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Tilings of a Riemann surface and cubic Pisot numbers

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ABSTRACT. Using the reducible algebraic polynomial $x^5 - x^4 - 1 = (x^2 - x + 1) \cdot (x^3 - x - 1)$, we study two types of tiling substitutions τ^* and σ^*: τ^* generates a tiling of a plane based on five prototiles of polygons, and σ^* generates a tiling of a Riemann surface, which consists of two copies of the plane, based on ten prototiles of parallelograms. Finally we claim that τ^*-tiling of $\mathscr{P}$ equals a re-tiling of σ^*-tiling of $\mathscr{R}$ through the canonical projection of the Riemann surface to the plane.

0. Introduction

Starting from the following substitution σ:

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases}, \tag{0.1}$$

we studied the tiling substitution τ^* called *the dual tiling substitution of* σ in the paper [5], [6], and found a quasi-periodic tiling of a plane $\mathscr{P}$ with five polygonal prototiles in Figure 1, whose tiles are called τ^*-*tiles* (see Figure 2 and Figure 18).

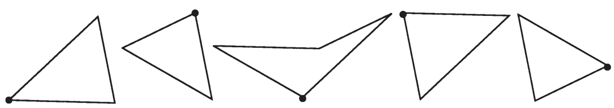

Fig. 1. Five polygonal prototiles of the tiling generated by τ^*.

In this paper, we introduce a new tiling substitution σ^*, called *the wedge tiling substitution*, that is, so-called *the extension of the dual substitution of* σ in [11].

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In the section 2, we claim that the wedge tiling substitution σ^* produces not only quasi-periodic tilings of the plane $\mathscr{P}$ but also a tiling of a Riemann surface $\mathscr{R}$ of degree 2, with ten prototiles of parallelograms in Figure 5 (see Figure 3 and Figure 6). The Riemann surface $\mathscr{R}$ is generated by two copies of $\mathscr{P}$, which is biholomorphic to the Riemann surface of $\sqrt{z}$ in the complex plane. The tiles of such a new tiling are called σ^*-*tiles*.

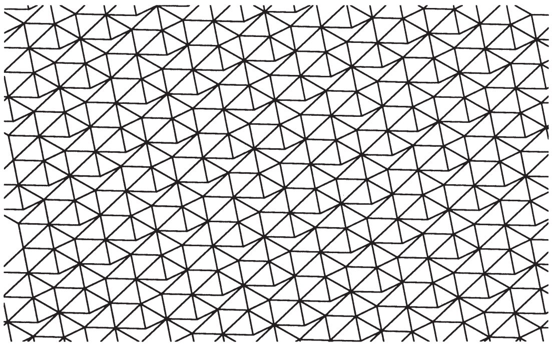


Fig. 2. The tiling generated by τ^* of $\mathscr{P}$.

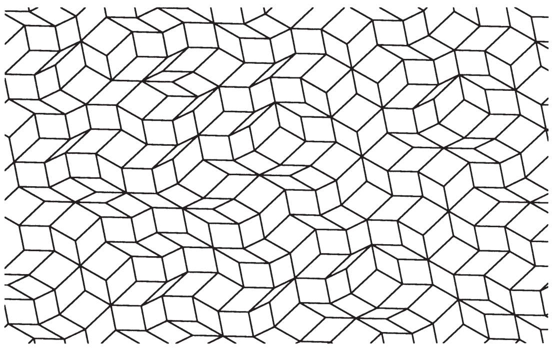


Fig. 3. The tiling generated by σ^* of $\mathscr{R}$.

In the section 3, we review the dual tiling substitution τ^*. The relationship of two tilings generated by σ^* and τ^* is discussed in the section 4. We claim that the tiling substitution τ^* also generates a tiling of the Riemann surface $\mathscr{R}$, that two tilings of $\mathscr{R}$ through σ^* and τ^* are the refinements of some common tiling of $\mathscr{R}$ and that the tiling of $\mathscr{R}$ through τ^* equals that of $\mathscr{P}$ through τ^* by the canonical projection of $\mathscr{R}$ to $\mathscr{P}$.

There are many articles on how to construct quasi-periodic tilings by polygons and prototiles with the fractal boundary from substitutions or numeration systems and their applications (see [4], [6], [10], [12], [3], [8], etc.).

The reason why we study in detail the special substitution σ like (0.1) appeared in Pisot β-expansions (see [1], [2], [5]) is that the characteristic polynomial of the incidence matrix of σ is reducible over $\mathbf{Q}$:

$$x^5 - x^4 - 1 = (x^2 - x + 1)(x^3 - x - 1) \tag{0.2}$$

and that the maximal solution of the polynomial is a Pisot unit of degree 3. On such a class, it is unclear and still open how we can produce the polygonal tiling associated with the substitution σ. And this question is interesting from the viewpoint of tiling theory, fractal analysis and numeration system. Finally we remark that all the assertions in this paper can be extended to the class of

$$x^5 - \mathrm{K}x^4 - (\mathrm{K}+1)x - 1 = 0, \qquad \mathrm{K} \geq 0,\ \mathrm{K} \in \mathbf{Z}$$

(see [5]).

1. Wedge tiling substitution σ^*

For the substitution σ in (0.1), its incidence matrix L_σ is

$$L_\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and its characteristic polynomial is given by (0.2). L_σ is primitive and unimodular (see [7] for definition and notation). Eigenvalues λ_i, $(i = 1, 2, \ldots, 5)$ of L_σ are promised as follows:

- λ_3 is the maximal solution of $x^3 - x - 1 = 0$, which is the Perron-Frobenius eigenvalue of L_σ and satisfies $\lambda_3 > 1$;
- λ_1 and λ_2 are the conjugates of λ_3, which satisfy $\lambda_1 = \overline{\lambda_2}$ and $0 < |\lambda_1| = |\lambda_2| < 1$;
- λ_4 and λ_5 are the solutions of $x^2 - x + 1 = 0$, which satisfy $\lambda_4 = \overline{\lambda_5}$ and $|\lambda_4| = |\lambda_5| = 1$.

Thus λ_3 is a Pisot number, and σ is unimodular primitive substitution of Pisot type, but not irreducible (see [7]).

For the (complex) eigenvectors $\boldsymbol{u}_i$ corresponding to λ_i $(i = 1, 2, \ldots, 5)$, real vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_5$ are given by

$$\boldsymbol{v}_1 := \frac{\boldsymbol{u}_2 + \boldsymbol{u}_1}{2}, \quad \boldsymbol{v}_2 := \frac{\boldsymbol{u}_2 - \boldsymbol{u}_1}{2i}, \quad \boldsymbol{v}_3 := \boldsymbol{u}_3, \quad \boldsymbol{v}_4 := \frac{\boldsymbol{u}_5 + \boldsymbol{u}_4}{2}, \quad \boldsymbol{v}_5 := \frac{\boldsymbol{u}_5 - \boldsymbol{u}_4}{2i}.$$

Then, we can take $\boldsymbol{v}_3 > 0$, have $\det V > 0$ for the 5×5 matrix $V := [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5]$ by renumbering if necessary, and have the following relation:

$$L_\sigma V = V \begin{bmatrix} \mathrm{Re}\,\lambda_1 & -\mathrm{Im}\,\lambda_1 & 0 & 0 & 0 \\ \mathrm{Im}\,\lambda_1 & \mathrm{Re}\,\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Re}\,\lambda_4 & -\mathrm{Im}\,\lambda_4 \\ 0 & 0 & 0 & \mathrm{Im}\,\lambda_4 & \mathrm{Re}\,\lambda_4 \end{bmatrix}.$$

Let $\mathscr{P} := \mathrm{Span}_{\mathbf{R}}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ be the oriented subspace spanned by (ordered) vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, which is the L_σ-invariant contractive plane. Then we have the following direct decomposition:

$$\mathbf{R}^5 = \mathscr{P} \oplus \mathrm{Span}_{\mathbf{R}}(\boldsymbol{v}_3) \oplus \mathrm{Span}_{\mathbf{R}}(\boldsymbol{v}_4, \boldsymbol{v}_5),$$

and can define the projection map $\pi : \mathbf{R}^5 \to \mathscr{P}$ with respect to this direct decomposition.

Using the projection π, we can find that the canonical basis $\{\boldsymbol{e}_i \mid i = 1, 2, \ldots, 5\}$ of $\mathbf{R}^5$ satisfies the following equations on $\mathscr{P}$ (Figure 4):

$$\pi\boldsymbol{e}_3 + \pi\boldsymbol{e}_4 = \pi\boldsymbol{e}_1,$$

$$\pi\boldsymbol{e}_4 + \pi\boldsymbol{e}_5 = \pi\boldsymbol{e}_1 + \pi\boldsymbol{e}_2,$$

$$\pi\boldsymbol{e}_5 = \pi\boldsymbol{e}_2 + \pi\boldsymbol{e}_3$$

(see [5] for detail).

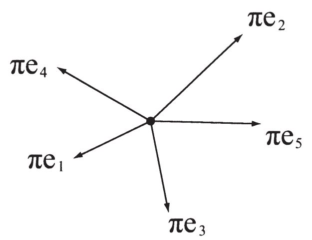


Fig. 4. $\pi\boldsymbol{e}_i$, $i = 1, 2, \ldots, 5$.

Let us consider the oriented parallelograms on $\mathscr{P}$ generated by $\pi\boldsymbol{e}_i$ and $\pi\boldsymbol{e}_j$ ($i \neq j$, $i, j = 1, 2, \ldots, 5$), and denote them by $i \wedge j$, that is,

$$i \wedge j := \{s\pi\boldsymbol{e}_i + t\pi\boldsymbol{e}_j \mid 0 \leq s, t \leq 1\}.$$

Then we may select an orientation of $\mathscr{P}$ such that all the parallelograms of the following set:

$$\Lambda_{\sigma^*} := \{2 \wedge 1, 1 \wedge 3, 4 \wedge 1, 1 \wedge 5, 3 \wedge 2, 2 \wedge 4, 5 \wedge 2, 4 \wedge 3, 3 \wedge 5, 5 \wedge 4\}$$

induce the positive orientation of $\mathscr{P}$ (see Figure 4 and Figure 5).

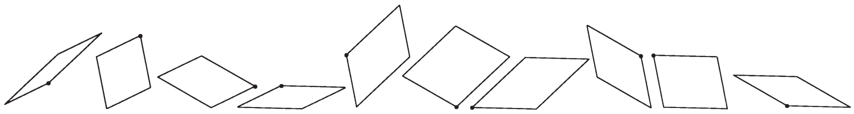


Fig. 5. Ten prototiles of parallelograms of the tiling generated by σ^*.

For $\boldsymbol{x} \in \mathbf{Z}^5$, $(\boldsymbol{x}, i \wedge j)$ denotes the oriented parallelogram generated by $\pi\boldsymbol{e}_i$ and $\pi\boldsymbol{e}_j$ located at $\pi\boldsymbol{x}$, that is,

$$(\boldsymbol{x}, i \wedge j) := \{\pi\boldsymbol{x} + s\pi\boldsymbol{e}_i + t\pi\boldsymbol{e}_j \mid 0 \le s, t \le 1\}.$$

Let $\mathscr{F}_{\sigma^*}$ be a $\mathbf{Z}$-free module generated by $\{(\boldsymbol{x}, i \wedge j) \mid \boldsymbol{x} \in \mathbf{Z}^5, i \wedge j \in \Lambda_{\sigma^*}\}$.

Definition 1. A homomorphism $\sigma^* : \mathscr{F}_{\sigma^*} \to \mathscr{F}_{\sigma^*}$ is defined by

$$\sigma^* : \begin{cases} (\boldsymbol{x}, 2 \wedge 1) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 1 \wedge 5) \\ (\boldsymbol{x}, 1 \wedge 3) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 5 \wedge 2) \\ (\boldsymbol{x}, 4 \wedge 1) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 3 \wedge 5) \\ (\boldsymbol{x}, 1 \wedge 5) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 5 \wedge 4) \\ (\boldsymbol{x}, 3 \wedge 2) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 2 \wedge 1) + (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2) \\ (\boldsymbol{x}, 2 \wedge 4) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 1 \wedge 3) + (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 3 \wedge 5) \\ (\boldsymbol{x}, 5 \wedge 2) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 4 \wedge 1) + (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 4) \\ (\boldsymbol{x}, 4 \wedge 3) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 3 \wedge 2) \\ (\boldsymbol{x}, 3 \wedge 5) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 2 \wedge 4) \\ (\boldsymbol{x}, 5 \wedge 4) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 4 \wedge 3) \end{cases}, \tag{1.3}$$

for generators of $\mathscr{F}_{\sigma^*}$ and is extended on $\mathscr{F}_{\sigma^*}$ homomorphically (see Figure 6). σ^* is called *a wedge tiling substitution* on $\mathscr{F}_{\sigma^*}$.

Definition 2. If an element γ of $\mathscr{F}_{\sigma^*}$ is represented, by using the generators λ_j of $\mathscr{F}_{\sigma^*}$ as follows

$$\gamma = \lambda_1 + \cdots + \lambda_k \qquad (\mathrm{Int}\,\lambda_i \cap \mathrm{Int}\,\lambda_j = \phi \text{ if } i \ne j),$$

γ is called a *patch* of $\mathscr{F}_{\sigma^*}$ on $\mathscr{P}$. We define $|\gamma|$ as the union of λ_j's, that is, $|\gamma| := \bigcup_{j=1}^k \lambda_j$. We set $\mathscr{T}(\gamma) := \{\lambda_1, \dots, \lambda_k\}$, which is the geometric representation of the patch γ. A patch γ is called *connected* if $|\gamma|$ is connected and the intersection of λ_i and λ_j $(i \ne j)$ is empty or some common edge. Let γ_1, γ_2 be two patches. We write $\gamma_1 \preceq \gamma_2$, if $\mathscr{T}(\gamma_1) \subset \mathscr{T}(\gamma_2)$, in addition we write

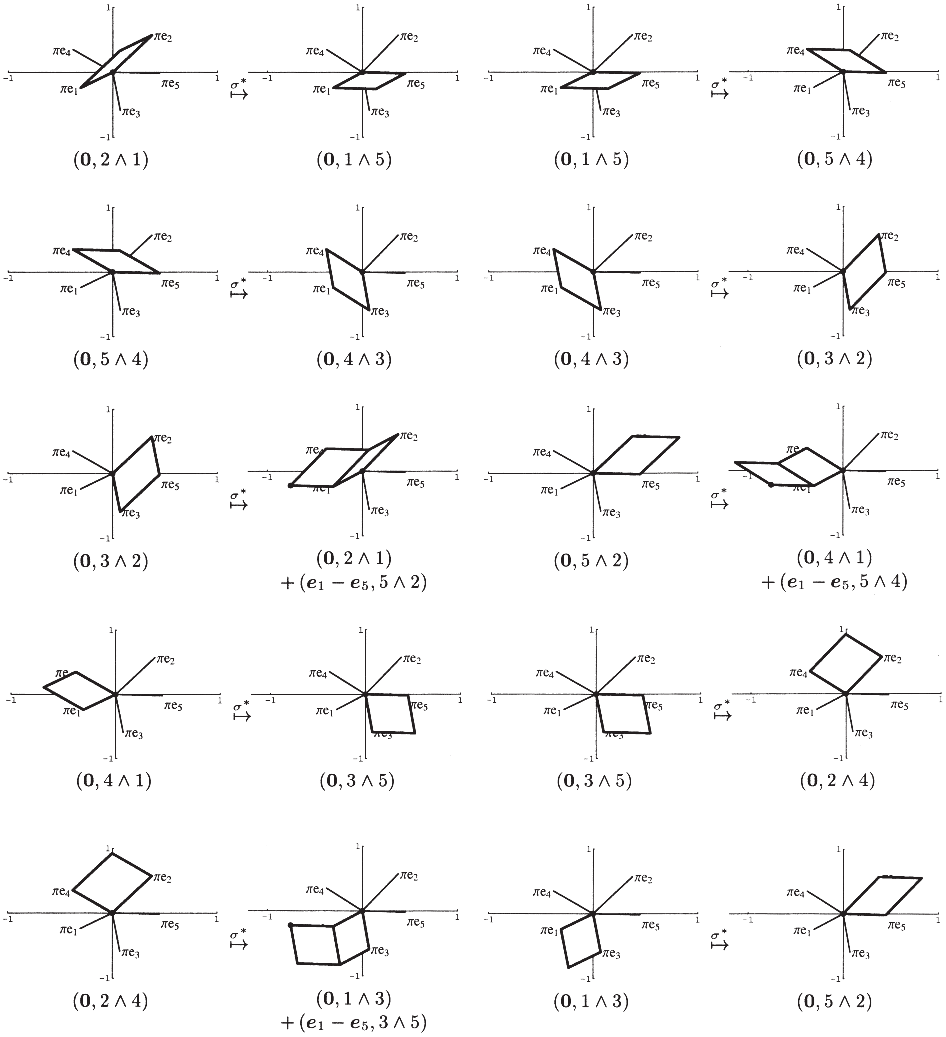
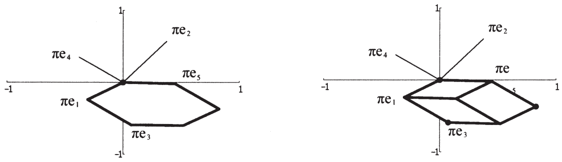


Fig. 6. $(\mathbf{0}, i \wedge j)$ and $\sigma^*(\mathbf{0}, i \wedge j)$ for $i \wedge j \in \Lambda_{\sigma^*}$.

$\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. Similar notation will be used for any tiling substitutions.

We distinguish the set $|\gamma|$ from the patch γ (or $\mathscr{T}(\gamma)$). Roughly speaking, the patch γ (or $\mathscr{T}(\gamma)$) is the partition of the set $|\gamma|$ (see Figure 7).

For the discussion of the tiling by σ^*, we introduce a segment substitution σ_1^* of the tiling substitution σ^*. For $\boldsymbol{x} \in \mathbf{Z}^5$ and $i \in \{1, 2, 3, 4, 5\}$, $(\boldsymbol{x}, i)$ is the oriented segment generated by $\pi \boldsymbol{e}_i$ located at $\pi \boldsymbol{x}$, that is,



$|\gamma|$ $\quad\quad$ $\gamma := (\mathbf{0}, 1 \wedge 5) + (\boldsymbol{e}_3, 5 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1)$ (or $\mathcal{T}(\gamma)$)

Fig. 7. The comparison between $|\gamma|$ and the patch γ (or $\mathcal{T}(\gamma)$).

$$(\boldsymbol{x}, i) := \{\pi\boldsymbol{x} + t\pi\boldsymbol{e}_i \mid 0 \le t \le 1\}.$$

Let $\mathscr{F}_{\sigma_1^*}$ be a $\mathbf{Z}$-free module generated by $\{(\boldsymbol{x}, i) \mid \boldsymbol{x} \in \mathbf{Z}^5, i = 1, 2, 3, 4, 5\}$. It is to be noted that the geometrical meaning of $-(\boldsymbol{x}, i)$ is the reverse oriented segment of $(\boldsymbol{x}, i)$.

Definition 3. If an element δ of $\mathscr{F}_{\sigma_1^*}$ is represented, by using the generators μ_j of $\mathscr{F}_{\sigma_1^*}$ with $\operatorname{Int}\mu_i \cap \operatorname{Int}\mu_j = \phi$ $(i \neq j)$ as follows:

$$\delta = \mu_1 + \cdots + \mu_k, \qquad \mu_j \cap \mu_{j+1} = \{\text{one point}\} \quad \text{for } j \in \{1, 2, \dots, k-1\},$$

δ is called a *broken segment* of $\mathscr{F}_{\sigma_1^*}$ on $\mathscr{P}$. We set $|\delta| := \bigcup_{j=1}^{k} \mu_j$ and $\mathscr{T}(\delta) := \{\mu_1, \dots, \mu_k\}$.

Definition 4. Let us define a homomorphism σ_1^* on $\mathscr{F}_{\sigma_1^*}$, which is called the *segment substitution*, from segments $(\boldsymbol{x}, i)$ to broken segments $\sigma_1^*(\boldsymbol{x}, i)$, by

$$\sigma_1^* : \begin{cases} (\boldsymbol{x}, 1) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 5) \\ (\boldsymbol{x}, 2) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 1) - (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 5) \\ (\boldsymbol{x}, 3) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 2) \\ (\boldsymbol{x}, 4) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 3) \\ (\boldsymbol{x}, 5) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 4) \end{cases}$$

for the generators of $\mathscr{F}_{\sigma_1^*}$ and is extended on $\mathscr{F}_{\sigma_1^*}$ homomorphically (see Figure 8).

Definition 5. A homomorphism $\partial : \mathscr{F}_{\sigma*} \to \mathscr{F}_{\sigma_1^*}$ is *the boundary map* given by

$$\partial(\boldsymbol{x}, i \wedge j) = (\boldsymbol{x}, i) + (\boldsymbol{x} + \boldsymbol{e}_i, j) - (\boldsymbol{x} + \boldsymbol{e}_j, i) - (\boldsymbol{x}, j).$$

for the generators $(\boldsymbol{x}, i \wedge j)$ of $\mathscr{F}_{\sigma*}$.

Then we have the following lemma.

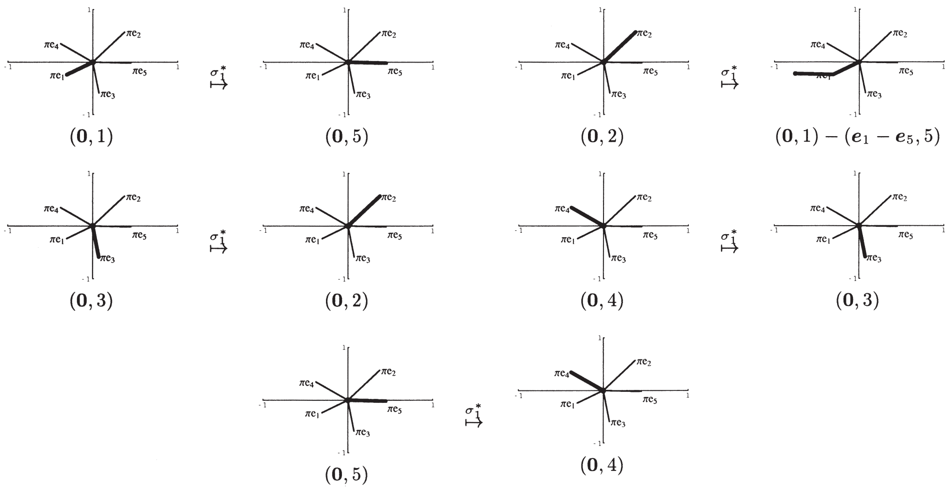


Fig. 8. $(\mathbf{0},i)$ and $\sigma_1^*(\mathbf{0},i)$ for $i \in \{1,2,\dots,5\}$.

LEMMA 1. *The following commutative relation holds:*

$$\begin{tikzcd} \mathscr{F}_{\sigma*} \arrow[r, "\sigma^*"] \arrow[d, "\partial"'] & \mathscr{F}_{\sigma*} \arrow[d, "\partial"] \\ \mathscr{F}_{\sigma_1^*} \arrow[r, "\sigma_1^*"] & \mathscr{F}_{\sigma_1^*} \end{tikzcd}$$

2. The tilings generated by σ^*

Let us define a connected patch $\mathscr{U} \in \mathscr{F}_{\sigma*}$ by

$$\begin{aligned} \mathscr{U} := {} & (\mathbf{0}, 1 \wedge 5) + (\mathbf{0}, 5 \wedge 2) + (\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 3 \wedge 5) \\ & + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 5 \wedge 4) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 4 \wedge 3) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 1 \wedge 3) \\ & + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 3 \wedge 2) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 2 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1). \end{aligned}$$

Then we obtain the following relations $\sigma^{*5n}(\mathscr{U}) \prec \sigma^{*5(n+1)}(\mathscr{U})$ $(n \in \mathbf{N})$ (see Figure 9), which induce the following theorem.

THEOREM 1. *$\sigma^{*5n}(\mathscr{U})$ $(n \in \mathbf{N})$ generate a quasi-periodic tiling of $\mathscr{P}$, that is,*

$$\bigcup_{n \in \mathbf{N}} \mathscr{T}(\sigma^{*5n}(\mathscr{U})) \qquad \textit{is a tiling of } \mathscr{P}.$$

The idea of the proof can be found in [9] and [5].

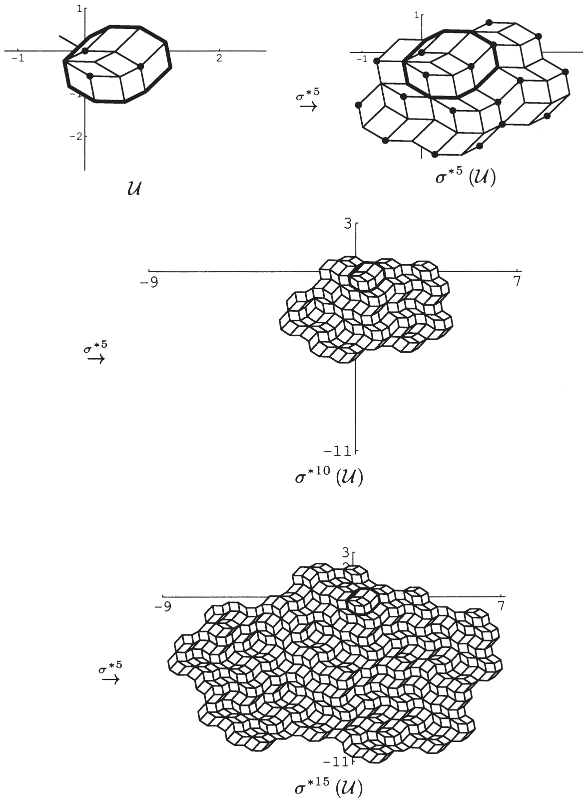


Fig. 9. $\sigma^{*n}(\mathscr{U})$, $n = 0, 5, 10, 15$.

Remark 1. *We can get several types of tilings generated by the wedge tiling substitution* σ^*. *For example, let us define a connected patch* $\mathscr{V} \in \mathscr{F}_{\sigma^*}$ *by*

$$\mathscr{V} := (\mathbf{0}, 1 \wedge 3) + (\mathbf{0}, 3 \wedge 5) + (\mathbf{0}, 5 \wedge 2) + (\mathbf{0}, 2 \wedge 4) + (\mathbf{0}, 4 \wedge 1).$$

Then we have another quasi-periodic tiling of $\mathscr{P}$, *that is,*

$$\bigcup_{n \in \mathbf{N}} \mathscr{T}(\sigma^{*5n}(\mathscr{V})) \qquad \textit{is a tiling of } \mathscr{P}$$

(see Figure 10*).*

We introduce $\Lambda'_{\sigma^*} := \{2 \wedge 1, 1 \wedge 5, 5 \wedge 4, 4 \wedge 3, 3 \wedge 2\}$ to consider a tiling of a Riemann surface. Let us define connected patches $\mathscr{W}^{(n)}_{i \wedge j}$ on $\mathscr{P}$ for $i \wedge j \in \Lambda'_{\sigma^*}$ and $n \in \mathbf{N}$ by

$$\mathscr{W}^{(n)}_{i \wedge j} := \sigma^{*5n}(\mathbf{0}, i \wedge j),$$

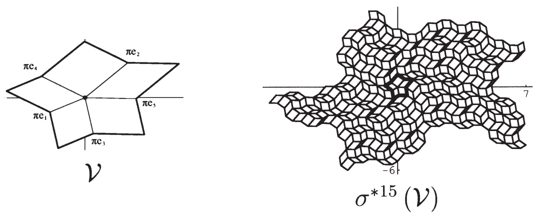


Fig. 10. $\mathscr{V}$ and $\sigma^{*15}(\mathscr{V})$.

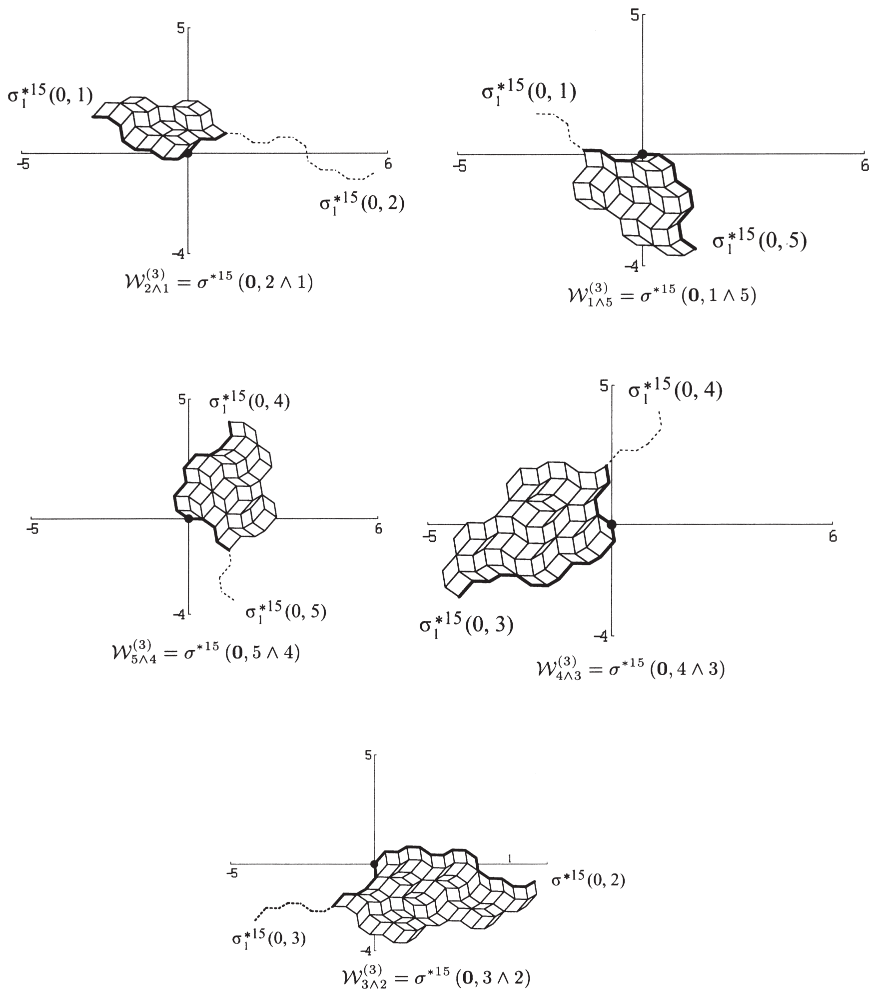


Fig. 11. Parts of the tiling of the Riemann surface $\mathscr{R}$.

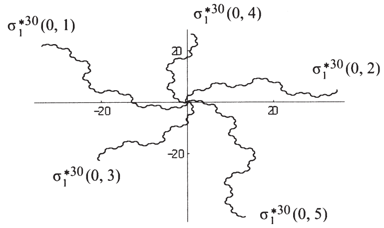


Fig. 12. Five broken lines L_j $(i = 1, 2, \ldots, 5)$.

and domains $D_{i\wedge j}^{(n)}$ in $\mathscr{P}$ by $D_{i\wedge j}^{(n)} := |\mathscr{W}_{i\wedge j}^{(n)}|$, then we know that

$$\mathscr{W}_{i\wedge j}^{(n)} \prec \mathscr{W}_{i\wedge j}^{(n+1)} \qquad (n \in \mathbf{N})$$

and that from Lemma 1, the boundary of $D_{i\wedge j}^{(n)}$ can be obtained by

$$\partial(D_{i\wedge j}^{(n)}) = |\partial(\mathscr{W}_{i\wedge j}^{(n)})| = |\sigma_1^{*5n}(\partial(\mathbf{0}, i\wedge j))|$$

(see Figure 11).

Let us define domains $D_{i\wedge j} \subset \mathscr{P}$ for $i\wedge j \in \Lambda'_{\sigma^*}$ by

$$D_{i\wedge j} := \lim_{n\to\infty} D_{i\wedge j}^{(n)},$$

and broken lines L_j $(j = 1, 2, \ldots, 5)$ by

$$L_j := \lim_{n\to\infty} |\sigma_1^{*5n}(\mathbf{0}, j)|.$$

Then the boundary of the domain $D_{i\wedge j}$ can be given by broken lines L_i and L_j, and five broken lines L_k $(k = 1, 2, \ldots, 5)$ divide the plane $\mathscr{P}$ into five connected domains (see Figure 12). Therefore $\bigcup_{i\wedge j\in\Lambda'_{\sigma^*}} D_{i\wedge j}$ is the double covering of the plane $\mathscr{P}$.

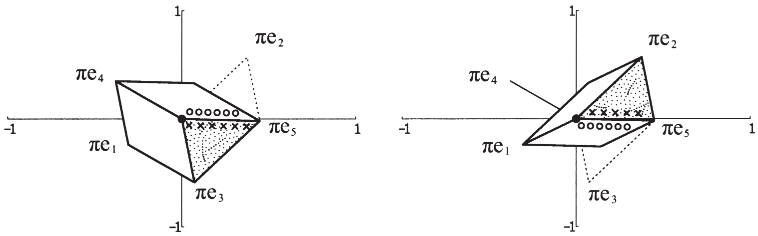
Now let us introduce a Riemann surface as follows: for two copies $\mathscr{P}_1$ and $\mathscr{P}_2$ of the L_σ-invariant plane $\mathscr{P}$, by cutting the broken line L_5 of $\mathscr{P}_1$ and $\mathscr{P}_2$, we can construct a Riemann surface $\mathscr{R} := \mathscr{P}_1 \underset{L_5}{\#} \mathscr{P}_2$. We denote the canonical projection by $\varpi : \mathscr{R} \to \mathscr{P}$.

Let us introduce a connected patch $\mathscr{W}_{\sigma^*}$ of $\mathscr{R}$ by

$$\mathscr{W}_{\sigma^*} := (\mathbf{0}, 2\wedge 1) + (\mathbf{0}, 1\wedge 5) + (\mathbf{0}, 5\wedge 4) + (\mathbf{0}, 4\wedge 3) + (\mathbf{0}, 3\wedge 2)$$

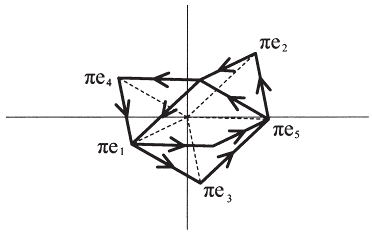
(see Figure 13).

We can also consider σ^* as the tiling substitution which tiles the Riemann surface $\mathscr{R}$, and $D_{i\wedge j}^{(n)}$ and $D_{i\wedge j}$ as the domains of $\mathscr{R}$. We see that the boundary of the domain $\bigcup_{i\wedge j\in\Lambda'_{\sigma^*}} D_{i\wedge j}^{(n)}$ is given by the closed curve $|\partial\sigma^{*5n}(\mathscr{W}_{\sigma^*})| = |\sigma_1^{*5n}(\partial\mathscr{W}_{\sigma^*})|$ and the distance from the origin to the boundary of the domain



of $\mathcal{P}_1$ of $\mathcal{P}_2$

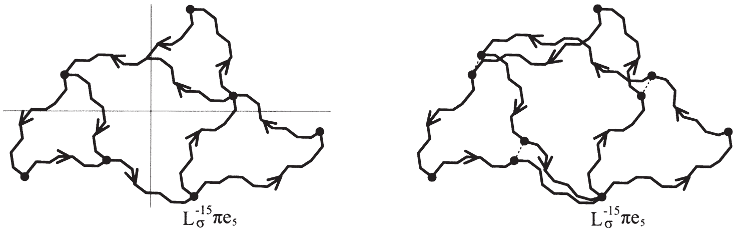
Fig. 13. The structure of the patch $\mathscr{W}_{\sigma^*}$ of $\mathscr{R}$.



$$\partial\mathcal{W}_{\sigma^*} = (\boldsymbol{e}_5,4) - (\boldsymbol{e}_4,5) + (\boldsymbol{e}_4,3) - (\boldsymbol{e}_3,4) + (\boldsymbol{e}_3,2) - (\boldsymbol{e}_2,3) + (\boldsymbol{e}_2,1) - (\boldsymbol{e}_1,2) + (\boldsymbol{e}_1,5) - (\boldsymbol{e}_5,3)$$

$$\partial\mathcal{W}_{\sigma^*} = (\boldsymbol{e}_5,4) - (\boldsymbol{e}_4,5) + (\boldsymbol{e}_4,3) - (\boldsymbol{e}_3,4) + (\boldsymbol{e}_3,2) - (\boldsymbol{e}_2,3) + (\boldsymbol{e}_2,1) - (\boldsymbol{e}_1,2) + (\boldsymbol{e}_1,5) - (\boldsymbol{e}_5,3)$$

Fig. 14. The left-hand side of figures is the boundary of $\mathscr{W}_{\sigma^*}$ of $\mathscr{R}$ and the right hand side of figures is the slight modification of the left-hand side to clarify that the winding number of the boundary curve around the origin is two.



$\partial\sigma^{*15}(\mathcal{W}_{\sigma^*})$ $\partial\sigma^{*15}(\mathcal{W}_{\sigma^*})$

Fig. 15. The left-hand side of figures is the boundary of $\sigma^{*15}(\mathscr{W}_{\sigma^*})$ of $\mathscr{R}$ and the right hand side of figures is the slight modification of the left-hand side to clarify that the winding number of the boundary curve around the origin is two.

$\bigcup_{i \wedge j \in \Lambda'_{\sigma^*}} D^{(n)}_{i \wedge j}$ tends to ∞ as $n \to \infty$ (see Figure 15). Hence, the union $\bigcup_{i \wedge j \in \Lambda'_{\sigma^*}} D_{i \wedge j}$ is the Riemann surface $\mathscr{R}$.

Therefore, we obtain the following theorem.

Theorem 2. *The wedge tiling substitution σ^* generates a tiling of the Riemann surface $\mathscr{R}$, that is,*

$$\bigcup_{n \in \mathbf{N}} \mathscr{T}(\sigma^{*5n}(\mathscr{W}_{\sigma^*})) \qquad \textit{is a tiling of } \mathscr{R}$$

(see Figure 11*).*

Remark 2. *We denote the tiling of $\mathscr{R}$ obtained in Theorem* 2 *by $\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$, which is called σ^*-tiling of $\mathscr{R}$. From the fact that $\sigma^{*n}(\mathscr{W}_{\sigma^*}) \prec \sigma^{*(n+1)}(\mathscr{W}_{\sigma^*})$ for any $n \in \mathbf{N}$, we get*

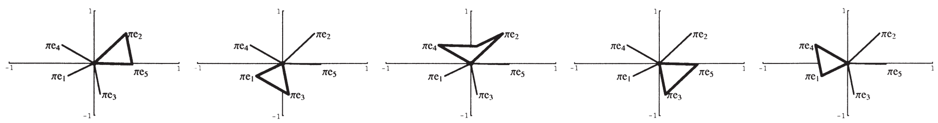
$$\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*}) = \bigcup_{n \in \mathbf{N}} \mathscr{T}(\sigma^{*n}(\mathscr{W}_{\sigma^*}))$$

3. The tiling substitution τ^* in the sense of duality of the substitution σ

To study the relation of tilings generated by σ^* and τ^*, we go back to the definition of τ^*, so-called *the dual tiling substitution* of σ in [5].

We define $(\mathbf{0}, i^*)$ $(i = 1, 2, \ldots, 5)$ as the polygons in Figure 16, and set

$$\Lambda_{\tau^*} := \{(\mathbf{0}, i^*) \mid i = 1, 2, \ldots, 5\}.$$



$(\mathbf{0}, 1^*)$ $(\mathbf{0}, 2^*)$ $(\mathbf{0}, 3^*)$ $(\mathbf{0}, 4^*)$ $(\mathbf{0}, 5^*)$

Fig. 16. $(\mathbf{0}, i^*)$, $i = 1, 2, \ldots, 5$.

Let $\mathscr{F}_{\tau^*}$ be a $\mathbf{Z}$-free module generated by $\{(\boldsymbol{x}, i^*) \mid \boldsymbol{x} \in \mathbf{Z}^5, i \in \{1, 2, \ldots, 5\}\}$, where $(\boldsymbol{x}, i^*)$ is the oriented polygon obtained from $(\mathbf{0}, i^*)$ located at $\pi\boldsymbol{x}$.

Definition 6. Let us define the *dual tiling substitution* τ^* on $\mathscr{F}_{\tau^*}$ by

$$\tau^* : \begin{cases} (\boldsymbol{x}, 1^*) \mapsto (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*) + (L_\sigma^{-1}\boldsymbol{x}, 5^*) \\ (\boldsymbol{x}, 2^*) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 1^*) \\ (\boldsymbol{x}, 3^*) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 2^*) \\ (\boldsymbol{x}, 4^*) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 3^*) \\ (\boldsymbol{x}, 5^*) \mapsto (L_\sigma^{-1}\boldsymbol{x}, 4^*) \end{cases},$$

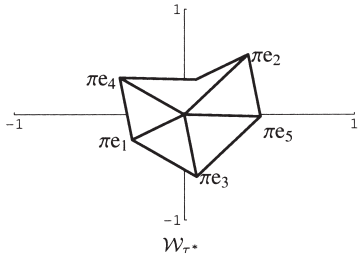


Fig. 17. $\mathscr{W}_{\tau^*}$.

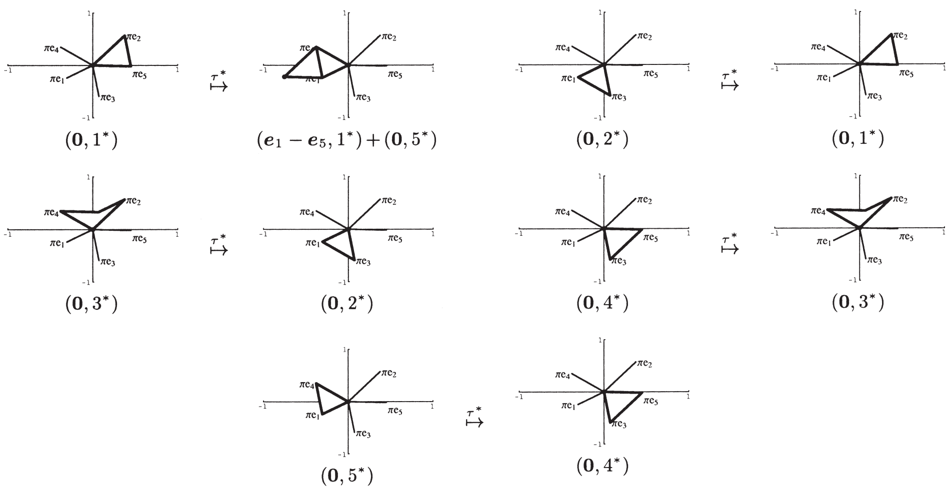


Fig. 18. $(\mathbf{0}, i^*)$ and $\tau^*(\mathbf{0}, i^*)$, $i = 1, 2, \dots, 5$.

for any generators of $\mathscr{F}_{\tau^*}$, and τ^* is extended homomorphically on $\mathscr{F}_{\tau^*}$ (see Figure 18).

We define a connected patch $\mathscr{W}_{\tau^*}$ by

$$\mathscr{W}_{\tau^*} = \sum_{i=1}^{5} (\mathbf{0}, i^*)$$

(see Figure 17). Then we have the following theorem in [5].

THEOREM 3. *The following statements hold:*

(1) *The dual tiling substitution τ^* generates a tiling of $\mathscr{P}$, that is,*

$$\bigcup_{n \in \mathbf{N}} \mathscr{T}(\tau^{*n}(\mathscr{W}_{\tau^*})) \qquad \textit{is a tiling of } \mathscr{P}$$

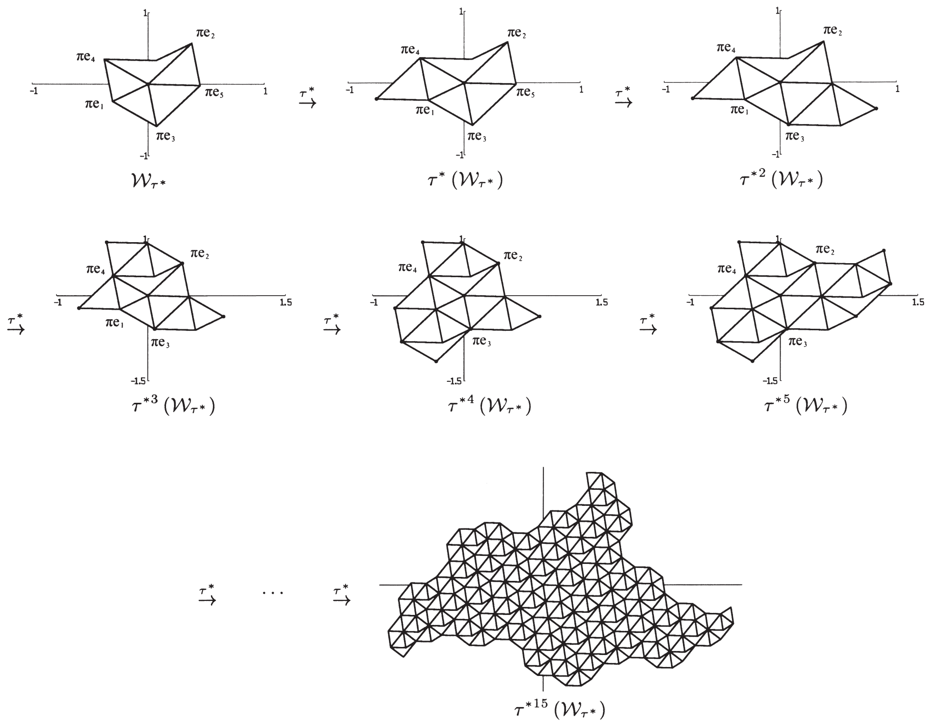


Fig. 19. $\tau^{*n}(\mathscr{W}_{\tau^*}),\ n = 0, 1, \ldots, 5, 15.$

(see Figure 19*). We denote such a tiling by*

$$\mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*}) := \bigcup_{n \in \mathbf{N}} \mathscr{T}(\tau^{*n}(\mathscr{W}_{\tau^*})),$$

which is called τ^*-tiling of $\mathscr{P}$;

(2) *Using $U_n = |\tau^{*n}(\mathscr{W}_{\tau^*})|$, let us define X and X_i as the renormalization*

$$X := \lim_{n \to \infty} L_\sigma^n U_n$$

$$X_i := \lim_{n \to \infty} L_\sigma^n \bigcup_{(x, j^*) \in \tau^{*n}(0, i^*)} (\boldsymbol{x}, j^*)$$

(see Figure 20*). Then the sets X and X_i, $i = 1, 2, \ldots, 5$ satisfy the following set equations:*

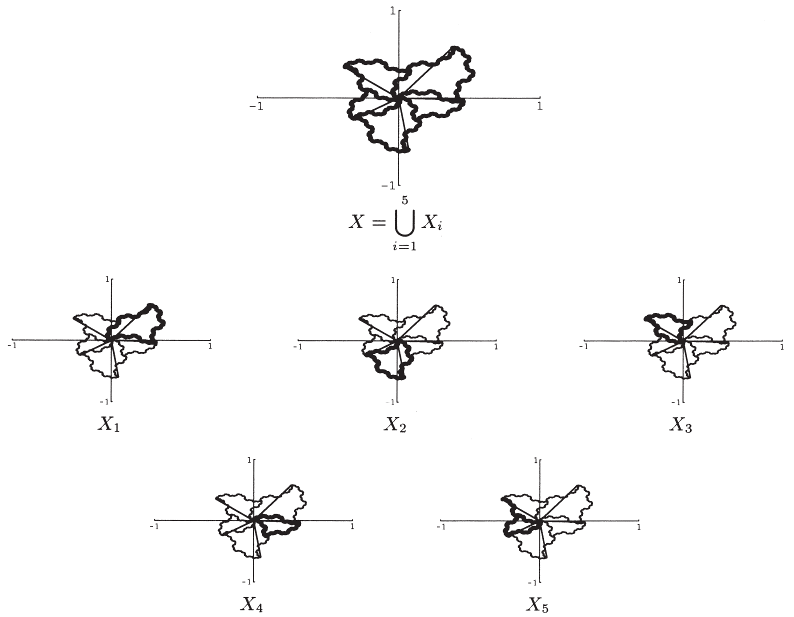


Fig. 20. $X = \bigcup_{i=1}^{5} X_i$ and X_i, $i = 1, 2, \ldots, 5$.

$$X = \bigcup_{i=1}^{5} X_i (\subset \mathscr{P}) \qquad (\textit{non-overlapping})$$

$$L_\sigma^{-1} X_1 = X_5 \cup (\boldsymbol{e}_1 - \boldsymbol{e}_5 + X_1)$$

$$L_\sigma^{-1} X_2 = X_1$$

$$L_\sigma^{-1} X_3 = X_2$$

$$L_\sigma^{-1} X_4 = X_3$$

$$L_\sigma^{-1} X_5 = X_4$$

where $\boldsymbol{x} + Y = \{\pi(\boldsymbol{x}) + \boldsymbol{y} \mid \boldsymbol{y} \in Y\}$.

4. The relationship between two tilings $\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$ and $\mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*})$

We have two tiling substitutions σ^* and τ^* such that σ^* generates the tiling of the Riemann surface $\mathscr{R}$, that is, $\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$, on the other hand, τ^* generates the tiling $\mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*})$ of $\mathscr{P}$. In this section, we study the relationship between two tilings $\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$ and $\mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*})$.

Definition 7. Let us introduce a homomorphism $\Phi : \mathscr{F}_{\sigma^*} \to \mathscr{F}_{\tau^*}$, which is called *pre-blockcoding map*, as follows: Φ is defined as

$$\Phi : \begin{cases} (\boldsymbol{x}, 2 \wedge 1) \mapsto (\boldsymbol{x}, 3^*) + (\boldsymbol{x}, 5^*) - (\boldsymbol{x} + \boldsymbol{e}_4, 4^*) \\ (\boldsymbol{x}, 1 \wedge 5) \mapsto (\boldsymbol{x}, 2^*) + (\boldsymbol{x}, 4^*) - (\boldsymbol{x} + \boldsymbol{e}_3, 3^*) \\ (\boldsymbol{x}, 5 \wedge 4) \mapsto (\boldsymbol{x}, 1^*) + (\boldsymbol{x}, 3^*) - (\boldsymbol{x} + \boldsymbol{e}_2, 2^*) \\ (\boldsymbol{x}, 4 \wedge 3) \mapsto (\boldsymbol{x}, 5^*) + (\boldsymbol{x}, 2^*) \\ (\boldsymbol{x}, 3 \wedge 2) \mapsto (\boldsymbol{x}, 1^*) + (\boldsymbol{x}, 4^*) \\ (\boldsymbol{x}, 5 \wedge 2) \mapsto (\boldsymbol{x}, 1^*) + (\boldsymbol{x} + \boldsymbol{e}_2, 4^*) \\ (\boldsymbol{x}, 4 \wedge 1) \mapsto (\boldsymbol{x}, 5^*) + (\boldsymbol{x} + \boldsymbol{e}_4, 2^*) \\ (\boldsymbol{x}, 3 \wedge 5) \mapsto (\boldsymbol{x}, 4^*) + (\boldsymbol{x} + \boldsymbol{e}_3, 1^*) \\ (\boldsymbol{x}, 2 \wedge 4) \mapsto (\boldsymbol{x}, 3^*) + (\boldsymbol{x} + \boldsymbol{e}_4, 1^*) + (\boldsymbol{x} + \boldsymbol{e}_2, 5^*) \\ (\boldsymbol{x}, 1 \wedge 3) \mapsto (\boldsymbol{x}, 2^*) + (\boldsymbol{x} + \boldsymbol{e}_3, 5^*) \end{cases},$$

for any generators of $\mathscr{F}_{\sigma^*}$, and is extended on $\mathscr{F}_{\sigma^*}$ homomorphically.

The map Φ geometrically gives us a rewriting rule from the parallelograms of $\mathscr{F}_{\sigma^*}$ to the polygons of $\mathscr{F}_{\tau^*}$ (see Figure 21). Note that a negative polygon $-(\boldsymbol{x}, i^*)$ geometrically means the negative oriented polygon and it is colored gray in this paper.

Then we have the following proposition.

Proposition 1. *The following commutative relation holds:*

$$\Phi \circ \sigma^* = \tau^* \circ \Phi \qquad \text{on } \mathscr{F}_{\sigma^*}$$

(see Figure 22 and Figure 23).

Proof. For $(\boldsymbol{0}, 2 \wedge 1)$, we see that

$$\Phi(\sigma^*(\boldsymbol{0}, 2 \wedge 1)) = \Phi(\boldsymbol{0}, 1 \wedge 5) = (\boldsymbol{0}, 2^*) + (\boldsymbol{0}, 4^*) - (\boldsymbol{e}_3, 3^*).$$

On the other hand, we see that

$$\begin{aligned} \tau^*(\Phi(\boldsymbol{0}, 2 \wedge 1)) &= \tau^*((\boldsymbol{0}, 3^*) + (\boldsymbol{0}, 5^*) - (\boldsymbol{e}_4, 4^*)) \\ &= (\boldsymbol{0}, 2^*) + (\boldsymbol{0}, 4^*) - (L_\sigma^{-1}\boldsymbol{e}_4, 3^*) \\ &= (\boldsymbol{0}, 2^*) + (\boldsymbol{0}, 4^*) - (\boldsymbol{e}_3, 3^*). \end{aligned}$$

Therefore, we have $\Phi(\sigma^*(\boldsymbol{0}, 2 \wedge 1)) = \tau^*(\Phi(\boldsymbol{0}, 2 \wedge 1))$. Other cases are discussed analogously. □

Corollary 1. *Each $n \in \mathbf{N}$, we have the following commutative relation:*

$$\Phi \circ \sigma^{*n} = \tau^{*n} \circ \Phi \qquad \text{on } \mathscr{F}_{\sigma^*}$$

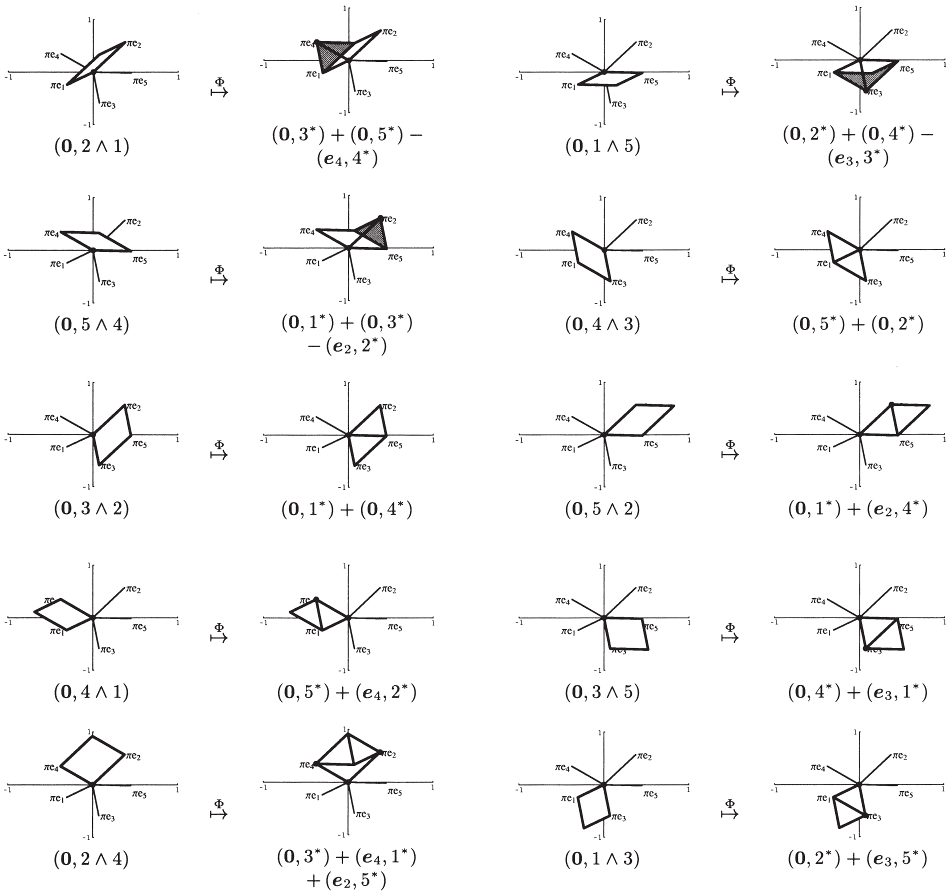


Fig. 21. The rewriting rule Φ for $(\mathbf{0}, i \wedge j) \in \Lambda_{\sigma^*}$.

We shall pay attention to the tiles $(\boldsymbol{x}, 2 \wedge 1)$, $(\boldsymbol{x}, 1 \wedge 5)$, $(\boldsymbol{x}, 5 \wedge 4)$, which Φ maps to the "patches" including negative tiles.

LEMMA 2. *The following holds for any* $(\mathbf{0}, i \wedge j) \in \Lambda_{\sigma^*}$*:*
(1) *If* $(\boldsymbol{x}, 2 \wedge 1) \prec \sigma^{*n}(\mathbf{0}, i \wedge j)$ *for some* $n \geq 5$, *then*

$$(\boldsymbol{x}, 2 \wedge 1) + (\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2) \prec \sigma^{*n}(\mathbf{0}, i \wedge j).$$

(2) *If* $(\boldsymbol{x}, 1 \wedge 5) \prec \sigma^{*n}(\mathbf{0}, i \wedge j)$ *for some* $n \geq 5$, *then*

$$(\boldsymbol{x}, 1 \wedge 5) + (\boldsymbol{x} + \boldsymbol{e}_3, 5 \wedge 4) + (\boldsymbol{x} + \boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1) \prec \sigma^{*n}(\mathbf{0}, i \wedge j).$$

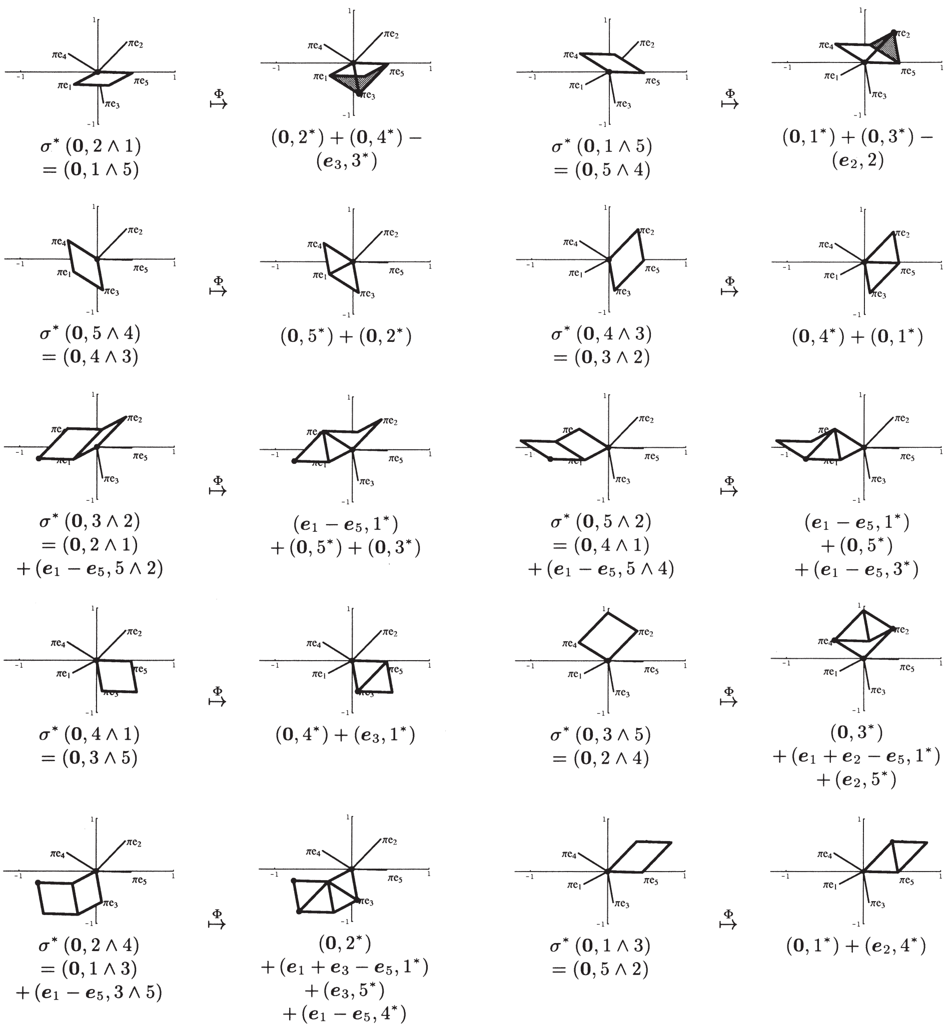


Fig. 22. $\sigma^*(\mathbf{0}, i\wedge j)$ and $\Phi(\sigma^*(\mathbf{0}, i\wedge j))$, $i\wedge j \in \Lambda_{\sigma^*}$ in Proposition 1.

(3) *If* $(\mathbf{x}, 5\wedge 4) \prec \sigma^{*n}(\mathbf{0}, i\wedge j)$ *for some* $n \geq 5$, *then*

$$(\mathbf{x}, 5\wedge 4) + (\mathbf{x} + \mathbf{e}_2, 4\wedge 3) + (\mathbf{x} + \mathbf{e}_4 - \mathbf{e}_3, 3\wedge 5) \prec \sigma^{*n}(\mathbf{0}, i\wedge j)$$
$$\text{or} \qquad (\mathbf{x}, 5\wedge 4) + (\mathbf{x} + \mathbf{e}_5 - \mathbf{e}_1, 4\wedge 1) \prec \sigma^{*n}(\mathbf{0}, i\wedge j).$$

Proof. For (1). If $(\mathbf{x}, 2\wedge 1) \prec \sigma^{*n}(\mathbf{0}, i\wedge j)$, then by the definition σ^*, there exists uniquely $(\mathbf{z}, 3\wedge 2) \prec \sigma^{*n-1}(\mathbf{0}, i\wedge j)$ such that $\sigma^*(\mathbf{z}, 3\wedge 2) \succ (\mathbf{x}, 2\wedge 1)$. On the other hand, we know that

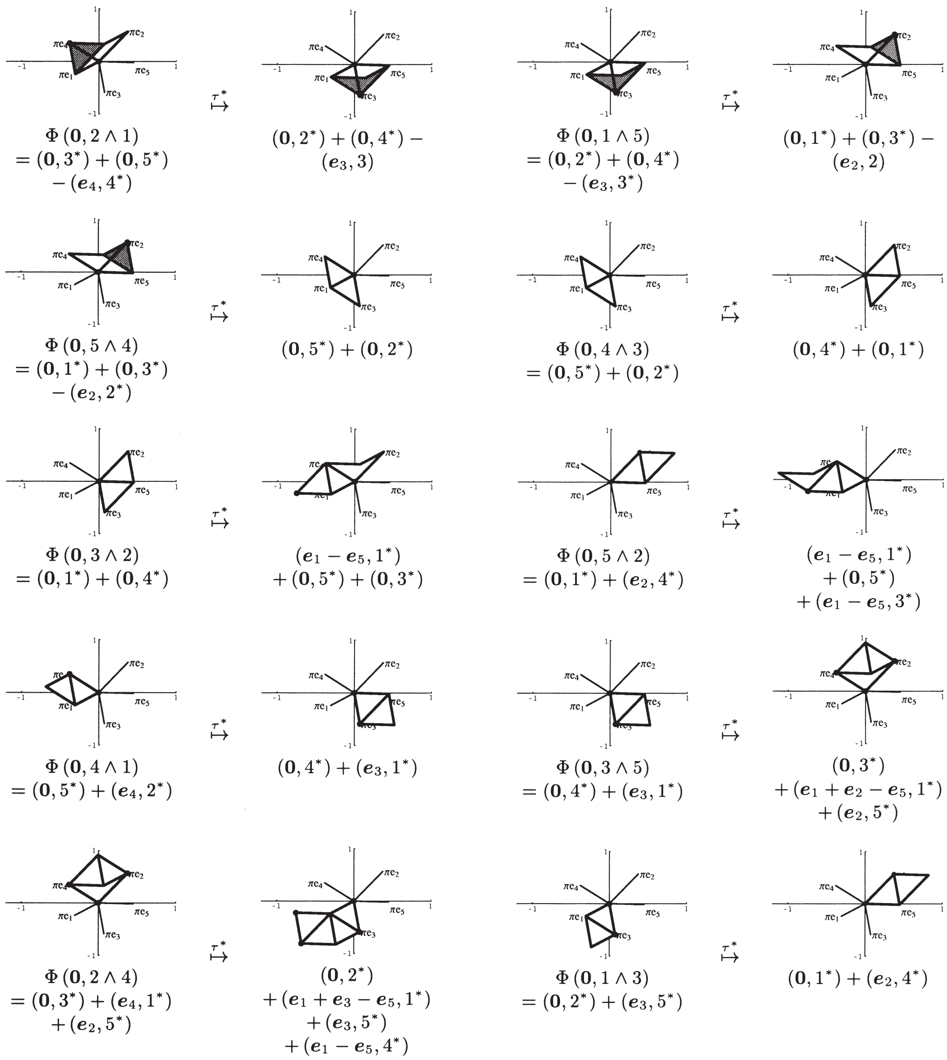


Fig. 23. $\Phi(\mathbf{0}, i \wedge j)$ and $\tau^*(\Phi(\mathbf{0}, i \wedge j))$, $i \wedge j \in \Lambda_{\sigma^*}$ in Proposition 1.

$$\sigma^*(\boldsymbol{z}, 3 \wedge 2) = (\boldsymbol{x}, 2 \wedge 1) + (\boldsymbol{x} + \boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2)$$

and

$$\sigma^*(\boldsymbol{z}, 3 \wedge 2) \prec \sigma^{*n}(\mathbf{0}, i \wedge j).$$

For (2). If $(\boldsymbol{x}, 1 \wedge 5) \prec \sigma^{*n}(\mathbf{0}, i \wedge j)$, then there exists $(\boldsymbol{w}, 3 \wedge 2) \prec \sigma^{*n-2}(\mathbf{0}, i \wedge j)$ uniquely such that $\sigma^{*2}(\boldsymbol{w}, 3 \wedge 2) \succ (\boldsymbol{x}, 1 \wedge 5)$. On the other hand,

$$\sigma^{*2}(\boldsymbol{w}, 3 \wedge 2) = (\boldsymbol{x}, 1 \wedge 5) + (\boldsymbol{x} + \boldsymbol{e}_3, 5 \wedge 4) + (\boldsymbol{x} + \boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1) \prec \sigma^{*n}(\boldsymbol{0}, i \wedge j).$$

Therefore, we obtain (2).

For (3). If $(\boldsymbol{x}, 5 \wedge 4) \prec \sigma^{*n}(\boldsymbol{0}, i \wedge j)$, then there exists

(i) $(\boldsymbol{w}, 3 \wedge 2) \prec \sigma^{*n-3}(\boldsymbol{0}, i \wedge j)$ such that

$$\sigma^{*3}(\boldsymbol{w}, 3 \wedge 2) = (\boldsymbol{x}, 5 \wedge 4) + (\boldsymbol{x} + \boldsymbol{e}_2, 4 \wedge 3) + (\boldsymbol{x} + \boldsymbol{e}_4 - \boldsymbol{e}_3, 3 \wedge 5)$$

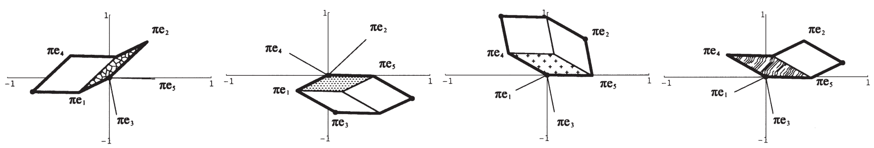
or

(ii) $(\boldsymbol{z}, 5 \wedge 2) \prec \sigma^{*n-1}(\boldsymbol{0}, i \wedge j)$ such that

$$\sigma^{*}(\boldsymbol{z}, 5 \wedge 2) = (\boldsymbol{x}, 5 \wedge 4) + (\boldsymbol{x} + \boldsymbol{e}_5 - \boldsymbol{e}_1, 4 \wedge 1).$$

Therefore, we obtain (3). □

REMARK 3. *Lemma 2 says that if* $(\boldsymbol{x}, 2 \wedge 1)$, $(\boldsymbol{x}, 1 \wedge 5)$ *or* $(\boldsymbol{x}, 5 \wedge 4)$ *can be found in* $\sigma^{*n}(\boldsymbol{0}, i \wedge j)$, $n \geq 5$, *then the connected patches including* $(\boldsymbol{x}, 2 \wedge 1)$, $(\boldsymbol{x}, 1 \wedge 5)$ *or* $(\boldsymbol{x}, 5 \wedge 4)$ *given by the parallel translations of Figure* 24 *are found in* $\sigma^{*n}(\boldsymbol{0}, i \wedge j)$ *(see Figure* 25*). Such connected patches at* $\boldsymbol{x} = \boldsymbol{0}$ *will be named by* $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{V}_3$, $\mathscr{V}_{11}$ *respectively.*



$(\boldsymbol{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2)$

$(\boldsymbol{0}, 1 \wedge 5) + (\boldsymbol{e}_3, 5 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1)$

$(\boldsymbol{0}, 5 \wedge 4) + (\boldsymbol{e}_2, 4 \wedge 3) + (\boldsymbol{e}_4 - \boldsymbol{e}_3, 3 \wedge 5)$

$(\boldsymbol{0}, 5 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_1, 4 \wedge 1)$

Fig. 24. The connected patches including $(\boldsymbol{0}, 2 \wedge 1)$, $(\boldsymbol{0}, 1 \wedge 5)$, and $(\boldsymbol{0}, 5 \wedge 4)$ respectively.

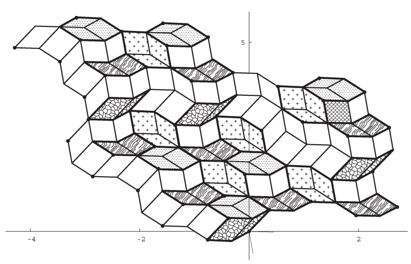


Fig. 25. The distribution of the connected patches including $(\boldsymbol{x}, 2 \wedge 1)$, $(\boldsymbol{y}, 1 \wedge 5)$, and $(\boldsymbol{z}, 5 \wedge 4)$ in $\sigma^{*20}(\boldsymbol{0}, 2 \wedge 1)$.

Let us define connected patches of parallelograms as follows:

$$\mathscr{V}_1 := (\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2) = \sigma^{*5}(\mathbf{0}, 2 \wedge 1),$$

$$\mathscr{V}_2 := (\mathbf{0}, 1 \wedge 5) + (\boldsymbol{e}_3, 5 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_4, 4 \wedge 1) = \sigma^{*5}(\mathbf{0}, 1 \wedge 5),$$

$$\mathscr{V}_3 := (\mathbf{0}, 5 \wedge 4) + (\boldsymbol{e}_2, 4 \wedge 3) + (\boldsymbol{e}_4 - \boldsymbol{e}_3, 3 \wedge 5) = \sigma^{*5}(\mathbf{0}, 5 \wedge 4),$$

$$\mathscr{V}_4 := (\mathbf{0}, 4 \wedge 3),$$

$$\mathscr{V}_5 := (\mathbf{0}, 3 \wedge 2),$$

$$\mathscr{V}_6 := (\mathbf{0}, 4 \wedge 1),$$

$$\mathscr{V}_7 := (\mathbf{0}, 3 \wedge 5),$$

$$\mathscr{V}_8 := (\mathbf{0}, 2 \wedge 4),$$

$$\mathscr{V}_9 := (\mathbf{0}, 1 \wedge 3),$$

$$\mathscr{V}_{10} := (\mathbf{0}, 5 \wedge 2),$$

$$\mathscr{V}_{11} := (\mathbf{0}, 5 \wedge 4) + (\boldsymbol{e}_5 - \boldsymbol{e}_1, 4 \wedge 1),$$

(see Figure 26).

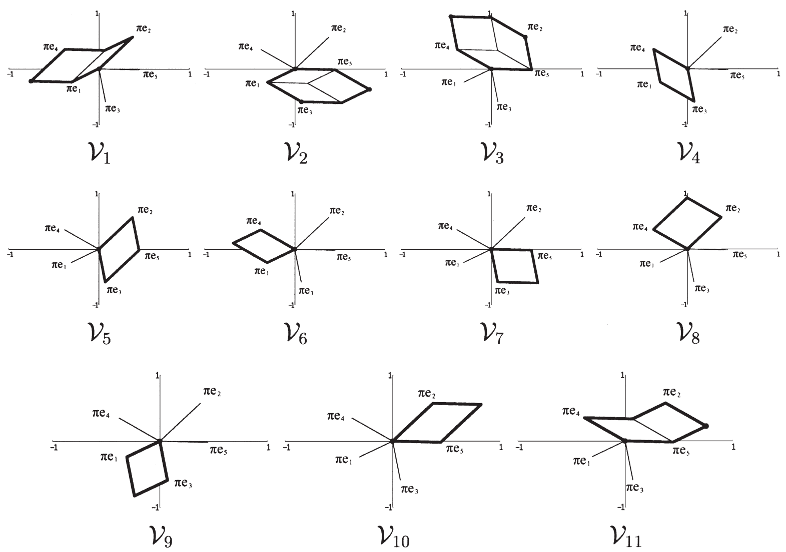


Fig. 26. The family of patches $\mathscr{V}_j$.

Lemma 3. *For each patch* $\mathscr{V}_j$ $(j = 1, 2, \dots 11)$, *the patch* $\sigma^*(\mathscr{V}_j)$ *can be decomposed by the parallel translations of* $\{\mathscr{V}_j \mid j = 1, 2, \dots 11\}$ *as follows:*

$$\sigma^* : \begin{cases} \mathscr{V}_1 \mapsto \mathscr{V}_2 \\ \mathscr{V}_2 \mapsto \mathscr{V}_3 \\ \mathscr{V}_3 \mapsto \mathscr{V}_4 + (\boldsymbol{e}_4 - \boldsymbol{e}_2 + \mathscr{V}_5) + (\boldsymbol{e}_3 - \boldsymbol{e}_2 + \mathscr{V}_8) \\ \mathscr{V}_4 \mapsto \mathscr{V}_5 \\ \mathscr{V}_5 \mapsto \mathscr{V}_1 \\ \mathscr{V}_6 \mapsto \mathscr{V}_7 \\ \mathscr{V}_7 \mapsto \mathscr{V}_8 \\ \mathscr{V}_8 \mapsto \mathscr{V}_9 + (\boldsymbol{e}_1 - \boldsymbol{e}_5 + \mathscr{V}_7) \\ \mathscr{V}_9 \mapsto \mathscr{V}_{10} \\ \mathscr{V}_{10} \mapsto \boldsymbol{e}_1 - \boldsymbol{e}_5 + \mathscr{V}_{11} \\ \mathscr{V}_{11} \mapsto \mathscr{V}_4 + (\boldsymbol{e}_4 - \boldsymbol{e}_5 + \mathscr{V}_7) \end{cases}$$

where $\boldsymbol{x} + \mathscr{V}_j := \{\pi\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{y} \in \mathscr{V}_j\}$ *for any* $\boldsymbol{x} \in \mathbf{Z}^5$.

Proof. See Figure 27.

By Lemma 2 and Lemma 3, we have the following proposition.

Proposition 2. *For each patch* $\mathscr{V}_j$ $(j = 1, 2, \dots 11)$, *the connected patch* $\sigma^{*(5+n)}(\mathscr{V}_j)$ $(n \in \mathbf{N} \cup \{0\})$ *can be decomposed by the parallel translations of* $\{\mathscr{V}_j \mid j = 1, 2, \dots 11\}$.

Definition 8. Let $\mathfrak{T}_j := \mathfrak{T}_j(\mathscr{M}, \Lambda_j)$ $(j = 1, 2)$ be two tilings of a Riemann surface $\mathscr{M}$ with protosets Λ_j. If any tile $\alpha \in \mathfrak{T}_1$ can be decomposed by tiles of $\mathfrak{T}_2$, that is, there exist finite tiles $\beta_1, \dots, \beta_k \in \mathfrak{T}_2$ such that

$$\alpha = |\beta_1 + \cdots + \beta_k|, \qquad \operatorname{Int} \beta_i \cap \operatorname{Int} \beta_j = \phi \quad (i \neq j),$$

then $\mathfrak{T}_2$ is called *a refinement* of $\mathfrak{T}_1$, which is denoted by $\mathfrak{T}_1 \underset{rf}{\preceq} \mathfrak{T}_2$. If $\mathfrak{T}_1 \neq \mathfrak{T}_2$ and $\mathfrak{T}_1 \underset{rf}{\preceq} \mathfrak{T}_2$, then we write $\mathfrak{T}_1 \underset{rf}{\prec} \mathfrak{T}_2$. A refinement of patches is also defined analogously.

Now we shall construct two new tilings of the Riemann surface $\mathscr{R}$. Setting $\gamma_j := |\mathscr{V}_j|$ $(j = 1, 2, \dots, 11)$ and $\Gamma := \{\gamma_1, \gamma_2, \dots, \gamma_{11}\}$, we denote by $\mathscr{F}(\Gamma)$ a $\mathbf{Z}$-free module generated by $\{\boldsymbol{x} + \gamma_j \mid \boldsymbol{x} \in \mathbf{Z}^5, j = 1, 2, \dots, 11\}$, where $\boldsymbol{x} + \gamma_j := \{\pi\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{y} \in \gamma_j\}$. A homomorphism $\eta : \mathscr{F}(\Gamma) \to \mathscr{F}(\Gamma)$ can be defined by

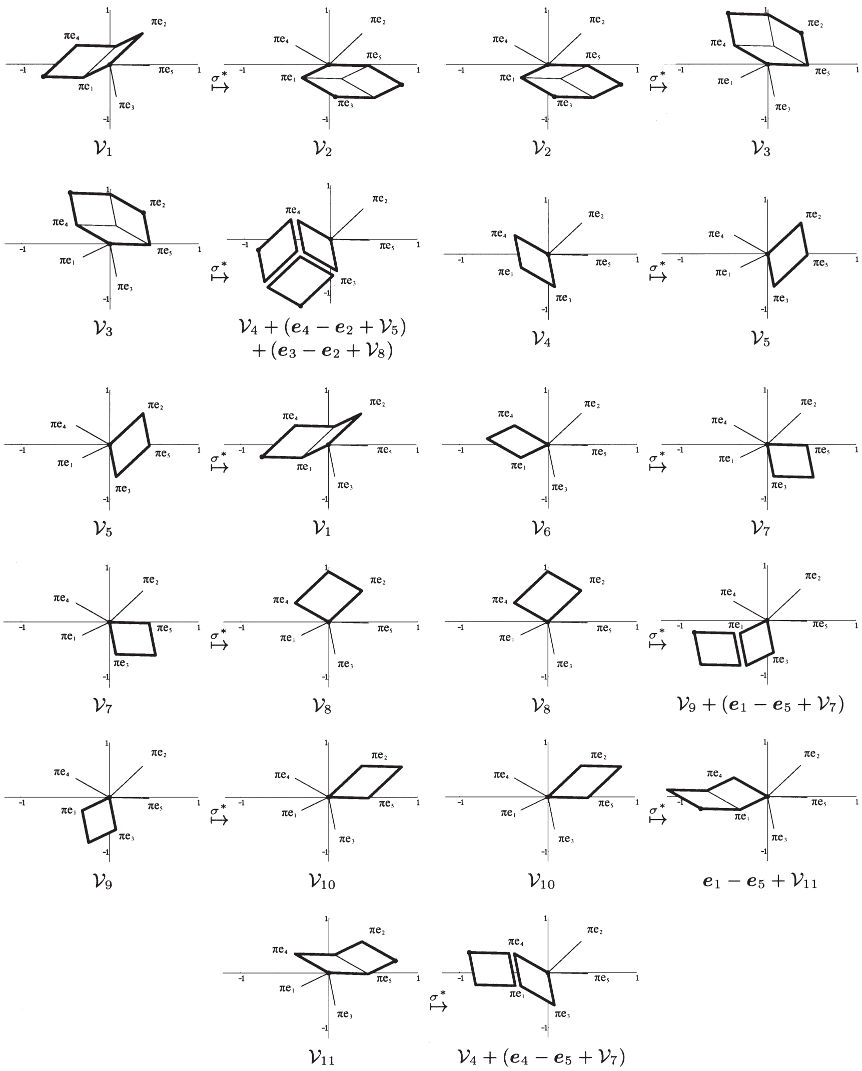


Fig. 27. The geometrical representation of $\sigma^*(\mathscr{V}_j)$ $(j = 1, 2, \ldots, 11)$.

$$\eta : \begin{cases} \gamma_1 \mapsto \gamma_2 \\ \gamma_2 \mapsto \gamma_3 \\ \gamma_3 \mapsto \gamma_4 + (\boldsymbol{e}_4 - \boldsymbol{e}_2 + \gamma_5) + (\boldsymbol{e}_3 - \boldsymbol{e}_2 + \gamma_8) \\ \gamma_4 \mapsto \gamma_5 \\ \gamma_5 \mapsto \gamma_1 \\ \gamma_6 \mapsto \gamma_7 \\ \gamma_7 \mapsto \gamma_8 \\ \gamma_8 \mapsto \gamma_9 + (\boldsymbol{e}_1 - \boldsymbol{e}_5 + \gamma_7) \\ \gamma_9 \mapsto \gamma_{10} \\ \gamma_{10} \mapsto \boldsymbol{e}_1 - \boldsymbol{e}_5 + \gamma_{11} \\ \gamma_{11} \mapsto \gamma_4 + (\boldsymbol{e}_4 - \boldsymbol{e}_5 + \gamma_7) \end{cases},$$

$\eta(\boldsymbol{x} + \gamma_j) := L_\sigma^{-1}(\boldsymbol{x}) + \eta(\gamma_j)$ $(\boldsymbol{x} \in \mathbf{Z}^5, j = 1, 2, \ldots, 11)$, and extended on $\mathscr{F}(\Gamma)$ homomorphically.

A homomorphism $\Theta : \mathscr{F}(\Gamma) \to \mathscr{F}_{\sigma^*}$ is defined by

$$\Theta(\boldsymbol{x} + \gamma_j) := \boldsymbol{x} + \mathscr{V}_j$$

for any generators of $\mathscr{F}(\Gamma)$. Then we have the following proposition by the definitions of η, Θ and Proposition 2.

Proposition 3. *The following commutative relation holds for any $n \in \mathbf{N}$:*

$$\Theta \circ \eta^n = \sigma^{*n} \circ \Theta \qquad \text{on } \mathscr{F}(\Gamma).$$

By Proposition 3, we have for each $j \in \{1, 2, \ldots, 5\}$

$$\eta^{5+n}(\gamma_j) \underset{rf}{\prec} \sigma^{*(5+n)}(\Theta(\gamma_j)) \qquad (n \in \mathbf{N} \cup \{0\}).$$

Taking as a connected patch on Riemann surface $\mathscr{R}$

$$\mathscr{W}_\eta := \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,$$

we get

$$\eta^n(\mathscr{W}_\eta) \prec \eta^{n+1}(\mathscr{W}_\eta) \qquad (n \in \mathbf{N} \cup \{0\}).$$

Therefore we obtain the following theorem:

Theorem 4. *The tiling substitution η generates a tiling $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \mathscr{W}_\eta)$ of $\mathscr{R}$, and $\mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$ is a refinement of $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \mathscr{W}_\eta)$, that is,*

(1) $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \mathscr{W}_\eta) := \lim_{n\to\infty} \mathscr{T}(\eta^n(\mathscr{W}_\eta))$ *is a tiling of $\mathscr{R}$,*

(2) $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \mathscr{W}_\eta) \underset{rf}{\prec} \mathfrak{T}(\mathscr{R}, \Lambda_{\sigma^*}, \sigma^*, \mathscr{W}_{\sigma^*})$.

Next we shall construct another new tiling of Riemann surface $\mathscr{R}$.

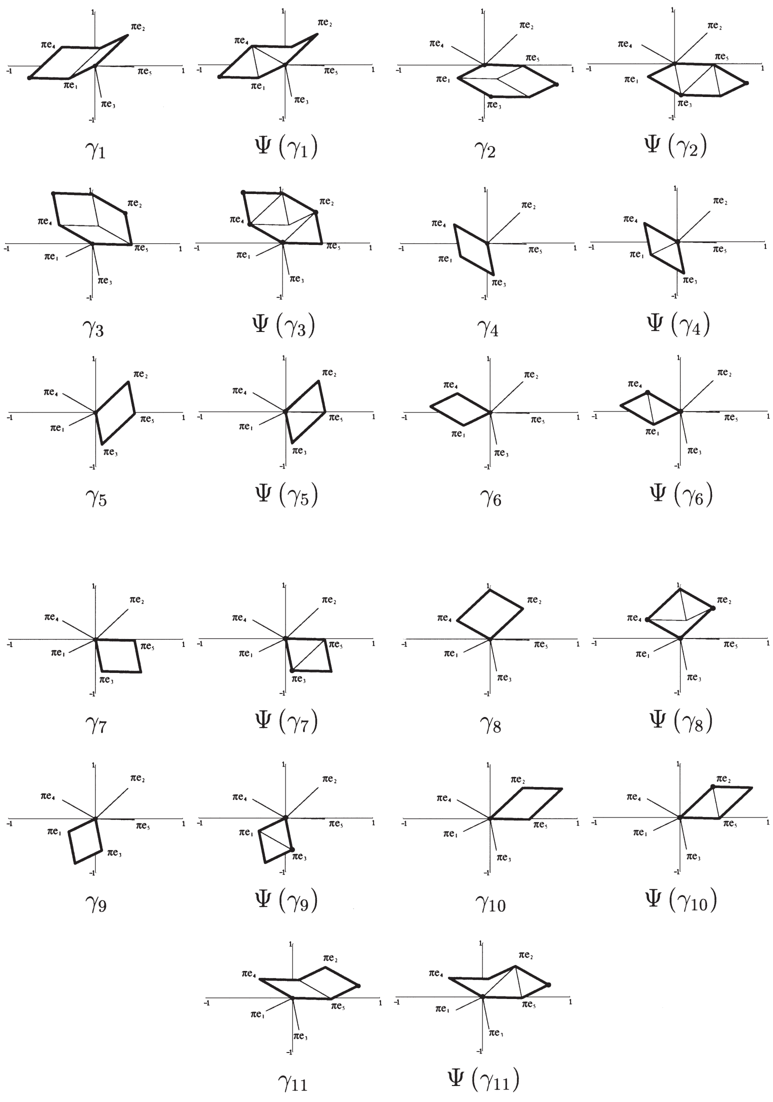


Fig. 28. γ_j and $\Psi(\gamma_j)$.

DEFINITION 9. A homomorphism $\Psi : \mathscr{F}(\Gamma) \to \mathscr{F}_{\tau^*}$ is defined by

$$\Psi(\boldsymbol{x} + \gamma_j) := \Phi(\boldsymbol{x} + \mathscr{V}_j)$$

for any generators $\boldsymbol{x} + \gamma_j$ of $\mathscr{F}(\Gamma)$. Ψ is called a *blockcoding* map.

For example, for $\gamma_1 = |(\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2)|$, $\Psi(\gamma_1)$ is given by $(\mathbf{0}, 3^*) + (\mathbf{0}, 5^*) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*)$ after the cancellation of $-(\boldsymbol{e}_4, 4^*) + (\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_5, 4^*)$ (see Figure 28).

REMARK 4. *Figure 28 says also that the negative tiles in $\Phi(\boldsymbol{x}, 2 \wedge 1)$, $\Phi(\boldsymbol{x}, 1 \wedge 5)$ and $\Phi(\boldsymbol{x}, 5 \wedge 4)$ disappeared in $\Psi(\gamma_i)$ $(i = 1, 2, \ldots, 11)$. Each $\Psi(\gamma_i)$ consists of only positive τ^*-tiles.*

PROPOSITION 4. *The following commutative relation holds for any $n \in \mathbf{N}$:*

$$\Psi \circ \eta^n = \tau^{*n} \circ \Psi \qquad \text{on } \mathscr{F}(\Gamma)$$

PROOF. For $\gamma_1 = |(\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2)|$, we have

$$\begin{aligned}\tau^*(\Psi(\gamma_1)) &= \tau^*(\Phi((\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 5 \wedge 2))) \\ &= \tau^*((\mathbf{0}, 3^*) + (\mathbf{0}, 5^*) + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*)) \\ &= (\mathbf{0}, 2^*) + (\mathbf{0}, 4^*) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 1^*) + (-\boldsymbol{e}_4 + \boldsymbol{e}_5, 5^*).\end{aligned}$$

On the other hand, we know

$$\begin{aligned}\Psi(\eta^*(\gamma_1)) &= \Psi((\mathbf{0}, 1 \wedge 5) + (-\boldsymbol{e}_4 + \boldsymbol{e}_5, 4 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 5 \wedge 4)) \\ &= (\mathbf{0}, 2^*) + (\mathbf{0}, 4^*) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 1^*) + (-\boldsymbol{e}_4 + \boldsymbol{e}_5, 5^*)\end{aligned}$$

We see that γ_i $(i = 2, 3, \ldots, 11)$ hold analogously. □

We can also consider τ^* as the tiling substitution on the Riemann surface $\mathscr{R}$. By Proposition 4, we get each $j \in \{1, 2, \ldots, 5\}$

$$\eta^{5+n}(\gamma_j) \underset{rf}{\prec} \tau^{*(5+n)}(\Psi(\gamma_j)) \qquad (n \in \mathbf{N} \cup \{0\}).$$

Let us take the following connected patch on $\mathscr{R}$:

$$\tilde{\mathscr{W}}_{\tau^*} := \sum_{j=1}^{5} \Psi(\gamma_j).$$

Then we have

$$\tau^{*n}(\tilde{\mathscr{W}}_{\tau^*}) \underset{rf}{\prec} \tau^{*(n+1)}(\tilde{\mathscr{W}}_{\tau^*}) \qquad (n \in \mathbf{N} \cup \{0\}).$$

Therefore we get the following theorem.

THEOREM 5. *The dual tiling substitution τ^* generates the tiling $\mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*})$ of $\mathscr{R}$, and $\mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*})$ is a refinement of $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \Lambda(\Gamma))$, that is,*

(1) $\mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*}) := \lim_{n \to \infty} \mathscr{T}(\tau^{*n}(\tilde{\mathscr{W}}_{\tau^*}))$ *is a tiling of $\mathscr{R}$,*

(2) $\mathfrak{T}(\mathscr{R}, \Gamma, \eta, \Lambda(\Gamma)) \underset{rf}{\prec} \mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*})$.

Moreover we have the following theorem.

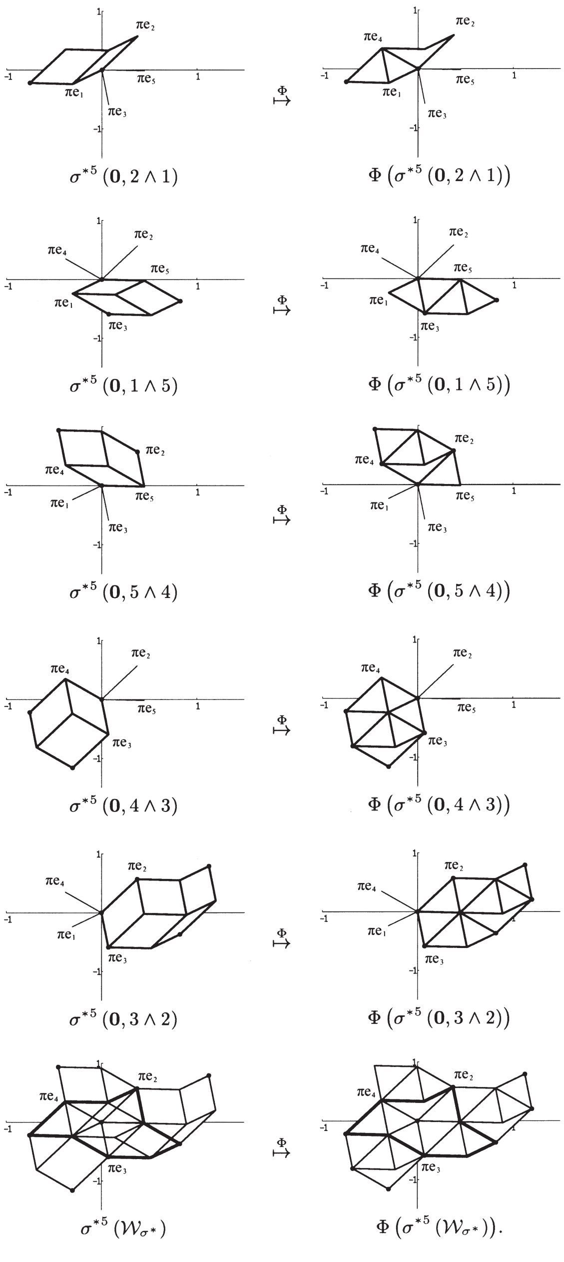


Fig. 29. $\sigma^{*5}(\mathbf{0}, i\wedge j)$, $\Phi(\sigma^{*5}(\mathbf{0}, i\wedge j))$ $(i\wedge j \in \Lambda'_{\sigma^*})$, and $\sigma^{*5}(\mathscr{W}_{\sigma^*})$, $\Phi(\sigma^{*5}(\mathscr{W}_{\sigma^*}))$.

THEOREM 6. *Let* $\varpi : \mathscr{R} \to \mathscr{P}$ *be the canonical projection. Then we have*

$$\varpi(\mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*})) = \mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*}),$$

that is, for any tile $(\boldsymbol{x}, j^*)$ *of* $\mathfrak{T}(\mathscr{P}, \Lambda_{\tau^*}, \tau^*, \mathscr{W}_{\tau^*})$, *there exist only two tiles* $(\boldsymbol{x}, j^*)_1$, $(\boldsymbol{x}, j^*)_2$ *of* $\mathfrak{T}(\mathscr{R}, \Lambda_{\tau^*}, \tau^*, \tilde{\mathscr{W}}_{\tau^*})$ *such that*

$$\varpi(\boldsymbol{x}, j^*)_1 = \varpi(\boldsymbol{x}, j^*)_2 = (\boldsymbol{x}, j^*)$$

(see Figure 29*).*

PROOF. The τ^*-tilings of $\mathscr{R}$ and $\mathscr{P}$ are generated by the same tiling rule τ^*. Thus by Proposition 4 we get the conclusion. □

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