homomorphism $f : F \to M$. Now there exists a left A-module N with left A-homomorphisms $w : M \to N$, $v : P \to N$ such that $wf = vg$. By [19, Theorem 4.7.4], w is an epimorphism which implies that ${}_AN$ is cyclic. Since ${}_AP$ is C-projective, there exists a left A-homomorphism $h : P \to M$ such that $wh = v$. Now there exists a left A-module D and left A-homomorphisms $s : D \to P$, $t : D \to M$ such that $wt = vs$. Since $wf = vg$, there exists an isomorphism $m : F \to D$ such that $f = tm$ and $g = sm$. Then s is an epimorphism (because of w). Since $v = wh$, by [19, Theorem 4.7.6(2)], ${}_AD = \ker s \oplus C$ for some submodule C of D. If $d \in F$, $m(d) = 1 + c$, $1 \in \ker s$, $c \in C$, and $d = m^{-1}(l) + m^{-1}(c) \in m^{-1}(\ker s) + m^{-1}(C)$. Since $m^{-1}(\ker s) \cap m^{-1}(C) = o$, then $F = m^{-1}(\ker s) \oplus m^{-1}(C)$. But $m^{-1}(\ker s) = \ker g$, which implies that $F \approx \ker g \oplus (F/\ker g)$, whence ${}_AP \approx {}_AF/\ker g$ is projective. □

Recall that a left A-module M is YJ-injective if, given any $o \neq a \in A$, there exists *a* positive integer n such that $a^n \neq o$ and every left A-homomorphism of Aa^n into M extends to one of A into M. A is called left YJ-injective if ${}_AA$ is YJ-injective. YJ-injectivity is similarly defined on the right side.

Theorem 3.3. *The following conditions are equivalent:*

(1) *A is VNR, left hereditary;*

(2) *A is a left p-injective ring whose left ideals are C-projective;*

(3) *A is a left YJ-injective ring whose left ideals are C-projective.*

Proof. It is clear that (1) implies (2) while (2) implies (3).

Assume (3). Then every principal left ideal of A is projective by Proposition 3.1. Since A is left YJ-injective, A is VNR by [9, Theorem 2.9]. In that case, every left ideal of A is projective by Proposition 3.2 and thus (3) implies (1). □

§ 4. Semi-simple Artinian and self-injective regular rings

In [20, Theorem 3.3], it is proved that if A has a classical left quotient ring Q, then Q is semi-simple Artinian if and only if every divisible torsion free left A-module is injective. The next result (which is a sequel to [40, Theorem 5]), slightly weakens Levy's condition.

Proposition 4.1. *Let A be a ring having a classical left quotient ring Q. If every divisible torsion free left A-module is C-injective, then Q is semi-simple Artinian.*

Proof. Let $K = Qc$ be a non-zero cyclic left Q-module, ${}_QE$ the injective hull of ${}_QK$. By a well-known result of B. Osofsky [27, p. 186], it is sufficient to prove that $K = E$ for Q to be semi-simple Artinian. Now K is a torsion free divisible left A-module and therefore a C-injective left A-module. Since ${}_AAc$ is essential in ${}_AK$ and by Proposition 1.1, ${}_AAc$ has an injective hull in ${}_AK$, then ${}_AK$ is the injective hull of ${}_AAc$. Therefore ${}_AE = {}_AK \oplus_A P$ and ${}_AP$ is divisible (because ${}_AE$ is divisible). For any $z \in P$ and any $q \in Q$, we show that $qz \in P$. Indeed, with $q = b^{-1}d, b, d \in A$, b being a non-zero-divisor, if $v = qz = b^{-1}dz$, since $P = bP$, with $w = dz \in P$, $bv = w = bu$ for some $u \in P$ which implies that $v = u \in P$. We have shown that P is a left Q-module. Then ${}_QE = {}_QK \oplus_Q P$ which proves that ${}_QK = {}_QE$ is injective. □

Theorem 4.2. *The following conditions are equivalent:*
(1) *A is semi-simple Artinian;*
(2) *Every left A-module is C-projective;*
(3) *Every singular left A-module is C-projective;*
(4) *Every simple left A-module is C-projective;*
(5) *Every left A-module is C-injective;*
(6) *Every cyclic left A-module is C-injective;*
(7) *Every cyclic semi-simple left A-module is C-injective.*

Proof. It is clear that (1) implies (2) while (2) implies (3).

Assume (3). Then every simple singular left A-module is C-projective. If $S = As$ is a simple non-singular left A-module, then $S \approx A/l(s)$ and $l(s)$ cannot be an essential left ideal of A. Therefore the maximal left ideal $l(s)$ is a direct summand of ${}_AA$ which implies that ${}_AA/l(s)$ is projective. Therefore, every simple left A-module is C-projective and (3) implies (4).

Assume (4). Then every simple left A-module is projective by Proposition 3.1.

Therefore every maximal left ideal of A is a direct summand of ${}_AA$ and (4) implies (1).

It is also obvious that (1) implies (5), (5) implies (6) and (6) implies (7). Finally, (7) implies (1) by [23, Theorem 3.2] and Proposition 1.1. □

Now we turn to a connection between C-injectivity and C-projectivity.

THEOREM 4.3. *The following conditions are equivalent:*

(1) *A is left self-injective regular;*

(2) *For every finitely generated left A-module M, ${}_AM/Z(M)$ is C-injective and C-projective;*

(3) *For every cyclic left A-module M, ${}_AM/Z(M)$ is C-injective and C-projective.*

PROOF. Assume (1). Let M be a finitely generated left A-module. Since $Z = 0$, by [33, Theorem 4], ${}_AM/Z(M)$ is non-singular. By [41, Corollary 10], ${}_AM/Z(M)$ is both injective and projective. Therefore (1) implies (2).

(2) implies (3) evidently.

Assume (3). Since ${}_AA/Z$ is C-projective, then by Proposition 3.1, ${}_AA/Z$ is projective which implies that ${}_AZ$ is a direct summand of ${}_AA$. Since Z cannot contain a non-zero idempotent, $Z = 0$. Therefore the C-injectivity of ${}_AA$ implies that ${}_AA$ is injective by Proposition 1.1. Thus (3) implies (1). □

We add a final remark.

REMARK 3. (1) A is a left V-ring if and only if every simple left A-module is C-injective.

(2) The following statements are equivalent:

(a) Every simple left A-module is either injective or projective;

(b) Every simple left A-module is either C-injective or C-projective. (Apply Propositions 1.1 and 3.1.)

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