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Normal Gorenstein del Pezzo surfaces with quasi-lines

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Abstract. In this paper, we give a classification of normal del Pezzo surfaces X with at most three quasi-lines and determine the geometric structure of the complement of quasi-lines on X. Moreover, we give the complete list of compactifications X of $\mathbf{C}^2$ with quasi-lines as boundaries.

1. Introduction

A normal projective Gorenstein surface X over $\mathbf{C}$ is called a normal del Pezzo surface if the anti-canonical divisor $-K_X$ is ample. We assume that $\mathrm{Sing}(X) \neq \emptyset$.

Let $\varphi : M \to X$ be the minimal resolution of X with the exceptional set $\Delta = \bigcup_i \Delta_i = \varphi^{-1}(\mathrm{Sing}(X))$, where each Δ_i is an irreducible component of Δ. Then Brenton [2] and Hidaka-Watanabe [4] proved the following:

Proposition 1.1. *Let X and M be as above. Then one of the following two cases occurs.*

(i) *M is a rational surface and* $\mathrm{Sing}(X)$ *consists of rational double points and each Δ_i is a (-2)-curve. In particular, $K_M \sim \varphi^* K_X$.*

(ii) *M is a $\mathbf{P}^1$-bundle over an elliptic curve $\mathbf{T}$ with the negative section $\Delta = \varphi^{-1}(\mathrm{Sing}(X)) \simeq \mathbf{T}$. In particular,* $\mathrm{Sing}(X) = \{x_1\}$ *(one point) and $K_M \sim \varphi^* K_X - \Delta$.*

By using the above proposition, we can obtain the following:

Lemma 1.2. *Assume that M is a rational surface. Then an irreducible curve C on M with $(C^2) < 0$ is either a (-1)-curve or a (-2)-curve. Moreover, each (-2)-curve on M is an irreducible component of Δ.*

An irreducible curve ℓ on X is called a quasi-line if its proper transform on M is a (-1)-curve. From Proposition 1.1, we can easily see that M is a rational surface if X contains quasi-lines. We remark that $(K_X \cdot \ell) = -1$ for any quasi-line ℓ on X. Let N_X be the number of quasi-lines on X. Our aim is to give a complete classification of normal del Pezzo surface X with quasi-lines and determine the geometric structure of the complement of $N(\leq N_X)$-

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quasi-lines on X for the case of $1 \leq N_X \leq 3$. Our main results are on as follows:

THEOREM 1.3. *Let X be a normal del Pezzo surface over* $\mathbf{C}$ *of degree d. We assume that* $\mathrm{Sing}(X) \neq \emptyset$ *and* $1 \leq N_X \leq 3$. *Let $\varphi : M \to X$ be the minimal resolution of X with the exceptional set $\Delta = \bigcup_i \Delta_i$, where each Δ_i is an irreducible component. Then* $\mathrm{Sing}(X)$ *consists of rational double points and we have the following: Here we denote the singularities of X by the types of the corresponding Dynkin diagrams.*

(I) *If X has only one quasi-line ℓ, then we have $1 \leq d \leq 6$, and the types of singularities are uniquely determined up to deformation as follows:*
(1) $d = 1 \Rightarrow \mathrm{Sing}(X) = E_8$,
(2) $d = 2 \Rightarrow \mathrm{Sing}(X) = E_7$,
(3) $d = 3 \Rightarrow \mathrm{Sing}(X) = E_6$,
(4) $d = 4 \Rightarrow \mathrm{Sing}(X) = D_5$,
(5) $d = 5 \Rightarrow \mathrm{Sing}(X) = A_4$,
(6) $d = 6 \Rightarrow \mathrm{Sing}(X) = A_1 + A_2$.
The configurations of curves $\hat{\ell} \cup \Delta$ on M are as in Table I, *where $\hat{\ell}$ is the proper transform of ℓ. In particular, we obtain that* $X - \ell \simeq \mathbf{C}^2$.

(II) *If X has exactly two distinct quasi-lines ℓ_1, ℓ_2, then we have $1 \leq d \leq 7$, and the types of singularities are uniquely determined up to deformation as follows:*
(1) $d = 1 \Rightarrow \mathrm{Sing}(X) = D_8$,
(2) $d = 2 \Rightarrow \mathrm{Sing}(X) = A_7$ *or* $A_1 + D_6$,
(3) $d = 3 \Rightarrow \mathrm{Sing}(X) = A_1 + A_5$,
(4) $d = 4 \Rightarrow \mathrm{Sing}(X) = D_4$ *or* $2A_1 + A_3$,
(5) $d = 5 \Rightarrow \mathrm{Sing}(X) = A_3$,
(6) $d = 6 \Rightarrow \mathrm{Sing}(X) = A_2$ *or* $2A_1$,
(7) $d = 7 \Rightarrow \mathrm{Sing}(X) = A_1$.
The configurations of curves $\widehat{\ell_1} \cup \widehat{\ell_2} \cup \Delta$ on M are as in Table II, *where $\widehat{\ell_1}$ and $\widehat{\ell_2}$ are the proper transforms of ℓ_1 and ℓ_2, respectively. In particular,* $X - (\ell_1 \cup \ell_2) \simeq \mathbf{C}^2$ *or* $\mathbf{C} \times \mathbf{C}^*$. *In Table* II, *the first and second columns are the lists of X such that $(X, \ell_1 \cup \ell_2)$ is the compactification of* $\mathbf{C}^2$ *and* $\mathbf{C} \times \mathbf{C}^*$, *respectively.*

Table I (Compactification of $\mathbf{C}^2$)

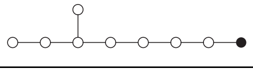
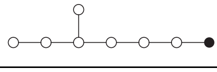
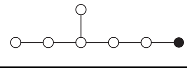
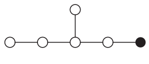
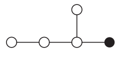
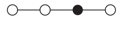
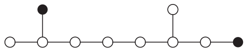
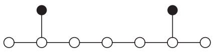
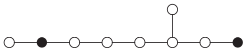
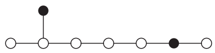
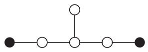
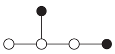
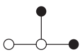
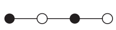
E_8 	E_7 	E_6 
D_5 	A_4 	$A_1 + A_2$ 

Table II (The type of singularities on X and the corresponding dual graph)

d	Compactification of $\mathbf{C}^2$	Compactification of $\mathbf{C} \times \mathbf{C}^*$
1		D_8 
2		A_7 
		$A_1 + D_6$ 
3		$A_1 + A_5$ 
4	D_4 	$2A_1 + A_3$
5	A_3 	
6	A_2 	
	$2A_1$ 	
7	A_1 	

(III) *If X has exactly three distinct quasi-lines ℓ_1, ℓ_2, ℓ_3, then we have $1 \leq d \leq 6$, and the types of singularities are uniquely determined up to deformation as follows:*

(1) $d = 1 \Rightarrow \mathrm{Sing}(X) = A_8$ *or* $A_1 + E_7$,

(2) $d = 2 \Rightarrow \mathrm{Sing}(X) = D_6$ *or* $A_2 + A_5$,

(3) $d = 3 \Rightarrow \mathrm{Sing}(X) = A_5$, D_5 *or* $3A_2$,

(4) $d = 4 \Rightarrow \mathrm{Sing}(X) = A_4$ *or* $A_1 + A_3$,

(5) $d = 5 \Rightarrow \mathrm{Sing}(X) = A_1 + A_2$,

(6) $d = 6 \Rightarrow \mathrm{Sing}(X) = A_1$.

The configurations of curves $\widehat{\ell_1} \cup \widehat{\ell_2} \cup \widehat{\ell_3} \cup \varDelta$ on M except of the type $A_1 + E_7$ are as in Table III, *where $\widehat{\ell_1}$, $\widehat{\ell_2}$ and $\widehat{\ell_3}$ are the proper transforms of ℓ_1, ℓ_2 and ℓ_3, respectively. In particular, $X - \bigcup_{i=1}^{3} \ell_i \simeq \mathbf{C}^2$, $\mathbf{C} \times \mathbf{C}^*$ or $(\mathbf{C}^*)^2$. In Table* III, *the first, second and third columns are the lists of X such that $(X, \bigcup_{i=1}^{3} \ell_i)$ is the compactification of $\mathbf{C}^2$, $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$, respectively. The configuration of curves $\widehat{\ell_1} \cup \widehat{\ell_2} \cup \widehat{\ell_3} \cup \varDelta$ on M of the type $A_1 + E_7$ is as in Table* IV.

Table III (The type of singularities on X and the corresponding dual graph)

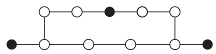
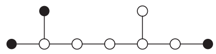
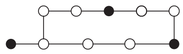
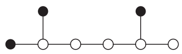
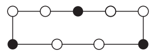
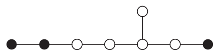
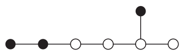
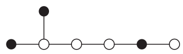
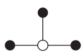
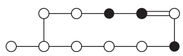
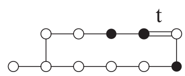
d	Compactification of $\mathbf{C}^2$	Compactification of $\mathbf{C} \times \mathbf{C}^*$	Compactification of $(\mathbf{C}^*)^2$
1			A_8 
2		D_6 	$A_2 + A_5$ 
3		A_5 	$3A_2$ 
		D_5 	
4		A_4 	
		$A_1 + A_3$ 	
5		$A_1 + A_2$	
6	A_1 		

Table IV

$A_1 + E_7$		

In this paper, the circle • (resp. ∘) denotes a (-1)-curve (resp. a (-2)-curve). Two components are joined by a straight line, double lines and double lines with a symbol t if the corresponding two curves meet at a point, at two distinct points and tangentially at a point, respectively.

THEOREM 1.4. *Let X be a normal del Pezzo surface with* $\mathrm{Sing}(X) \neq \emptyset$. *Let $\ell_1, \dots, \ell_N$ be quasi-lines on X such that $X - \bigcup_{i=1}^{N} \ell_i$ is biholomorphic to $\mathbf{C}^2$. Then $b_2(X) = N$ and $N \leq 3$.*

This paper is organized as follows. In Section 2, we give several preliminaries which will be used in Sections 3 and 4. In Section 3, we study a normal del Pezzo surface X with at most three quasi-lines. In Section 4, we determine the geometric structure of the complement of quasi-lines on X.

Notation

Throughtout this paper, we use the following symbols.
$\mathrm{Sing}(X)$: the singular locus of X
$K_X := \overline{K_{X-\mathrm{Sing}(X)}}$: the canonical divisor on X
K_M: the canonical divisor on M
$d := (K_X)^2$: the degree of X
N_X: the number of quasi-lines on X
$\sim$: the linear equivalence of divisors
$b_2(*)$: the second Betti number of $*$
$(z_0 : z_1 : z_2)$: the homogeneous coordinate system of $\mathbf{P}^2$.

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2. Preliminaries

In this section, we use the notations as in Section 1. We assume that $\mathrm{Sing}(X) \neq \emptyset$ and $1 \leq N_X \leq 3$. Then M is a rational surface. We remark that $M \not\simeq \mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{F}_1$ or $\mathbf{F}_2$. By Demazure [3] and Hidaka-Watanabe [4], we get the following:

Proposition 2.1. *There exists a set $\Sigma_r = \{P_1, \dots, P_r\}$ of $r(\leq 8)$-points on $\mathbf{P}^2$ which are in almost general position (See Definition 3.2 of Hidaka-Watanabe [4]) such that M is isomorphic to $V(\Sigma_r)$, where $V(\Sigma_r)$ is the rational surface obtained by the blowing-up of $\mathbf{P}^2$ with center Σ_r.*

Proposition 2.2. *There exists a smooth cubic curve Γ on $\mathbf{P}^2$ which passes through all points of Σ_r.*

Now, we put $\Sigma_j := \{P_1, \dots, P_j\} \subset \Sigma_r$ $(j \leq r)$. Let $\gamma_j : V(\Sigma_j) \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ with center Σ_j. Then we have a map $\pi_j : V(\Sigma_j) \to V(\Sigma_{j-1})$ such that $\gamma_j = \gamma_{j-1} \circ \pi_j$ $(2 \leq j \leq r)$. Thus we obtain the sequence of blowing-ups

$$M = V(\Sigma_r) \xrightarrow{\pi_r} V(\Sigma_{r-1}) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} V(\Sigma_1) \xrightarrow{\pi_1} \mathbf{P}^2,$$

where $\pi_1 = \gamma_1$. We put $\pi := \pi_1 \circ \cdots \circ \pi_r$. The map $\pi : M \to \mathbf{P}^2$ is called the blowing-up of $\mathbf{P}^2$ with center Σ_r.

Then we can show the following:

Corollary 2.3. *$K_M \sim -\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the proper transform of Γ on M. In particular, $\tilde{\Gamma}$ is an elliptic curve on M.*

Corollary 2.4. $\tilde{\Gamma}^2 = 9 - r > 0$. *In particular, we have* $1 \leq r \leq 8$.

The following is due to Brenton [1].

Proposition 2.5. $b_2(M) = b_2(X) + b_2(\Delta)$, *where* $b_2(\Delta)$ *is equal to the number of irreducible components of* Δ, *that is, the number of* (-2)-*curves on* M.

Then we have the following:

Lemma 2.6. $b_2(\Delta) \leq r$.

Proof. By using $b_2(M) = b_2(V(\Sigma_r)) = b_2(\mathbf{P}^2) + r = 1 + r$, we have $b_2(X) = 1 + r - b_2(\Delta)$. Since $b_2(X) \geq 1$, we have the assertion. $\square$

Lemma 2.7. *Let* Σ_r *be a set of* $r(\leq 8)$-*points on* $\mathbf{P}^2$ *which is allowed to contain infinitely near points and* $\pi : V(\Sigma_r) \to \mathbf{P}^2$ *the blowing-up of* $\mathbf{P}^2$ *with center* Σ_r.

(1) *If* C *is a* (-1)-*curve on* $V(\Sigma_r)$ *and the image* $\pi(C) =: C_0$ *is a curve on* $\mathbf{P}^2$, *then* C_0 *is one of the following:*

(i) *a line passing through two points of* Σ_r, *where* $r \geq 2$,

(ii) *a conic passing through five points of* Σ_r, *where* $r \geq 5$,

(iii) *a cubic passing through seven points of* Σ_r *such that one of them is a double point, where* $r \geq 7$,

(iv) *a quartic passing through all points of* Σ_8 *such that three of them are double points,*

(v) *a quintic passing through all points of* Σ_8 *such that six of them are double points,*

(vi) *a sextic passing through all points of* Σ_8 *such that seven of them are double points and one is a triple point.*

(2) *If* C *is a* (-2)-*curve on* $V(\Sigma_r)$ *and the image* $\pi(C) =: C_0$ *is a curve on* $\mathbf{P}^2$, *then* C_0 *is one of the following:*

(i) *a line passing through three points of* Σ_r, *where* $r \geq 3$,

(ii) *a conic passing through six points of* Σ_r, *where* $r \geq 6$,

(iii) *a cubic passing through all points of* Σ_8 *such that one of them is a double point.*

Proof. We denote the degree of C_0 by $\delta(\geq 1)$. Let $m_i = \mathrm{mult}_{P_i}\, C_0 \geq 0$ be the multiplicity of C_0 at P_i, where $m_i = 0$ means that $P_i \notin C_0$. We remark that $m_i \geq m_j$ if P_j is an infinitely near point of P_i. By the genus formula on the rational plane curve, we have

$$\frac{\delta^2 - 3\delta + 2}{2} = \sum_{i=1}^{r} \frac{m_i(m_i - 1)}{2},$$

that is, $\delta^2 - 3\delta + 2 = \sum_{i=1}^r m_i^2 - \sum_{i=1}^r m_i$. If C is a (-1)-curve, then

$$-1 = C^2 = C_0^2 - \sum_{i=1}^r m_i^2 = \delta^2 - \sum_{i=1}^r m_i^2.$$

Thus we have the system of equations

$$\sum_{i=1}^r m_i^2 = \delta^2 + 1 \qquad \text{and} \qquad \sum_{i=1}^r m_i = 3\delta - 1. \tag{1}$$

On the other hand, if C is a (-2)-curve, then

$$-2 = C^2 = C_0^2 - \sum_{i=1}^r m_i^2 = \delta^2 - \sum_{i=1}^r m_i^2.$$

Thus we also have the system of equations

$$\sum_{i=1}^r m_i^2 = \delta^2 + 2 \qquad \text{and} \qquad \sum_{i=1}^r m_i = 3\delta. \tag{2}$$

Hence it comes down to a question of the solutions for the systems of equations (1) and (2).

Now, we may assume that $m_1 \geq m_2 \geq \cdots \geq m_k \geq 1$ and $m_{k+1} = \cdots = m_r = 0$ without loss of generality, where $k \leq r$.

First, we shall solve the system of equations (1). If $\delta = 1$, then $\sum_{i=1}^k m_i^2 = 2$ and $\sum_{i=1}^k m_i = 2$. Hence $k = 2$ and $m_1 = m_2 = 1$. If $\delta = 2$, then $\sum_{i=1}^k m_i^2 = 5$ and $\sum_{i=1}^k m_i = 5$. Hence $k = 5$ and $m_1 = \cdots = m_5 = 1$. If $\delta \geq 3$, we have

$$k\left(\sum_{i=1}^k m_i^2\right) - \left(\sum_{i=1}^k m_i\right)^2 = \sum_{1 \leq i < j \leq k} (m_i - m_j)^2, \tag{3}$$

we have

$$k(\delta^2 + 1) - (3\delta - 1)^2 = (k - 9)\delta^2 + 6\delta + (k - 1) = \sum_{1 \leq i < j \leq k} (m_i - m_j)^2 \geq 0,$$

that is,

$$3 \leq \delta \leq \frac{3 + \sqrt{k(10 - k)}}{9 - k}.$$

From this inequality, we see $k = 7$ or 8.

If $k = 7$, then $\delta = 3$. Since $\sum_{i=1}^7 m_i^2 = 10$ and $\sum_{i=1}^7 m_i = 8$, we have $m_1 = 2$, $m_2 = \cdots = m_7 = 1$.

In the case $k = 8$, then $3 \le \delta \le 7$. (i) If $\delta = 7$, since $\sum_{i=1}^{8} m_i^2 = 50$ and $\sum_{i=1}^{8} m_i = 20$, we have $\sum_{1 \le i<j \le 8}(m_i - m_j)^2 = 8 \cdot 50 - 20^2 = 0$, that is, $m_1 = \cdots = m_8 = 5/2$. This leads to a contradiction. (ii) If $\delta = 6$, then $\sum_{i=1}^{8} m_i^2 = 37$ and $\sum_{i=1}^{8} m_i = 17$. Moreover, $\sum_{1 \le i<j \le 8}(m_i - m_j)^2 = 8 \cdot 37 - 17^2 = 7$. Since $\sum_{i=1}^{8} m_i \ge 8m_8$, $m_8 = 1$ or 2. If $m_8 = 1$, $\sum_{i=1}^{7} m_i^2 = 36$ and $\sum_{i=1}^{7} m_i = 16$. Then $\sum_{1 \le i<j \le 7}(m_i - m_j)^2 = 7 \cdot 36 - 16^2 = -4$, which leads to a contradiction. If $m_8 = 2$, $\sum_{i=1}^{7} m_i^2 = 33$ and $\sum_{i=1}^{7} m_i = 15$. Then $\sum_{1 \le i<j \le 7}(m_i - m_j)^2 = 7 \cdot 33 - 15^2 = 6$. Hence we have $\sum_{1 \le i \le 7}(m_i - 1)^2 = \sum_{1 \le i<j \le 8}(m_i - m_j)^2 - \sum_{1 \le i<j \le 7}(m_i - m_j)^2 = 1$, that is, $m_1 = 3$, $m_2 = \cdots = m_7 = 2$. (iii) If $\delta = 5$, then $\sum_{i=1}^{8} m_i^2 = 26$ and $\sum_{i=1}^{8} m_i = 14$. From $\sum_{i=1}^{8} m_i \ge 8m_8$, we have $m_8 = 1$. Then $\sum_{i=1}^{7} m_i^2 = 25$ and $\sum_{i=1}^{7} m_i = 13$. Moreover, from $\sum_{i=1}^{7} m_i \ge 7m_7$, we have $m_7 = 1$, which implies $\sum_{i=1}^{6} m_i^2 = 24$ and $\sum_{i=1}^{6} m_i = 12$. Hence we have $\sum_{1 \le i<j \le 6}(m_i - m_j)^2 = 6 \cdot 24 - 12^2 = 0$, that is, $m_1 = \cdots = m_6 = 2$. (iv) If $\delta = 4$, then $\sum_{i=1}^{8} m_i^2 = 17$ and $\sum_{i=1}^{8} m_i = 11$. If $m_4 \ge 2$, then $\sum_{i=1}^{8} m_i \ge 4 \cdot 2 + 4 = 12$. This leads to a contradiction. Thus we have $m_4 = \cdots = m_8 = 1$. Then $\sum_{i=1}^{3} m_i^2 = 12$ and $\sum_{i=1}^{3} m_i = 6$. Hence we have $m_1 = m_2 = m_3 = 2$. (v) If $\delta = 3$, then $\sum_{i=1}^{8} m_i^2 = 10$ and $\sum_{i=1}^{8} m_i = 8$. There are no solutions for this system of equations.

Therefore all solutions of the system of equations (1) are obtained as follows up to all possible permutations of the m_i's:

$$\delta = 1 \text{ and } m_1 = m_2 = 1, m_3 = \cdots = m_r = 0 \text{ for } r \ge 2,$$

$$\delta = 2 \text{ and } m_1 = \cdots = m_5 = 1, m_6 = \cdots = m_r = 0 \text{ for } r \ge 5,$$

$$\delta = 3 \text{ and } m_1 = 2, m_2 = \cdots = m_7 = 1, m_8 = 0 \text{ for } r \ge 7,$$

$$\delta = 4 \text{ and } m_1 = m_2 = m_3 = 2, m_4 = \cdots = m_8 = 1 \text{ for } r = 8,$$

$$\delta = 5 \text{ and } m_1 = \cdots = m_6 = 2, m_7 = m_8 = 1 \text{ for } r = 8,$$

$$\delta = 6 \text{ and } m_1 = 3, m_2 = \cdots = m_8 = 2 \text{ for } r = 8.$$

By the argument similar to the above, all solutions for the system of equations (2) are obtained as follows up to all possible permutations of the m_i's:

$$\delta = 1 \text{ and } m_1 = m_2 = m_3 = 1, m_4 = \cdots = m_r = 0, \text{ for } r \ge 3,$$

$$\delta = 2 \text{ and } m_1 = \cdots = m_6 = 1, m_7 = m_8 = 0 \text{ for } r \ge 6,$$

$$\delta = 3 \text{ and } m_1 = 2, m_2 = \cdots = m_7 = m_8 = 1 \text{ for } r = 8.$$

Thus the lemma holds. $\square$

By an elementary calculation, we can obtain the following:

LEMMA 2.8. *Let Σ_r be a set of r-points on $\mathbf{P}^2$ which is allowed to contain infinitely near points. Then we have the following:*

(1) *Let $\{P_1, P_2, P_3\}$ be a set of three points of Σ_r for $r \geq 3$. If all points of them are on a line L, then*
- (i) *no line except L passes through two of the points P_i,*
- (ii) *no conic passes through all of the points P_i,*
- (iii) *no cubic passes through all of the points P_i such that one of them is a double point,*
- (iv) *no quartic passes through all of the points P_i such that two of them are double points,*
- (v) *no quintic passes through all of the points P_i such that all of them are double points,*
- (vi) *no sextic passes through all of the points P_i such that two of them are double points and one is a triple point.*

(2) *Let $\{P_1, \dots, P_6\}$ be a set of six points of Σ_r for $r \geq 6$. If all points of them are on a smooth conic C, then*
- (i) *no line passes through three of the points P_i,*
- (ii) *no conic other than C passes through five of the points P_i,*
- (iii) *no cubic passes through all of the points P_i such that one of them is a double point,*
- (iv) *no quartic passes through all of the points P_i such that three of them are double points,*
- (v) *no quintic passes through all of the points P_i such that five of them are double points,*
- (vi) *no sextic passes through all of the points P_i such that five of the points P_i are double points and one is a triple point.*

(3) *If all points of $\Sigma_8 = \{P_1, \dots, P_8\}$ are on an irreducible cubic C with P_1 as a double point, then*
- (i) *no line passes through P_1 and other two of the points P_i,*
- (ii) *no conic passes through P_1 and other five of the points P_i,*
- (iii) *no cubic other than C passes through P_1 and other six of the points P_i such that P_1 is a double point,*
- (iv) *no cubic other than C passes through all of the points P_i such that one of them is a double point,*
- (v) *no quartic passes through all of the points P_i such that P_1 and other two of them are double points,*
- (vi) *no quintic passes through all of the points P_i such that P_1 and other five of them are double points,*
- (vii) *no sextic passes through all of the points P_i such that seven of them are double points and one is a triple point.*

Proof. (1) (iii) Let $\{P_1, P_2, P_3\}$ be a set of three points of Σ_r and L be a line which passes through all of points of them. Then we have the sequence of blowings-up

$$V(\Sigma_3) \xrightarrow{\pi_3} V(\Sigma_2) \xrightarrow{\pi_2} V(\Sigma_1) \xrightarrow{\pi_1} \mathbf{P}^2,$$

where $V(\Sigma_1)$ is the blowing up of $\mathbf{P}^2$ with center P_1 in $\mathbf{P}^2$ and $V(\Sigma_{j+1})$ is the blowing up of $V(\Sigma_j)$ with center P_{j+1} in $V(\Sigma_j)$. We set $E_j := \pi_j^{-1}(P_j)$ in $V(\Sigma_j)$. Assume that there exists a cubic D which passes through all of the points P_i such that P_1 is a double point. We denote the proper transform of L and D on $V(\Sigma_j)$ by $L^{(j)}$ and $D^{(j)}$, respectively. Then

$$(L^{(1)}, D^{(1)}) = (\pi_1^*L, \pi_1^*D) + 2E_1^2 = (L, D) + 2E_1^2 = 3 - 2 = 1$$

on $V(\Sigma_1)$ since $L^{(1)} \sim \pi_1^*L - E_1$ and $D^{(1)} \sim \pi_1^*D - 2E_1$,

$$(L^{(2)}, D^{(2)}) = (\pi_2^*L^{(1)}, \pi_2^*D^{(1)}) + E_2^2 = (L^{(1)}, D^{(1)}) + E_2^2 = 1 - 1 = 0$$

on $V(\Sigma_2)$ since $L^{(2)} \sim \pi_2^*L^{(1)} - E_2$ and $D^{(2)} \sim \pi_2^*D^{(1)} - E_2$. This implies that $L^{(2)} \cap D^{(2)} = \emptyset$, that is, $P_3 \notin D^{(2)}$ on $V(\Sigma_2)$, which is a contradiction. Similar arguments show the assertions (2), (3). $\square$

3. Classification of normal del Pezzo surfaces with at most three quasi-lines

Let us retain the above notations. Now, we fix the set Σ_r of r-points $(1 \le r \le 8)$ on $\mathbf{P}^2$ which are in almost general position. Let Γ be an elliptic curve passing through all points of Σ_r. We put $\Sigma_0 \subset \mathbf{P}^2$ the set of points of Σ_r which are not infinitely near points, that is, $\Sigma_0 = \Sigma_r - \{\text{infinitely near points}\}$. From the relation

$$\begin{aligned} N_X &:= \text{the number of quasi-lines on } X \\ &= \text{the number of } (-1)\text{-curves on } M \\ &\ge \text{the number of points of } \Sigma_0 \\ &=: |\Sigma_0|, \end{aligned}$$

we have the following:

(1) $N_X = 1 \Rightarrow |\Sigma_0| = 1$.
(2) $N_X = 2 \Rightarrow |\Sigma_0| \le 2$.
(3) $N_X = 3 \Rightarrow |\Sigma_0| \le 3$.

Case 1. The case $|\Sigma_0| = 1$

In this case, Σ_r consists of a point P_1 on $\mathbf{P}^2$ and its infinitely near points $P_2, \dots, P_r$. Let E_i be the exceptional curve of the first kind associated with the blowing-up with center P_i, where $P_{i+1} \in E_i$ $(1 \le i \le r-1)$. We denote the proper transform of E_i on M by the same notation E_i. Then E_i's $(1 \le i \le r-1)$ and E_r are (-2)-curves and a (-1)-curve on M, respectively. Let L be the tangent line to Γ at P_1 and put $\tilde{L}$ the proper transform of L on M.

Case 1.1. The case of $N_X = 1$

In this case, there exists only one (-1)-curve on M. If $r = 2$, then $N_X \neq 1$ since $\tilde{L}$ is a (-1)-curve on M. In case of $r \ge 3$, P_1 is a flex point of Γ. If it is not so, then $\tilde{L}$ is a (-1)-curve on M, that is, $N_X \neq 1$. From Lemma 2.6, we obtain that $E_1, \dots, E_{r-1}$, $\tilde{L}$ are all of (-2)-curves on M. Moreover, by Lemma 2.7, we observe that there exist no (-1)-curves on M except for E_r. Hence, the types of singularities of X with $N_X = 1$ are determined as follows:

$$r = 3 \Rightarrow \mathrm{Sing}(X) = A_1 + A_2,$$

$$r = 4 \Rightarrow \mathrm{Sing}(X) = A_4,$$

$$r = 5 \Rightarrow \mathrm{Sing}(X) = D_5,$$

$$r = 6 \Rightarrow \mathrm{Sing}(X) = E_6,$$

$$r = 7 \Rightarrow \mathrm{Sing}(X) = E_7,$$

$$r = 8 \Rightarrow \mathrm{Sing}(X) = E_8.$$

Remark 3.1. *All normal del Pezzo surfaces with* $\mathrm{Sing}(X) \neq \emptyset$ *and* $N_X = 1$ *are the six listed in Table* I.

Case 1.2. The case of $N_X = 2$

In this case, there exist exactly two (-1)-curves on M. If $r = 2$, then $N_X = 2$ since $\tilde{L}$ is a (-1)-curve on M. In case of $r \ge 3$, by the result in Case 1.1, P_1 is not a flex point of Γ and hence $\tilde{L}$ is a (-1)-curve on M. If $r = 3, 4$, from Lemma 2.7 and Lemma 2.8, it follows that $E_1, \dots, E_{r-1}$ (resp. E_r, $\tilde{L}$) are all of (-2)-curves (resp. (-1)-curves) on M. In case of $r \ge 5$, there exists a unique smooth conic C passing through five points $P_1, \dots, P_5$. We denote by $\tilde{C}$ the proper transform of C on M. If $r = 5$, then $N_X \neq 2$ since $\tilde{C}$ is a (-1)-curve on M. In case of $r \ge 6$, C must pass through the point P_6 and then $\tilde{C}$ is a (-2)-curve on M. From Lemma 2.6, we obtain that $E_1, \dots, E_{r-1}$, $\tilde{C}$ are all of (-2)-curves on M. Moreover, by Lemma 2.7 and Lemma 2.8, we have that there exist no (-1)-curves on M except for E_r, $\tilde{L}$. Hence, the types of singularities of X are determined as follows:

$$r = 2 \Rightarrow \mathrm{Sing}(X) = A_1,$$

$$r = 3 \Rightarrow \mathrm{Sing}(X) = A_2,$$

$$r = 4 \Rightarrow \mathrm{Sing}(X) = A_3,$$

$$r = 6 \Rightarrow \mathrm{Sing}(X) = A_1 + A_5,$$

$$r = 7 \Rightarrow \mathrm{Sing}(X) = A_7,$$

$$r = 8 \Rightarrow \mathrm{Sing}(X) = D_8.$$

For example, the configurations of $\{P_1, L, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ L = \{z_1 = 0\}, \\ C = \{z_0^2 - z_1 z_2 = 0\}. \end{cases}$$

Case 1.3. The case of $N_X = 3$

In this case, there exist exactly three (-1)-curves on M. By the results in Case 1.1 and Case 1.2, we may consider the case where P_1 is not a flex point of Γ and $r \geq 5$. Then $\tilde{L}$ is a (-1)-curve on M. There exists a unique smooth conic C passing through five points $P_1, \dots, P_5$. We put $\tilde{C}$ the proper transform of C on M. If $r = 5$, then $\tilde{C}$ is a (-1)-curve on M. Therefore, from Lemma 2.7, we obtain that there exist no (-2)-curves on M except for $E_1, \dots, E_4$ and no (-1)-curves on M except for E_5, $\tilde{L}$, $\tilde{C}$. Hence, $N_X = 3$. In case of $r \geq 6$, C does not pass through the point P_6 and then $\tilde{C}$ is a (-1)-curve on M. If $r = 6$, from Lemma 2.7, it follows that $E_1, \dots, E_5$ (resp. E_6, $\tilde{L}$, and $\tilde{C}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. Hence, $N_X = 3$. In case of $r \geq 7$, there exists uniquely an irreducible cubic D passing through seven points $P_1, \dots, P_7$ such that P_1 is a double point. We denote by $\tilde{D}$ the proper transform of D on M. We remark that the irreducible cubic D has P_1 as a node since Σ_r is in almost general position on $\mathbf{P}^2$. If $r = 7$, then $N_X \neq 3$ since $\tilde{D}$ is a (-1)-curve on M. If $r = 8$, then D passes through the point P_8, so $\tilde{D}$ is a (-2)-curve on M. From Lemma 2.6, we obtain that there exist no (-2)-curves on M except for $E_1, \dots, E_7$, $\tilde{D}$. Furthermore, by Lemma 2.7 and Lemma 2.8, we have that there exist no (-1)-curves on M except for E_8, $\tilde{L}$, $\tilde{C}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$r = 5 \Rightarrow \mathrm{Sing}(X) = A_4,$$

$$r = 6 \Rightarrow \mathrm{Sing}(X) = A_5,$$

$$r = 8 \Rightarrow \mathrm{Sing}(X) = A_8.$$

For example, the configurations of $\{P_1, L, C, D\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ L = \{z_1 = 0\}, \\ C = \left\{ z_0^2 + \dfrac{1}{2} z_1^2 + \dfrac{1}{\sqrt{2}} z_0 z_1 + \sqrt{2} z_1 z_2 = 0 \right\}, \\ D = \{z_0^3 + \sqrt{2} z_0 z_1 z_2 - z_1^2 z_2 = 0\}. \end{cases}$$

Case 2. The case of $|\Sigma_0| = 2$

Now, we assume that Σ_r consists of (distinct) two points $P_1 (= P_1^1)$ and $P_2 (= P_2^1)$ on $\mathbf{P}^2$ and their infinitely near points $P_1^2, \dots, P_1^{r_1}$ and $P_2^2, \dots, P_2^{r_2}$, respectively, where $r = r_1 + r_2$. Let E_i^j be the exceptional curve of the first kind associated with the blowing-up with center P_i^j, where $P_i^{j+1} \in E_i^j$ $(1 \le i \le 2, 1 \le j \le r_i - 1)$. We denote the proper transform of E_i^j on M by the same notation E_i^j. Then E_i^j's $(1 \le i \le 2, 1 \le j \le r_i - 1)$ and $E_1^{r_1}$, $E_2^{r_2}$ are respectively (-2)-curves and (-1)-curves on M. Let L_0 be the line passing through two points P_1 and P_2. We put $\widetilde{L_0}$ the proper transform of L_0 on M. If $r = 2$, namely, $(r_1, r_2) = (1, 1)$, there exist no (-2)-curves on M. This implies that X is smooth. Thus we may consider the case of $r \ge 3$.

Case 2.1. The case of $N_X = 2$

In this case, there exist exactly two (-1)-curves on M. Hence L_0 must be a tangent line to Γ, that is, $P_1^2 \in L_0$ or $P_2^2 \in L_0$. Then $\widetilde{L_0}$ is a (-2)-curve on M. Now, we may assume that $P_2^2 \in L_0$. Let L_1 be a tangent line to Γ at P_1 and put $\widetilde{L_1}$ the proper transform of L_1 on M.

(1) The case of $r_1 = 1$. In case of $2 \le r_2 \le 4$, from Lemma 2.7 and Lemma 2.8, we obtain that $E_2^1, \dots, E_2^{r_2 - 1}$, and $\widetilde{L_0}$ (resp. E_1^1 and $E_2^{r_2}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. In case of $r_2 \ge 5$, there exists uniquely a smooth conic C passing through five points $P_2^1, \dots, P_2^5$. We denote by $\tilde{C}$ the proper transform of C on M. If $r_2 = 5$, then $\tilde{C}$ is a (-1)-curve on M, that is, $N_X \ne 2$. In case of $r_2 \ge 6$, C must pass through the point P_2^6. Then $\tilde{C}$ is a (-2)-curve on M. By Lemma 2.6, one sees that there exist no (-2)-curves on M except for $E_2^1, \dots, E_2^{r_2 - 1}$, $\widetilde{L_0}$, $\tilde{C}$. Furthermore, from Lemma 2.7 and Lemma 2.8, we obtain that there exist no (-1)-curves on M except for E_1^1, $E_2^{r_2}$, that is, $N_X = 2$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (1, 2) \Rightarrow \mathrm{Sing}(X) = 2A_1,$$

$$(r_1, r_2) = (1, 3) \Rightarrow \mathrm{Sing}(X) = A_3,$$

$$(r_1, r_2) = (1, 4) \Rightarrow \mathrm{Sing}(X) = D_4,$$

$$(r_1, r_2) = (1, 6) \Rightarrow \mathrm{Sing}(X) = A_1 + D_6,$$

$$(r_1, r_2) = (1, 7) \Rightarrow \mathrm{Sing}(X) = D_8.$$

For example, the configurations of $\{P_1, P_2, L_0, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (1:0:0), \\ L_0 = \{z_1 = 0\}, \\ C = \{z_0^2 - z_1 z_2 = 0\}. \end{cases}$$

(2) The case of $r_1 = 2$. In this case, $N_X \neq 2$ since $\widetilde{L_1}$ is a (-1)-curve on M.

In case of $r_1 \geq 3$, P_1 must be a flex point of Γ and then $\widetilde{L_1}$ is a (-2)-curve on M. From Lemma 2.6, we have that $E_1^1, \ldots, E_1^{r_1-1}$, $E_2^1, \ldots, E_2^{r_2-1}$, $\widetilde{L_0}$, and $\widetilde{L_1}$ exhaust all of (-2)-curves on M.

(3) The case of $r_1 = 3$. In case of $2 \leq r_2 \leq 4$, by Lemma 2.7 and Lemma 2.8, it follows that there exist no (-1)-curves on M except for E_1^3, $E_2^{r_2}$, that is, $N_X = 2$. If $r_2 = 5$, then there exists uniquely a smooth conic C passing through five points $P_2^1, \ldots, P_2^5$. We denote by $\tilde{C}$ the proper transform of C on M. Then we have $N_X \neq 2$ since $\tilde{C}$ is a (-1)-curve on M. Therefore, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (3, 2) \Rightarrow \mathrm{Sing}(X) = 2A_1 + A_3,$$

$$(r_1, r_2) = (3, 3) \Rightarrow \mathrm{Sing}(X) = A_1 + A_5,$$

$$(r_1, r_2) = (3, 4) \Rightarrow \mathrm{Sing}(X) = A_1 + D_6.$$

(4) The case of $r_1 = 4$. Then since $2 \leq r_2 \leq 4$, by Lemma 2.7 and Lemma 2.8, we obtain that there exist no (-1)-curves on M except for E_1^4, $E_2^{r_2}$. Hence, we have $N_X = 2$ and the types of singularities of X are determined as follows:

$$(r_1, r_2) = (4, 2) \Rightarrow \mathrm{Sing}(X) = A_1 + A_5,$$

$$(r_1, r_2) = (4, 3) \Rightarrow \mathrm{Sing}(X) = A_7,$$

$$(r_1, r_2) = (4, 4) \Rightarrow \mathrm{Sing}(X) = D_8.$$

(5) The case of $r_1 = 5$. Then since $2 \leq r_2 \leq 3$, by Lemma 2.7 and Lemma 2.8, one can show that there exist no (-1)-curves on M except for E_1^5, $E_2^{r_2}$. Hence, we have $N_X = 2$ and the types of singularities of X are determined as follows:

$$(r_1, r_2) = (5, 2) \Rightarrow \mathrm{Sing}(X) = A_1 + D_6,$$

$$(r_1, r_2) = (5, 3) \Rightarrow \mathrm{Sing}(X) = D_8.$$

(6) The case of $r_1 = 6$**.** In this case, there exists a unique irreducible cubic D passing through seven points $P_1^1, \dots, P_1^6$, P_2^1 such that P_2 is a double point. We set $\tilde{D}$ the proper transform of D on M. Then we see $N_X \neq 2$ since $\tilde{D}$ is a (-1)-curve.

Case 2.2. The case of $N_X = 3$

(1) The case where $\widetilde{L_0}$ **is a** (-2)**-curve on** M**.** In this case, since L_0 is a tangent line to Γ, we may assume that $r_2 \geq 2$ and $P_2^2 \in L_0$. Let L_1 be the tangent line to Γ at P_1 and put $\widetilde{L_1}$ the proper transform of L_1 on M.

(1-1) The case of $r_1 = 1$**.** In case of $2 \leq r_2 \leq 4$, one has $N_X = 2$ by the result in (1) of Case 2.1. In case of $r_2 \geq 5$, there exists uniquely a smooth conic C passing through five points $P_2^1, \dots, P_2^5$. We denote by $\tilde{C}$ the proper transform of C on M. If $r_2 = 5$, then $\tilde{C}$ is a (-1)-curve on M. By Lemma 2.7 and Lemma 2.8, it follows that the curves $E_2^1, \dots, E_2^4$, $\tilde{C}$ (resp. E_1^1, E_2^5, $\widetilde{L_0}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. Thus we have $N_X = 3$. In case of $r_2 \geq 6$, by the result in (1) of Case 2.1, C must not pass through the point P_2^6. Then $\tilde{C}$ is a (-1)-curve on M. If $r_2 = 6$, by Lemma 2.7 and Lemma 2.8, we obtain that the curves $E_2^1, \dots, E_2^5$, $\widetilde{L_0}$ (resp. E_1^1, E_2^6, $\tilde{C}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. Hence we see $N_X = 3$. If $r_2 = 7$, then there exists a unique irreducible cubic D passing through seven points $P_2^1, \dots, P_2^7$ such that P_2^1 is a double point. We set $\tilde{D}$ the proper transform of D on M. Then we have $N_X \neq 3$ since $\tilde{D}$ is a (-1)-curve on M. Therefore the types of singularities of X are determined as follows:

$$(r_1, r_2) = (1, 5) \Rightarrow \mathrm{Sing}(X) = D_5,$$

$$(r_1, r_2) = (1, 6) \Rightarrow \mathrm{Sing}(X) = D_6.$$

(1-2) The case of $r_1 = 2$**.** In this case, $\widetilde{L_1}$ is a (-1)-curve on M. In case of $2 \leq r_2 \leq 4$, by Lemma 2.7 and Lemma 2.8, we obtain that there exist no (-1)-curves and no (-2)-curves on M except for E_1^2, $E_2^{r_2}$, $\widetilde{L_1}$ and E_1^1, $E_2^1, \dots, E_2^{r_2-1}$, $\widetilde{L_0}$, respectively. Then we see $N_X = 3$. In case of $r_2 \geq 5$, there exists a unique smooth conic C passing through five points $P_2^1, \dots, P_2^5$. We put $\tilde{C}$ the proper transform of C on M. If $r_2 = 5$, then $N_X \neq 3$ since $\tilde{C}$ is a (-1)-curve on M. If $r_2 = 6$ and C passes through the point P_2^6, then $\tilde{C}$ is a (-2)-curve on M. From Lemma 2.6, it follows that E_1^1, $E_2^1, \dots, E_2^5$, $\widetilde{L_0}$ and $\tilde{C}$ exhaust all of (-2)-curves on M. Moreover, from Lemma 2.7 and Lemma 2.8, we obtain that there exist no (-1)-curves on M except for E_2^2, E_2^6, $\widetilde{L_1}$, that is, $N_X = 3$. Hence, the types of singularities on X are determined as follows:

$$(r_1, r_2) = (2,2) \Rightarrow \mathrm{Sing}(X) = A_1 + A_2,$$

$$(r_1, r_2) = (2,3) \Rightarrow \mathrm{Sing}(X) = A_4,$$

$$(r_1, r_2) = (2,4) \Rightarrow \mathrm{Sing}(X) = D_5,$$

$$(r_1, r_2) = (2,6) \Rightarrow \mathrm{Sing}(X) = A_1 + E_7.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (0:1:0), \\ L_0 = \{z_0 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ C = \{z_0^2 + z_0 z_2 - z_1 z_2 = 0\} \text{ or } \{z_0^2 - z_1 z_2 = 0\}. \end{cases}$$

(1-3) The case of $r_1 = 3$**.** First, we consider the case where P_1^1 is a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-2)-curve on M. By Lemma 2.6, we obtain that there exist no (-2)-curves on M except for E_1^1, E_1^2, $E_2^1, \dots, E_2^{r_2-1}$, $\widetilde{L_0}$, $\widetilde{L_1}$. In case of $2 \le r_2 \le 4$, $N_X = 2$ by the result in (3) of Case 2.1. If $r_2 = 5$, then there exists a unique smooth conic C passing through five points $P_2^1, \dots, P_2^5$. We denote by $\tilde{C}$ the proper transform of C on M. Then $\tilde{C}$ is a (-1)-curve on M. From Lemma 2.7 and Lemma 2.8, we obtain that there exist no (-1)-curves on M except for E_2^3, E_2^5, $\tilde{C}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (3,5) \Rightarrow \mathrm{Sing}(X) = A_1 + E_7.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (0:1:0), \\ L_0 = \{z_0 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ C = \{z_0^2 + z_0 z_2 - z_1 z_2 = 0\} \text{ or } \{z_0^2 - z_1 z_2 = 0\}. \end{cases}$$

Next, we consider the case where P_1^1 is not a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-1)-curve on M. In case of $2 \le r_2 \le 4$, from Lemma 2.7 and Lemma 2.8, we have that E_1^1, E_1^2, $E_2^1, \dots, E_2^{r_2-1}$ and $\widetilde{L_0}$ (resp. E_1^3, $E_2^{r_2}$ and $\widetilde{L_1}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. Hence, $N_X = 3$. If $r_2 = 5$, then there exists a unique smooth conic C passing through five points $P_2^1, \dots, P_2^5$. We set $\tilde{C}$ the proper transform of C on M. Then $N_X \ne 3$ since $\tilde{C}$ is a (-1)-curve on M. Thus the types of singularities of X are determined as follows:

$$(r_1, r_2) = (3, 2) \Rightarrow \mathrm{Sing}(X) = A_1 + A_3,$$
$$(r_1, r_2) = (3, 3) \Rightarrow \mathrm{Sing}(X) = A_5,$$
$$(r_1, r_2) = (3, 4) \Rightarrow \mathrm{Sing}(X) = D_6.$$

(1-4) The case of $r_1 = 4$. First, we consider the case where P_1^1 is a flex point of Γ. In this case, one has $N_X = 2$ by the result in (4) of Case 2.1. Next, we consider the case where P_1^1 is not a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-1)-curve on M. Moveover, there exists uniquely a smooth conic C passing through five points $P_1^1, \dots, P_1^4$, P_2^1. We put $\tilde{C}$ the proper transform of C on M. Then we have $N_X \neq 3$ since $\tilde{C}$ is a (-1)-curve on M.

(1-5) The case of $r_1 = 5$. First, we consider the case where P_1^1 is a flex point of Γ. In this case, one has $N_X = 2$ by the result in (5) of Case 2.1. Next, we consider the case where P_1^1 is not a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-1)-curve on M. Furthermore, there exists uniquely a smooth conic C passing through five points $P_1^1, \dots, P_1^4$, P_2^1. We denote by $\tilde{C}$ the proper transform of C on M. Then C must pass through the point P_1^5 and hence $\tilde{C}$ is a (-2)-curve on M. From Lemma 2.6, we observe that $E_1^1, \dots, E_1^4$, $E_2^1, \dots, E_2^{r_2-1}$, $\widetilde{L_0}$ and $\tilde{C}$ exhaust all of (-2)-curves on M. Moreover, by Lemma 2.7 and Lemma 2.8, it follows that there exist no (-1)-curves on M except for E_1^5, $E_2^{r_2}$, $\widetilde{L_1}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (5, 2) \Rightarrow \mathrm{Sing}(X) = A_2 + A_5.$$
$$(r_1, r_2) = (5, 3) \Rightarrow \mathrm{Sing}(X) = A_8.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (0:1:0), \\ L_0 = \{z_0 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ C = \{z_0^2 + z_0 z_1 - z_1 z_2 = 0\}. \end{cases}$$

(1-6) The case of $r_1 = 6$. First, we consider the case where P_1^1 is a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-2)-curve on M. Furthermore, there exists a unique irreducible cubic D passing through seven points $P_1^1, \dots, P_1^6$, P_2^1 such that P_2^1 is a double point. We denote by $\tilde{D}$ the proper transform of D on M. From Lemma 2.6, it follows that $E_1^1, \dots, E_1^5$, E_2^1, $\widetilde{L_0}$ and $\widetilde{L_1}$ exhaust all of (-2)-curves on M. Moreover, by Lemma 2.7 and Lemma 2.8, we observe that there exist no (-1)-curves on M except for E_1^6, E_2^2, $\tilde{D}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (6, 2) \Rightarrow \mathrm{Sing}(X) = A_1 + E_7.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, D\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (0:1:0), \\ L_0 = \{z_0 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ D = \{z_0^3 - z_1 z_2^2 + z_0 z_1 z_2 = 0\}, \ \text{or} \ \{z_0^3 - z_0^2 z_1 - z_1 z_2^2 + 2 z_0 z_1 z_2 = 0\}. \end{cases}$$

Next, we consider the case where P_1^1 is not a flex point of Γ. In this case, $\widetilde{L_1}$ is a (-1)-curve on M. Then there exists uniquely a smooth conic C passing through five points $P_1^1, \dots, P_1^4$, P_2^1. We set $\tilde{C}$ the proper transform of C on M. Then C must pass through the point P_1^5, and hence $\tilde{C}$ is a (-2)-curve on M. From Lemma 2.6, it follows that $E_1^1, \dots, E_1^5$, E_2^1, $\widetilde{L_0}$ and $\tilde{C}$ exhaust all of (-2)-curves on M. Furthermore, by Lemma 2.7 and Lemma 2.8, we see that there exist no (-1)-curves on M except for E_2^6, E_2^2, $\widetilde{L_1}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (6, 2) \Rightarrow \mathrm{Sing}(X) = A_8.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, C\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (0:0:1), \\ P_2 = (0:1:0), \\ L_0 = \{z_0 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ C = \{z_0^2 - z_1 z_2 = 0\}. \end{cases}$$

(2) The case where $\widetilde{L_0}$ is a (-1)-curve on M. Then it follows that L_0 is not a tangent line to Γ at P_1^1. Let L_1 be the tangent line to Γ at P_1^1.

(2-1) The case of $r_2 = 1$. In this case, it follows that $r_1 \geq 3$ and P_1^1 is a flex point of Γ. Then $\widetilde{L_1}$ is a (-2)-curve on M. In case of $3 \leq r_1 \leq 5$, by Lemma 2.7 and Lemma 2.8, we obtain that $E_1^1, \dots, E_1^{r_1 - 1}$ and $\widetilde{L_1}$ (resp. $E_1^{r_1}$, E_2^1 and $\widetilde{L_0}$) exhaust all of (-2)-curves (resp. (-1)-curves) on M. Thus we have $N_X = 3$. In case of $r_1 \geq 6$, there exists a unique irreducible cubic D passing through seven points $P_1^1, \dots, P_1^6$, P_2^1 such that P_2^1 is a double point. We denote by $\tilde{D}$ the proper transform of D on M. If $r_1 = 6$, then we have $N_X \neq 3$ since $\tilde{D}$ is a (-1)-curve on M. If $r_1 = 7$, D must pass through the point P_2^7 and hence $\tilde{D}$ is a (-2)-curve on M. From Lemma 2.6, we observe that the (-2)-curves on M are eight curves $E_1^1, \dots, E_1^6$, $\widetilde{L_1}$, $\tilde{D}$. Furthermore, by Lemma 2.7 and Lemma 2.8, it follows that there exist no (-1)-curves on M except for E_1^7, E_2^1, $\widetilde{L_0}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (3, 1) \Rightarrow \mathrm{Sing}(X) = A_1 + A_2,$$

$$(r_1, r_2) = (4, 1) \Rightarrow \mathrm{Sing}(X) = A_4,$$

$$(r_1, r_2) = (5, 1) \Rightarrow \mathrm{Sing}(X) = D_5,$$

$$(r_1, r_2) = (7, 1) \Rightarrow \mathrm{Sing}(X) = A_1 + E_7.$$

For example, the configurations of $\{P_1, P_2, L_0, L_1, D\}$ on $\mathbf{P}^2$ are given by

$$\begin{cases} P_1 = (1:0:0), \\ P_2 = (0:1:0), \\ L_0 = \{z_2 = 0\}, \\ L_1 = \{z_1 = 0\}, \\ D = \{z_2^3 - z_0^2 z_1 + z_0 z_1 z_2 = 0\}, \text{ or } \{z_2^3 - z_0^2 z_1 - z_1 z_2^2 + 2 z_0 z_1 z_2 = 0\}. \end{cases}$$

Next, we assume that $r_2 \geq 2$. Then L_0 is not the tangent line to Γ. Let L_1 and L_2 be the tangent lines to Γ at P_1^1 and P_2^1, respectively. We put $\widetilde{L_1}$ and $\widetilde{L_2}$ the proper transforms on M of L_1 and L_2, respectively. We may assume that $r_1 \geq r_2$.

(2-2) The case of $r_2 = 2$. In this case, $N_X \neq 3$ since $\widetilde{L_2}$ is a (-1)-curve on M.

In case of $r_2 \geq 3$, it follows that both P_1^1 and P_2^1 must be flexes on Γ and $r_1 \geq 3$. Then $\widetilde{L_1}$ and $\widetilde{L_2}$ are (-2)-curves on M. By Lemma 2.6, we obtain that $E_1^1, \dots, E_1^{r_1-1}$, $E_2^1, \dots, E_2^{r_2-1}$, $\widetilde{L_1}$ and $\widetilde{L_2}$ exhaust all of (-2)-curves on M.

(2-3) The case of $r_2 = 3$. In case of $3 \leq r_1 \leq 4$, from Lemma 2.7 and Lemma 2.8, we observe that there exist no (-1)-curves on M except for $E_1^{r_1}$, E_2^3, $\widetilde{L_0}$, that is, $N_X = 3$. If $r_1 = 5$, then there exists a unique irreducible cubic D passing through seven points $P_1^1, \dots, P_1^5$, P_2^1, P_2^2 such that P_2^1 is a double point. We put $\tilde{D}$ the proper transform of D on M. Then $N_X \neq 3$ since $\tilde{D}$ is a (-1)-curve on M. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (3, 3) \Rightarrow \mathrm{Sing}(X) = 3A_2,$$

$$(r_1, r_2) = (4, 3) \Rightarrow \mathrm{Sing}(X) = A_2 + A_5.$$

(2-4) The case of $r_2 = 4$. In this case, by Lemma 2.7 and Lemma 2.8, we have that there exist no (-1)-curves on M except for E_1^4, E_2^4, $\widetilde{L_0}$, that is, $N_X = 3$. Hence, the types of singularities of X are determined as follows:

$$(r_1, r_2) = (4, 4) \Rightarrow \mathrm{Sing}(X) = A_8.$$

Case 3. The case of $|\Sigma_0| = 3$

Now, we may assume that Σ_r consists of (distinct) three points $P_1(= P_1^1)$, $P_2(= P_2^1)$ and $P_3(= P_3^1)$ on $\mathbf{P}^2$ and their infinitely near points $\{P_1^2, \dots, P_1^{r_1}\}$, $\{P_2^2, \dots, P_2^{r_2}\}$ and $\{P_3^2, \dots, P_3^{r_3}\}$, respectively, where $r = r_1 + r_2 + r_3$. Let E_i^j be the exceptional curve of the first kind associated with the blowing-up with center P_i^j, where $P_i^{j+1} \in E_i^j$ $(1 \leq i \leq 3, 1 \leq j \leq r_i - 1)$. We denote the proper transform of E_i^j on M by the same notation E_i^j. Then E_i^j's $(1 \leq i \leq 3, 1 \leq j \leq r_i - 1)$ are (-2)-curves on M and $\{E_1^{r_1}, E_2^{r_2}, E_3^{r_3}\}$ are (-1)-curves on M.

Case 3.1. The case where there exists a line passing through three points P_1, P_2, P_3

In this case, let L_0 be the line passing through three points P_1, P_2, P_3 and put $\widetilde{L_0}$ the proper transform of L_0 on M, which implies that $\widetilde{L_0}$ is a (-2)-curve on M. We may assume that $r_1 \geq r_2 \geq r_3$. Let L_1, L_2 and L_3 be tangent lines to Γ at P_1, P_2 and P_3, respectively. We denote by $\widetilde{L_1}$, $\widetilde{L_2}$ and $\widetilde{L_3}$ the proper transforms on M of L_1, L_2 and L_3, respectively. Then it turns out $r_i = 1$ or $r_i \geq 3$ for each i. Moreover, P_i is a flex point of Γ if $r_i \geq 3$, which implies that $\widetilde{L_i}$ is a (-2)-curve on M.

(1) The case of $r_1 = 1$. In this case, we have $N_X = 3$ and the types of singularities of X are determined as follows:

$$(r_1, r_2, r_3) = (1, 1, 1) \Rightarrow \mathrm{Sing}(X) = A_1.$$

(2) The case of $r_1 \geq 3$, $r_2 = r_3 = 1$. In this case, $\widetilde{L_1}$ is a (-2)-curve on M. In case of $3 \leq r_1 \leq 5$, by Lemma 2.7 and Lemma 2.8, we observe that all of (-2)-curves (resp. (-1)-curves) on M are $E_1^1, \dots, E_1^{r_1-1}$, $\widetilde{L_0}$, $\widetilde{L_1}$ (resp. $E_1^{r_1}$, E_2^1, E_3^1). If $r_1 = 6$, then there exists a unique irreducible cubic C passing through seven points $P_1^1, \dots, P_1^6$, P_2^1 such that P_2^1 is a double point. We put $\tilde{C}$ the proper transform of C on M. Then $N_X \neq 3$ since $\tilde{C}$ is a (-1)-curve on M. Hence the types of singularities of X are determined as follows:

$$(r_1, r_2, r_3) = (3, 1, 1) \Rightarrow \mathrm{Sing}(X) = A_1 + A_3,$$

$$(r_1, r_2, r_3) = (4, 1, 1) \Rightarrow \mathrm{Sing}(X) = A_5,$$

$$(r_1, r_2, r_3) = (5, 1, 1) \Rightarrow \mathrm{Sing}(X) = D_6.$$

(3) The case of $r_1 \geq 3$, $r_2 = 3$, $r_3 = 1$. In this case, $\widetilde{L_1}$ and $\widetilde{L_2}$ are (-2)-curves on M. From Lemma 2.6, it follows that $E_1^1, \dots, E_1^{r_1-1}$, E_2^1, E_2^2, $\widetilde{L_0}$, $\widetilde{L_1}$ and $\widetilde{L_2}$ exhaust all of (-2)-curves on M. Moreover, by Lemma 2.7 and Lemma 2.8, it follows that there exist no (-1)-curves on M except for $E_1^{r_1}$, E_2^3,

E_3^1, that is, $N_X = 3$. Hence, the types of singularities on X are determined as follows:

$$(r_1, r_2, r_3) = (3, 3, 1) \Rightarrow \mathrm{Sing}(X) = A_1 + A_5,$$

$$(r_1, r_2, r_3) = (4, 3, 1) \Rightarrow \mathrm{Sing}(X) = A_8.$$

Case 3.2. The case where there exist no lines passing through three points P_1, P_2, P_3

Now, let L_1, L_2 and L_3 be lines passing through two points $\{P_1, P_2\}$, $\{P_2, P_3\}$ and $\{P_1, P_3\}$, respectively. We put $\widetilde{L_1}$, $\widetilde{L_2}$ and $\widetilde{L_3}$ the proper transforms on M of L_1, L_2 and L_3, respectively. Then, for each i, it follows that $r_i \geq 2$ and L_i is the tangent line to Γ. Thus each $\widetilde{L_i}$ is a (-2)-curve on M.

We may assume that L_1, L_2 and L_3 are tangent to Γ at P_1, P_2 and P_3, respectively. So we consider four cases $(r_1, r_2, r_3) = (2, 2, 2), (3, 2, 2), (3, 3, 2), (4, 2, 2)$.

In cases of $(r_1, r_2, r_3) = (2, 2, 2), (3, 2, 2), (3, 3, 2)$, by Lemma 2.7 and Lemma 2.8, we observe that there exist no (-1)-curves on M except for $E_1^{r_1}$, $E_2^{r_2}$, $E_3^{r_3}$. In case of $(4, 2, 2)$, there exists uniquely a smooth conic C passing through five points $P_1^1, \dots, P_1^4$, P_2^1. We denote by $\tilde{C}$ the proper transform of C on M. Thus $N_X \neq 3$ since $\tilde{C}$ is a (-1)-curve on M. Therefore, the types of singularities of X are determined as follows:

$$(r_1, r_2, r_3) = (2, 2, 2) \Rightarrow \mathrm{Sing}(X) = 3A_2,$$

$$(r_1, r_2, r_3) = (3, 2, 2) \Rightarrow \mathrm{Sing}(X) = A_2 + A_5,$$

$$(r_1, r_2, r_3) = (3, 3, 2) \Rightarrow \mathrm{Sing}(X) = A_8.$$

Finally, if two normal del Pezzo surfaces X and X' with at most three quasi-lines have the same degree and type of singularities, we can see that their minimal resolutions M and M' have the same configuration of (-1)-curves and (-2)-curves.

Thus the assertions concerning the types of singularities on X and the configurations of $\hat{\ell} \cup \Delta$ in Theorem 1.3 are proved.

4. The structure of the complement of quasi-lines

Let X be a normal del Pezzo surface with $\mathrm{Sing}(X) \neq \emptyset$ and $N_X \geq 1$. We put $\ell := \bigcup_{j=1}^{N_X} \ell_j$, where each ℓ_j is a quasi-line on X. We assume that $X - \ell$ is biholomorphic to a two-dimensional affine variety $V = \mathbf{C}^2$, $\mathbf{C} \times \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$. Let $\varphi : M \to X$ be the minimal resolution of X and $\Delta = \bigcup_{i=1}^{s} \Delta_i = \varphi^{-1}(\mathrm{Sing}(X))$ the exceptional set, where each Δ_i is an irreducible component. We set $\hat{\ell} := \bigcup_{j=1}^{N_X} \hat{\ell}_j$, where each $\hat{\ell}_j$ is the proper transform of ℓ_j. Now, we can see that each singular point x_i of X lies on ℓ, which implies $M - (\hat{\ell} \cup \Delta) \overset{\varphi}{\simeq} X - \ell \simeq V$. Moreover, we observe that the curves on M with negative self-intersection numbers consist of the components of $\hat{\ell} \cup \Delta$. In particular, if $N_X \leq 3$, by successive applications of birational transformations of M, which are biregular on $M - (\hat{\ell} \cup \Delta)$, the pair $(M, \hat{\ell} \cup \Delta)$ except of the type $A_1 + E_7$ can be transformed into that of one of minimal normal compactifications of V in Morrow [5] and Suzuki [6]. This completes the proof of our Theorem 1.3.

Let us consider the case $V = \mathbf{C}^2$. We put $C := \hat{\ell} \cup \Delta$. Then the pair (M, C) is a compactification of $\mathbf{C}^2$. Then we have the following:

LEMMA 4.1. $b_2(X) = b_2(\hat{\ell}) = N_X$.

PROOF. First we shall prove that $H^2(M; \mathbf{Z}) \simeq H^2(C; \mathbf{Z})$. Let us consider the following exact sequence of cohomology groups over $\mathbf{Z}$ for pair (M, C)

$$\cdots \to H^i(M, C; \mathbf{Z}) \to H^i(M; \mathbf{Z}) \to H^i(C; \mathbf{Z}) \to H^{i+1}(M, C; \mathbf{Z}) \to \cdots.$$

By Poincaré duality,

$$H^i(M, C; \mathbf{Z}) \simeq H_i(M - C; \mathbf{Z}) \simeq H_i(\mathbf{C}^2; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & (i = 0) \\ 0 & (1 \leq i \leq 4) \end{cases}.$$

Thus we have $H^2(M; \mathbf{Z}) \simeq H^2(C; \mathbf{Z})$. Therefore, we have $b_2(M) = b_2(C)$.

Next we shall show that $b_2(C) = b_2(\hat{\ell} \cup \Delta) = b_2(\hat{\ell}) + b_2(\Delta)$. Let us consider the following Mayer-Vietoris exact sequence

$$\to H_i(\hat{\ell} \cap \Delta; \mathbf{Z}) \to H_i(\hat{\ell}; \mathbf{Z}) \oplus H_i(\Delta; \mathbf{Z}) \to H_i(\hat{\ell} \cup \Delta; \mathbf{Z}) \to H_{i-1}(\hat{\ell} \cap \Delta; \mathbf{Z}) \to \cdots.$$

Since $\hat{\ell} \cap \Delta$ consists of a finite set of points, we have $H_i(\hat{\ell} \cap \Delta; \mathbf{Z}) = 0$ for $i > 0$. Thus we observe $b_2(C) = b_2(\hat{\ell}) + b_2(\Delta)$. On the other hand, from Proposition 2.5, $b_2(M) = b_2(X) + b_2(\Delta)$. Hence it follows $b_2(X) = b_2(\hat{\ell}) = N_X$. □

Next we prove $N_X \leq 3$. For all $x_i \in \mathrm{Sing}(X)$, there exists a quasi-line ℓ_j on X such that $x_i \in \ell_j$. The negative curves on M, that is, (-1)-curves and (-2)-curves on M are components of $\Delta \cup \hat{\ell}$. Assume that $M - (\Delta \cup \hat{\ell}) \cong X - \ell \cong \mathbf{C}^2$. Let $\pi : M \to \mathbf{P}^2$ be the blowing-down of (-1)-curves. Then $\pi(\Delta \cup \hat{\ell})$ is a line L on $\mathbf{P}^2$. It follows that $\pi : M \to \mathbf{P}^2$ is a blowing-up with center at most three points on L. If $N_X \geq 4$, it implies that there exists a curve $C \neq L$ on $\mathbf{P}^2$ such that its proper transform of M is a component of $\hat{\ell}$, which is a contradiction. Therefore we have $N_X \leq 3$.

This proves our Theorem 1.4. □

References

[1] L. Brenton, Some algebraicity criteria for singular surfaces, Invent. math. **41** no. 2 (1977), 129–147.

[2] L. Brenton, On singular complex surfaces with negative canonical bundle with applications to singular compactifications of $\mathbf{C}^2$ and to 3-dimensional rational singularities, Math. Ann. **148** no. 2 (1980), 117–124.
[3] M. Demazure, Surfaces de Del Pezzo, Lecture Notes in Math., 777, Séminaire sur les Singularités des Surfaces, Springer-Verlag, 1980.
[4] F. Hidaka, K. Watanabe, Normal Gorenstein surfaces with ample anti-canonical divisor, Tokyo J. Math. Vol. 4, No. 2 (1981), 319–330.
[5] J. Morrow, Minimal normal compactifications of $\mathbf{C}^2$, Proceedings of the Conference of Complex Analysis, Rice Univ. Studies No. 59 (1973), 97–112.
[6] M. Suzuki, Compactifications of $\mathbf{C} \times \mathbf{C}^*$ and $(\mathbf{C}^*)^2$, Tohoku Math. J. **31** (1979) 453–468.

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