On asymptotic behavior of positive solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ with $\alpha < -1$

Ichiro Tsukamoto

(Received August 16, 2004)
(Revised September 21, 2006)

Abstract. In this paper we shall show asymptotic behavior of all positive solutions of the second order nonlinear differential equation written in the title. It will complete this task to obtain an analytical expression or an asymptotic form of every solution valid in a neighborhood of an end of its domain.

1. Introduction

Let us consider the second order nonlinear differential equation

$$(\text{E}) \qquad x'' = e^{\alpha\lambda t}x^{1+\alpha} \qquad (' = d/dt)$$

where t and x are real variables, and α and λ are real parameters. As stated in the famous Bellman's book, (E) was deduced from an important second order nonlinear differential equation containing the Emden equation in astrophysics and the Fermi-Thomas equation in atomic physics (cf. [1]). Moreover (E) is related to a positive radial solution of an elliptic partial differential equation (cf. [13]). In fact (E) has been considered in many papers (cf. [6], [7], [10] and so on). For example in [6] and [7], (E) was treated in more general form and existence of the solution continuable to ∞ was mainly discussed. On the other hand, in [10] and [14] an initial value problem of (E) with its own form was considered and asymptotic behavior of all positive solutions was investigated.

In this paper we shall consider (E) in a domain

$$-\infty < t < \infty, \qquad x > 0.$$

Since the cases $\alpha > 0$ and $-1 < \alpha < 0$ have been already treated in [10] and [14] respectively, we suppose

$$\alpha < -1, \qquad \lambda < 0$$

throughout this paper. The assumption $\lambda < 0$ will not harm any generality, since replacing t with $-t$ leads to the case $\lambda > 0$. Here let the initial condition of (E) be given as

(I) $$x(t_0) = a, \qquad x'(t_0) = b$$

where $-\infty < t_0 < \infty$, $a > 0$ and $-\infty < b < \infty$. Then for every (t_0, a, b) we shall show asymptotic behavior of a solution of an initial value problem (E) and (I) in terms of getting an asymptotic form or an analytical expression of the solution valid in a neighborhood of an end of its domain.

For stating our main conclusions, it is necessary to introduce a transformation

(T) $$x = p(t)y^{1/\alpha} \quad (\text{namely } y = p(t)^{-\alpha}x^\alpha), \qquad z = y'$$

where $p(t) = \lambda^{2/\alpha}e^{-\lambda t}$ $(\lambda^{2/\alpha} = (\lambda^2)^{1/\alpha})$ is a particular solution of (E). Since we consider only positive x, we always have $y > 0$. The transformation such as (T) was originally used in [8] and [9], and applied in [10] through [16]. Furthermore (T) transforms (E) into a first order rational differential equation

(R) $$\frac{dz}{dy} = \frac{(\alpha - 1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3)}{\alpha yz}.$$

Conversely (T) transforms (R) into (E). Using a parameter s, we rewrite (R) as a 2-dimensional dynamical system

(D)
$$\frac{dy}{ds} = \alpha yz$$

$$\frac{dz}{ds} = (\alpha - 1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3).$$

The singular points of (D) are $(0,0)$, $(1,0)$, and a solution of (R) always represents an orbit of (D) in the region $yz \neq 0$.

In order to solve (E), we shall get a solution of (D), from this a solution of (R) and through (T) a solution of (E). In this process, we shall sometimes use

(1.1) $$z = \alpha y(\lambda + x'/x)$$

which is obtained from (T). Setting $t = t_0$ in (1.1), we get from (I) an initial condition

(1.2) $$z(y_0) = z_0$$

of (R) where

$$y_0 = \lambda^{-2} e^{\alpha \lambda t_0} a^\alpha, \qquad z_0 = \alpha y_0 (\lambda + b/a).$$

Now if $-2 < \alpha < -1$, we claim that there exist three orbits Γ_1, Γ_2 and Γ_3 of (D) with the following properties:

(i)   If $(y(s), z(s)) \in \Gamma_1$, then we have

(1.3) $$\lim_{s \to \infty} (y(s), z(s)) = (0, 0), \qquad \lim_{s \to \infty} z(s)/y(s) = \alpha \lambda$$

together with

$$\lim_{s \to \infty} y(s)^{-1} v(s) = \lambda \qquad (v(s) = y(s)^{-1} z(s) - \alpha \lambda),$$

and $(y(s), z(s))$ tends to $(1, 0)$ as $s \to -\infty$, going around $(1, 0)$ clockwise infinitely many times.

(ii)   If $(y(s), z(s)) \in \Gamma_2$, then we obtain (1.3) and for some r $(-\infty \le r < s)$

$$\lim_{s \to r} y(s) = \infty, \qquad \lim_{s \to r} y(s)^{-3/2} z(s) = \alpha \lambda \sqrt{2/(\alpha + 2)}.$$

(iii)   If $(y(s), z(s)) \in \Gamma_3$, then for some r $(s < r \le \infty)$ we have

$$\lim_{s \to r} y(s) = \infty, \qquad \lim_{s \to r} y(s)^{-3/2} z(s) = -\alpha \lambda \sqrt{2/(\alpha + 2)}$$

and as $s \to -\infty$, $(y(s), z(s))$ tends to $(1, 0)$ like $(y(s), z(s)) \in \Gamma_1$.

The case $-2 < \alpha < -1$ The case $\alpha \le -2$
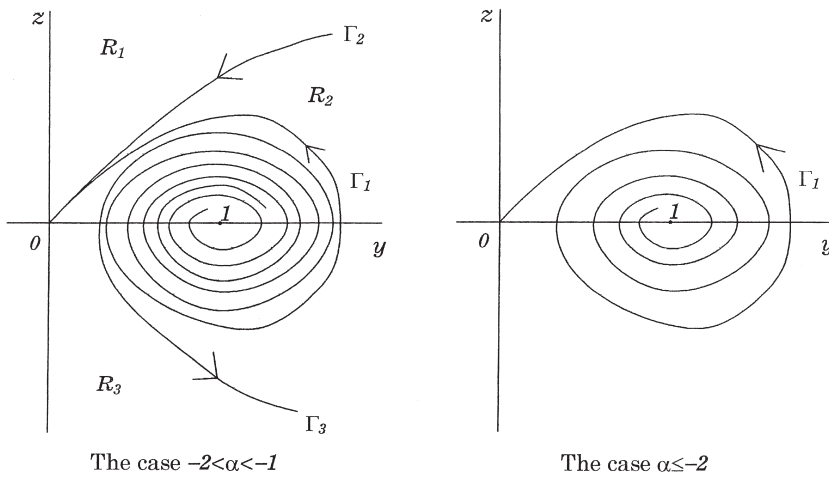
Figure 1

Furthermore if $\alpha \leq -2$, we claim that there exists an orbit Γ_1 of (D) with the property (i). Of course these will be shown in the following sections.

If we denote a solution of an initial value problem (E) and (I) as $x = x(t)$, then we can state our theorems.

THEOREM I.   *If* $-2 < \alpha < -1$, *then we conclude the following:*
(iv)   *If* (t_0, a, b) *satisfies* $(y_0, z_0) \in \Gamma_1$, *then* $x(t)$ *is defined for* $(-\infty, \infty)$. *Moreover* $x(t)$ *is represented as*

$$(1.4) \qquad x(t) = \lambda^{2/\alpha} e^{-\lambda t} \left[1 + 2Ce^{\lambda t} \cos(\lambda\sqrt{-1-\alpha}\, t + \delta) \right.$$

$$+ \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} C^m e^{m\lambda t} \{ a_{mn} \cos n(\lambda\sqrt{-1-\alpha}\, t + \delta)$$

$$\left. + b_{mn} \sin n(\lambda\sqrt{-1-\alpha}\, t + \delta) \} + \sum_{m=1}^{\infty} c_m C^{2m} e^{2m\lambda t} \right]$$

$$(C(\neq 0), \delta, a_{mn}, b_{mn} \text{ and } c_m \text{ are constants})$$

in the neighborhood of $t = \infty$, *and*

$$(1.5) \qquad\qquad x(t) = \lambda^{2/\alpha} C^{1/\alpha} \left\{ 1 + \sum_{n=1}^{\infty} a_n (Ce^{\alpha\lambda t})^n \right\}$$

$$(C(\neq 0) \text{ and } a_n \text{ are constants})$$

in the neighborhood of $t = -\infty$.
(v)   *If* (t_0, a, b) *satisfies* $(y_0, z_0) \in \Gamma_2$, *then* $x(t)$ *is defined for* $(-\infty, \omega_+)$ *where* $\omega_+ < \infty$. *Furthermore* $x(t)$ *is represented as*

$$(1.6) \qquad x(t) = \left\{ \frac{2(\alpha+2)}{\alpha^2} \right\}^{1/\alpha} e^{-\lambda\omega_+} (\omega_+ - t)^{-2/\alpha} \left\{ 1 + \sum_{n=1}^{\infty} c_n(\omega_+ - t)^n \right\}$$

$$(c_n \text{ are constants})$$

in the neighborhood of $t = \omega_+$, *and*

$$(1.7) \qquad\qquad x(t) = ct + d + \frac{(ct)^{1+\alpha}}{\alpha^2\lambda^2} e^{\alpha\lambda t}(1 + o(1))$$

$$(c(<0) \text{ and } d \text{ are constants})$$

as $t \to -\infty$.

(vi)   *If (t_0, a, b) satisfies $(y_0, z_0) \in \Gamma_3$, then $x(t)$ is defined for (ω_-, ∞) where $\omega_- > -\infty$.   Moreover $x(t)$ is represented as (1.4) in the neighborhood of $t = \infty$, and*

$$(1.8) \qquad x(t) = \left\{ \frac{2(\alpha + 2)}{\alpha^2} \right\}^{1/\alpha} e^{-\lambda \omega_-} (t - \omega_-)^{-2/\alpha} \left\{ 1 + \sum_{n=1}^{\infty} c_n (t - \omega_-)^n \right\}$$

(c_n *are constants*)

in the neighborhood of $t = \omega_-$.

Now, notice that in the case $-2 < \alpha < -1$ of Figure 1, R_1 denotes a region above Γ_2, R_2 a region surrounded by Γ_1, Γ_2 and Γ_3, and R_3 a region below Γ_1 and Γ_3.   Then we obtain

THEOREM II.   *If $-2 < \alpha < -1$, then we conclude the following:*
(vii)   *If (t_0, a, b) satisfies $(y_0, z_0) \in R_1$, then $x(t)$ is defined for $(-\infty, \omega_+)$ where $\omega_+ < \infty$.   Furthermore $x(t)$ is represented as*

$$(1.9) \qquad x(t) = K(\omega_+ - t)$$

$$\times \left\{ 1 + \sum_{l+m+n \geq 1} d_{lmn} (\omega_+ - t)^l (\omega_+ - t)^{-\alpha m/2} (\omega_+ - t)^{(\alpha+2)n/2} \right\}$$

($K(\neq 0)$ *and d_{lmn} are constants*)

in the neighborhood of $t = \omega_+$, and (1.7) as $t \to -\infty$.
(viii)   *If (t_0, a, b) satisfies $(y_0, z_0) \in R_2$, then $x(t)$ is defined for $(-\infty, \infty)$. Moreover $x(t)$ is represented as (1.4) in the neighborhood of $t = \infty$, and (1.7) as $t \to -\infty$.*
(ix)   *If (t_0, a, b) satisfies $(y_0, z_0) \in R_3$, then $x(t)$ is defined for (ω_-, ∞) where $\omega_- > -\infty$.   Moreover $x(t)$ is represented as (1.4) in the neighborhood of $t = \infty$, and*

$$(1.10) \qquad x(t) = K(t - \omega_-)$$

$$\times \left\{ 1 + \sum_{l+m+n \geq 1} d_{lmn} (t - \omega_-)^l (t - \omega_-)^{-\alpha m/2} (t - \omega_-)^{(\alpha+2)n/2} \right\}$$

($K(\neq 0)$ *and d_{lmn} are constants*)

in the neighborhood of $t = \omega_-$.

In the case $\alpha \leq -2$, we can state the theorem more easily since R_1, R_3, Γ_2 and Γ_3 do not appear.

THEOREM III.   *If $\alpha \leq -2$, then we conclude the following:*
(x)   *If (t_0, a, b) satisfies $(y_0, z_0) \in \Gamma_1$, then* (iv) *follows.*
(xi) *If not, then the conclusion of* (viii) *does.*

The case $(y_0, z_0) = (1, 0)$ is missing in these theorems.   However we clearly get $x(t) \equiv p(t)$ in this case.

2. Preliminaries

For proving our theorems, we prepare some lemmas.   First, let

$$Z_\pm(y) = (\alpha\lambda/(1 - \alpha))\, y\{1 \pm \sqrt{\alpha - (\alpha - 1)y}\}$$

and P denote a region $Z_-(y) < z < Z_+(y)$.   Then we get

LEMMA 2.1.   *In* (D),

$$dz/ds > 0,\ = 0,\ < 0$$

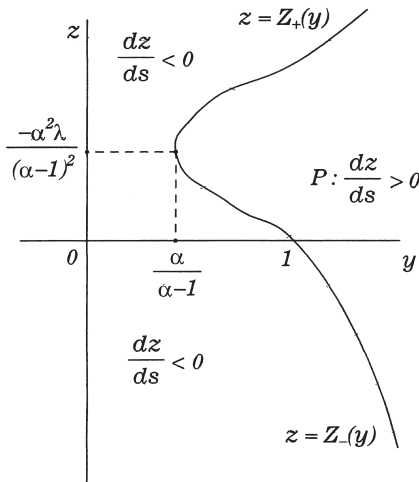*respectively if $(y, z) \in P$, $z = Z_\pm(y)$, $(y, z) \notin \bar{P}$.   Here $\bar{P}$ denotes the closure of P.*

Figure 2

If in (R) we change (y, z) for (η, ζ) and next (η, ζ) for (ξ, w) in terms of

(2.1) $y = 1/\eta, \qquad z = 1/\zeta, \qquad w = \eta^{-3/2}\zeta, \qquad \xi = \eta^{1/2},$

then we obtain a Briot-Bouquet differential equation

$$(2.2) \qquad \xi\frac{dw}{d\xi} = -\frac{\alpha+2}{\alpha}w + 4\lambda\xi w^2 - 2\alpha\lambda^2(\xi^2-1)w^3.$$

Now, put the Briot-Bouquet differential equation of the general form

$$(2.3) \qquad \xi\frac{dw}{d\xi} = f(\xi, w)$$

where $f(\xi, w)$ is a holomorphic function in the neighborhood of $(\xi, w) = (0, 0)$ with $f(0, 0) = 0$ and $f_w(0, 0) < 0$. Then we conclude

LEMMA 2.2. *If there exists a solution $w = w(\xi)$ of (2.3) and if 0 is an accumulation point of $w(\xi)$ as ξ tends to 0 so that $\arg \xi$ is bounded, then $w(\xi)$ is the unique holomorphic solution.*

This is Lemma 2.5 of [14] and the proof is omitted.

Let $x = x(t)$ be a solution of (E) and (ω_-, ω_+) denote the domain of $x(t)$. Moreover let y and z be functions obtained from $x(t)$ through (T). Then we get

LEMMA 2.3. *Suppose that $\alpha \neq -2$, $\tau = \omega_\pm$ and z does not vanish. Then if $y \to \infty$ as $t \to \tau$, we obtain $|\tau| < \infty$.*

PROOF. Since z does not vanish, z is a solution of (R). Therefore from $y \to \infty$ as $t \to \tau$ and (2.1) there exists a solution of (2.2) continuable to $\xi = 0$ (namely $y = \infty$). So let us examine the solution of (2.2) for every value to which the solution tends as $\xi \to 0$. For this, notice that the right-hand side of (2.2) vanishes at $(\xi, w) = (0, \pm\rho)$ where

$$\rho = (1/\alpha\lambda)\sqrt{(\alpha+2)/2}$$

and let w_γ denote a solution of (2.2) which has an accumulation point γ as $\xi \to 0$.

Suppose that $\gamma \neq 0, \pm\rho, \pm\infty$. Then from Painlevé's theorem (cf. [3]), ξ is a solution of $d\xi/dw = \xi/\chi(\xi, w)$ ($\chi(\xi, w)$ is the right-hand side of (2.2)) with an initial condition $\xi(\gamma) = 0$. Hence the uniqueness of the solution implies a contradiction $\xi \equiv 0$. Thus this case does not occur. Namely γ is the limit of w_γ and $\gamma = 0, \pm\rho, \pm\infty$. Notice that the discussion for obtaining $\xi \equiv 0$ will be often used without such a detailed description.

Next suppose $\gamma = 0$. Then if $-2 < \alpha < -1$, we have from (2.2)

$$w_\gamma = C\xi^{-(1+2/\alpha)}\left[1 + \sum_{m+n\geq 1} a_{mn}\xi^m\{C\xi^{-(1+2/\alpha)}\}^n\right]$$

where C is an arbitrary nonzero constant and the power series converges in the neighborhood of $\xi = 0$, since $-(\alpha + 2)/\alpha > 0$ and w divides the right-hand side of (2.2) (cf. Chapitre III of [5], especially a formula (1.17) of this). Returning to the original variables through (2.1) and (T), we obtain

$$Cy^{-1+1/\alpha}\left\{1 + \sum_{m+n\geq 1} a_{mn}y^{-m/2}(Cy^{(\alpha+2)/2\alpha})^n\right\}y' = 1$$

and integrating both sides of this with respect to t

$$\alpha Cy^{1/\alpha} + \sum_{m+n\geq 1} \tilde{a}_{mn}y^{-m/2+((\alpha+2)/2\alpha)n+1/\alpha} = t + D$$

where D is an integral constant. If $y \to \infty$ as $t \to \omega_-$, then the left-hand side vanishes as $t \to \omega_-$ and hence we have $D = -\omega_-$ and $\omega_- > -\infty$. Similarly if $y \to \infty$ as $t \to \omega_+$, then we obtain $D = -\omega_+$ and $\omega_+ < \infty$. Therefore we have

(2.4)
$$y^{1/\alpha}\left\{1 + \sum_{m+n\geq 1} b_{mn}y^{-m/2+((\alpha+2)/2\alpha)n}\right\}$$
$$= (t - \omega_-)/\alpha C \text{ and } (t - \omega_+)/\alpha C.$$

If $\alpha < -2$, then Lemma 2.2 implies that w_γ is the unique holomorphic solution as $-(\alpha + 2)/\alpha < 0$. Hence we get a contradiction $w_\gamma \equiv 0$ and this case does not occur.

Now suppose that $\gamma = \pm\rho$. Then we get $-2 < \alpha < -1$. Putting $\theta = w - \gamma$, we have

$$\xi\frac{d\theta}{d\xi} = \frac{2(\alpha + 2)}{\alpha^2\lambda}\xi + \frac{2(\alpha + 2)}{\alpha}\theta + \cdots$$

where $\cdots$ denotes a power series (a polynomial here) starting from terms whose degrees are greater than the degree of the previous terms. Since $2(\alpha + 2)/\alpha < 0$, Lemma 2.2 implies that θ is the unique holomorphic solution represented as

$$\theta = \sum_{n=1}^{\infty} a_n\xi^n.$$

Returning to the original variables, we get

(2.5)
$$z = y^{3/2}\left(\rho^{-1} + \sum_{n=1}^{\infty} b_ny^{-n/2}\right)$$

(2.6)
$$z = y^{3/2}\left(-\rho^{-1} + \sum_{n=1}^{\infty} b_ny^{-n/2}\right)$$

which imply $\gamma y^{-3/2}(1 + \cdots)y' = 1$. Integrating both sides with respect to t, it follows from the similar discussion done for getting (2.4) that

$$(2.7) \qquad -2\gamma y^{-1/2} - \sum_{n=1}^{\infty} \frac{2a_n}{n+1} y^{-(n+1)/2} = t - \omega_- \text{ and } t - \omega_+$$

where $\omega_- > -\infty$ and $\omega_+ < \infty$.

Finally suppose that $\gamma = \pm\infty$. Putting $w = 1/\theta$, we have $\theta \to 0$ as $\xi \to 0$ and

$$\frac{d\xi}{d\theta} = \frac{\alpha\xi\theta}{(\alpha+2)\theta^2 - 4\alpha\lambda\xi\theta + 2\alpha^2\lambda^2(\xi^2 - 1)}.$$

These imply a contradiction $\xi \equiv 0$. Consequently this case does not occur. Namely if $y \to \infty$ as $t \to \tau$, then in all cases to occur we obtain $|\tau| < \infty$. This completes the proof.

We get solutions of (E) from (2.2) only through (2.4) and (2.7), and therefore representations of the solutions as follows:

COROLLARY 2.4. *Suppose $\alpha \neq -2$ and let γ be an accumulation point of a solution w of (2.2) as $\xi \to 0$. Then if $-2 < \alpha < -1$ and $\gamma = 0$, we get either (1.10) in the neighborhood of $t = \omega_-$ or (1.9) in the neighborhood of $t = \omega_+$, and if $-2 < \alpha < -1$ and $\gamma = \pm\rho$, either (1.8) in the neighborhood of $t = \omega_-$ or (1.6) in the neighborhood of $t = \omega_+$. Otherwise a solution $x(t)$ of (E) cannot be obtained.*

In the alternatives of this corollary, notice that if $\xi \to 0$ as $t \to \omega_\pm$ we take the representation valid in the neighborhood of $t = \omega_\pm$ respectively.

PROOF OF COROLLARY 2.4. In (2.4) we put

$$\theta = y^{1/\alpha}, \qquad \tau = (t - \omega_-)/\alpha C \text{ or } (t - \omega_+)/\alpha C$$

and get

$$\theta\left\{1 + \sum_{m+n \geq 1} b_{mn}\theta^{-\alpha m/2}\theta^{(\alpha+2)n/2}\right\} = \tau.$$

Hence we have

$$\theta^{-\alpha/2}\left\{1 + \sum_{m+n \geq 1} c_{mn}\theta^{-\alpha m/2}\theta^{(\alpha+2)n/2}\right\} = \tau^{-\alpha/2}$$

$$\theta^{(\alpha+2)/2}\left\{1 + \sum_{m+n \geq 1} d_{mn}\theta^{-\alpha m/2}\theta^{(\alpha+2)n/2}\right\} = \tau^{(\alpha+2)/2}.$$

From the last two equations we obtain double power series of $\tau^{-\alpha/2}$ and $\tau^{(\alpha+2)/2}$ representing $\theta^{-\alpha/2}$ and $\theta^{(\alpha+2)/2}$. Substituting these into the first equation, we get

$$\theta = \tau\left\{1 + \sum_{m+n\geq 1} \tilde{b}_{mn}\tau^{-\alpha m/2}\tau^{(\alpha+2)n/2}\right\}.$$

Returning to the original variables and using (T), we have (1.9) or (1.10).

For obtaining (1.6) or (1.8), it suffices to apply the inverse function theorem directly to (2.7) and use (T). This completes the proof.

Furthermore we have

LEMMA 2.5. *If $\alpha = -2$ and z does not vanish, then y is bounded.*

PROOF. If y is unbounded, then (2.2) namely

$$(2.8) \qquad \qquad \xi\frac{dw}{d\xi} = 4\lambda\xi w^2 + 4\lambda^2(\xi^2 - 1)w^3$$

has a solution $w = w(\xi)$ continuable to $\xi = 0$ since z does not vanish.

Now if $w(\xi)$ accumulates to a nonzero number as $\xi \to 0$, then since the right-hand side of (2.8) does not vanish we get a contradiction $\xi \equiv 0$.

Next if $w(\xi)$ accumulates to 0 as $\xi \to 0$, then the theory of [4] implies that

$$w(\xi) = \{8\lambda^2(log\ \xi + C)\}^{-1/2}\left[1 + \sum_{1\leq 2j+k<2(N+1)} w_{jk}\xi^j\{8\lambda^2(log\ \xi + C)\}^{-k/2} + \Omega\right],$$

$$|\Omega| \leq K_N|log\ \xi|^{-N}$$

where C is an arbitrary constant, N is a positive integer and K_N is a constant, since w divides the right-hand side of (2.8) and $\xi \in \mathbf{R}$. Thus as $\xi \to 0$, we have a contradiction $w \notin \mathbf{R}$.

In both cases we have a contradiction and hence the proof is completed.

3. Existence of the orbit Γ_1

First let us consider solutions of (D) in the neighborhood of its singular point $(0,0)$. When a solution of (D) passes a line $z = \alpha\lambda y$, we get

$$(3.1) \qquad \qquad \frac{d}{ds}(z - \alpha\lambda y) = \alpha^2\lambda^2 y^3 > 0.$$

Moreover from Lemma 5 of [10] and a transformation $w = yz^{-1}$ we conclude that if a solution $z(y)$ of (R) converges to 0 as $y \to 0$, we have

$$\text{(3.2)} \qquad\qquad \lim_{y\to 0} \frac{z(y)}{y} = \alpha\lambda.$$

From the proof of Proposition 8 of [10], there exists uniquely a solution $z_1(y)$ of (R) represented as

$$\text{(3.3)} \qquad\qquad z_1(y) = \alpha\lambda y + \lambda y^2 - \frac{(\alpha+1)\lambda}{2\alpha^2} y^3 + \cdots$$

in the neighborhood of $y = 0$ such that (3.2) and

$$\text{(3.4)} \qquad\qquad \lim_{y\to 0} \frac{v(y)}{y} = \lambda \qquad (v(y) = y^{-1} z_1(y) - \alpha\lambda)$$

are valid. Hence from (T) and (3.3) we get $y' = \alpha\lambda y(1 + \cdots)$. Solving this, we have $y(1 + \cdots) = Ce^{\alpha\lambda t}$ which implies $t \to -\infty$ as $y \to 0$. Thus we obtain a representation (1.5) of a solution $x(t)$, and this is valid in the neighborhood of $t = -\infty$.

Moreover notice that $(y, z_1(y))$ is a solution of (D) whose orbit is $z = z_1(y)$. If y of $(y, z_1(y))$ increases, then s decreases since $dy/ds < 0$ in $z > 0$ from (D). Therefore since $\lambda y^2 < 0$ if $y \neq 0$, it follows from (3.1) that in the case $y > 0$, the orbit $z = z_1(y)$ and the line $z = \alpha\lambda y$ cannot cross and $z_1(y) < \alpha\lambda y$. Furthermore the line $z = \alpha\lambda y$ and a curve $z = Z_+(y)$ cross. Therefore as y increases, the orbit $z = z_1(y)$ increases from Lemma 2.1 and gets into the region P.
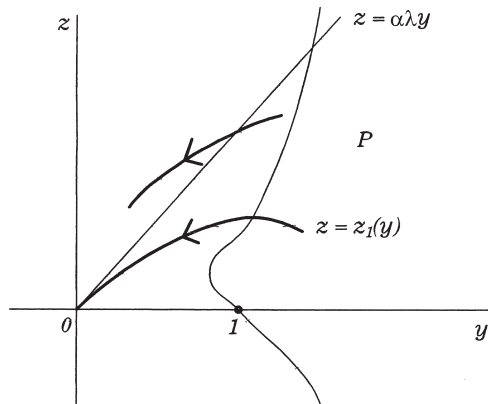
The arrows show directions of the orbits of (D) as s increases.

Figure 3

Next we consider solutions of (D) in the neighborhood of its singular point $(1,0)$. Putting $y = 1 + \eta$ and $z = \zeta$ in (D), we get

$$\frac{d\eta}{ds} = \alpha\zeta + \cdots, \qquad \frac{d\zeta}{ds} = \alpha^2\lambda^2\eta + 2\alpha\lambda\zeta + \cdots$$

in the neighborhood of $(\eta, \zeta) = (0,0)$. The coefficient matrix of the linear terms of this has eigenvalues

$$\mu = (1 + \sqrt{-1-\alpha}i)\alpha\lambda, \qquad \bar{\mu} = (1 - \sqrt{-1-\alpha}i)\alpha\lambda.$$

Therefore from Theorem A of [2] and its proof, we get

(3.5)
$$\eta = \alpha A e^{\mu s} + \alpha \bar{A} e^{\bar{\mu}s} + \sum_{m+n \geq 2} a_{mn}(Ae^{\mu s})^m (\bar{A}e^{\bar{\mu}s})^n$$

$$\zeta = \mu A e^{\mu s} + \bar{\mu} \bar{A} e^{\bar{\mu}s} + \sum_{m+n \geq 2} b_{mn}(Ae^{\mu s})^m (\bar{A}e^{\bar{\mu}s})^n$$

where A is an arbitrary nonzero constant, and a_{mn}, b_{mn} are constants such that $a_{mn} = \bar{a}_{nm}$, $b_{mn} = \bar{b}_{nm}$, since $\eta, \zeta \in \mathbf{R}$. Hence we obtain a solution

(3.6)
$$y = 1 + \alpha A e^{\mu s} + \alpha \bar{A} e^{\bar{\mu}s} + \sum_{m+n \geq 2} a_{mn}(Ae^{\mu s})^m (\bar{A}e^{\bar{\mu}s})^n$$

$$z = \mu A e^{\mu s} + \bar{\mu} \bar{A} e^{\bar{\mu}s} + \sum_{m+n \geq 2} b_{mn}(Ae^{\mu s})^m (\bar{A}e^{\bar{\mu}s})^n$$

of (D) which is valid in the neighborhood of $s = -\infty$, since $\eta \to 0$ as $s \to -\infty$ from (3.5). The orbit (3.6) tends to $(1,0)$ as $s \to -\infty$.

Now, let us investigate how (3.6) does so. For this, put $\operatorname{Im}\mu = v$ $(= \alpha\lambda\sqrt{-1-\alpha})$, $\delta = \arg A$, namely $A = |A|e^{i\delta}$, $\mu = \alpha\lambda + vi$. Then from (3.5) we get

$$\eta = 2\alpha|A|e^{\alpha\lambda s}\cos(vs+\delta) + O(e^{2\alpha\lambda s})$$

$$\zeta = 2|A|e^{\alpha\lambda s}\{\alpha\lambda\cos(vs+\delta) - v\sin(vs+\delta)\} + O(e^{2\alpha\lambda s})$$

as $s \to -\infty$. So, define a set

$$S_\varepsilon = \big\{s : s_n - \varepsilon < s < s_n + \varepsilon, \tau_n - \varepsilon < s < \tau_n + \varepsilon \ (n = 0, \pm 1, \pm 2, \ldots)\big\}$$

for a sufficiently small positive ε, where $s = s_n$, $s = \tau_n$ are solutions of

$$\cos(vs+\delta) = 0, \qquad \alpha\lambda\cos(vs+\delta) - v\sin(vs+\delta) = 0$$

respectively, namely

$$s_n = v^{-1}(\pi/2 - \delta + n\pi), \qquad \tau_n = v^{-1}(\operatorname{Tan}^{-1}\alpha\lambda/v - \delta + n\pi).$$

Then as $s \notin S_\varepsilon$ and $s \to -\infty$, $\cos(vs + \delta)$ and $\alpha\lambda \cos(vs + \delta) - v \sin(vs + \delta)$ do not accumulate to 0 and we obtain

$$\eta = \{2\alpha|A|e^{\alpha\lambda s} \cos(vs + \delta)\}\{1 + O(e^{\alpha\lambda s})\}$$

$$\zeta = 2|A|e^{\alpha\lambda s}\{\alpha\lambda \cos(vs + \delta) - v \sin(vs + \delta)\}\{1 + O(e^{\alpha\lambda s})\}$$

$$\frac{\zeta}{\eta} = \left\{\lambda - \frac{v}{\alpha} \tan(vs + \delta)\right\}\{1 + O(e^{\alpha\lambda s})\}.$$

Here if we put $\eta = r \cos \theta$, $\zeta = r \sin \theta$, then we have

$$r = O(e^{\alpha\lambda s}), \qquad \theta = \tan^{-1}\left\{\lambda - \frac{v}{\alpha} \tan(vs + \delta)\right\}\{1 + O(e^{\alpha\lambda s})\}.$$

Hence (3.6) tends to $(1, 0)$, going around this clockwise infinitely many times, since η, ζ are continuous in s.

Here we need the following:

LEMMA 3.1. (D) *has no periodic orbit in the region* $y > 0$.

PROOF. Suppose that (D) has a periodic orbit Γ_4 in the region $y > 0$. Then Γ_4 surrounds the singular point $(1, 0)$. However if $z \neq 0$, then (D) is equivalent to a 2-dimensional dynamical system

$$(3.7) \qquad \frac{dy}{dt} = z, \qquad \frac{dz}{dt} = \frac{(\alpha - 1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3)}{\alpha y}.$$

On the y axis $z = 0$, we get from (D)

$$\frac{dy}{ds} = 0, \qquad \frac{dz}{ds} = -\alpha^2\lambda^2(y^2 - y^3).$$

This implies that if $(y, z) \neq (0, 1)$, then an orbit (y, z) of (D) passes the y axis only as s attains discrete values. Similarly (y, z) has the same property as an orbit of (3.7). Hence orbits of (D) are those of (3.7). Therefore using t as a parameter, we represent Γ_4 as

$$y = y(t), \qquad z = z(t).$$

Since (3.7) is a dynamical system, $y(t)$ and $z(t)$ are defined on $-\infty < t < \infty$ and periodic functions. Moreover since $y(t)$ is the y coordinate of Γ_4, $y(t)$ periodically attains every value of a closed interval containing 1 as its inner point.

Now if we denote as $x = x(t)$ the solution of (E) got from applying (T) to Γ_4, then we have

$$(3.8) \qquad\qquad\qquad x(t) = p(t) y(t)^{1/\alpha}.$$

On the other hand, since $x(t)$ can be continued to ∞, it follows from Kamo's theorem (Theorem 4.3 (ii) of [6]) that we obtain

$$(3.9) \qquad\qquad\qquad x(t) \sim p(t) \qquad \text{as } t \to \infty.$$

However (3.8) and (3.9) are contradiction. This denies the existence of Γ_4 and the proof is complete.

LEMMA 3.2. *A solution of* (D) *passing a point of P tends to* $(1, 0)$ *as* $s \to -\infty$.

PROOF. Let $z = z(y)$ be an orbit of (D) passing a point of P. Then if $z(y)$ is bounded as $y \to \infty$, it implies a contradiction $\eta \equiv 0$ to put $y = 1/\eta$ in (R). Indeed, from (E) we have

$$\frac{d\eta}{dz} = -\frac{\alpha \eta^4 z}{(\alpha - 1)\eta^3 z^2 + 2\alpha\lambda\eta^2 z - \alpha^2\lambda^2(\eta - 1)}.$$

Moreover from Lemma 3.1, there does not exist a periodic orbit around $(1, 0)$. Hence from Poincaré-Bendixon's theorem the solution $(y, z(y))$ tends to $(1, 0)$ as $s \to -\infty$ and the proof is completed.

Recall here that the orbit $z = z_1(y)$ gets into P as s decreases. Then it follows from Lemma 3.2 that $(y, z_1(y))$ tends to $(1, 0)$ as $s \to -\infty$. Therefore $z_1(y)$ is represented as (3.6) in the neighborhood of $s = -\infty$. Substituting (3.6) into $z = y'$ and comparing coefficients as power series of A and $\bar{A}$, we get $s = t/\alpha + C$ (C is a constant) and a solution $x(t)$ of (E) such as

$$(3.10) \qquad x(t) = \lambda^{2/\alpha} e^{-\lambda t} \Bigg\{ 1 + A e^{(\mu/\alpha)t} + \bar{A} e^{(\bar{\mu}/\alpha)t}$$

$$+ \sum_{m+n \geq 2} x_{mn} (A e^{(\mu/\alpha)t})^m (\bar{A} e^{(\bar{\mu}/\alpha)t})^n \Bigg\}$$

which is valid in the neighborhood of $t = \infty$. Here $A e^{\mu C}$ is replaced with A. Since $x(t)$ attains real values, we have $x_{mn} = \bar{x}_{nm}$ and (1.4) where $C = |A|$ from (3.10).

Now let Γ_1 be an orbit of (D) which is represented as $z = z_1(y)$ in the neighborhood of $y = 0$. Then from the above discussion Γ_1 tends to $(1, 0)$ as $s \to -\infty$ in the way stated at the outset. Moreover take (t_0, a, b) of (I) satisfying $(y_0, z_0) \in \Gamma_1$ and let $x(t)$ be a solution of (E) and (I). Then through (T) we define a solution (y, z) of (D) passing (y_0, z_0) so that (y, z) also satisfies

(3.7). Therefore if (y, z) passes (y_0, z_0) at some t, then y is defined for $-\infty < t < \infty$ and the orbit of (y, z) is the whole of Γ_1. Hence from the above discussion we get the representations (1.4) and (1.5) of $x(t)$ as in (iv) of Theorem I.

4. On the case (t_0, a, b) with $(y_0, z_0) \notin \Gamma_1$

Now we take (t_0, a, b) of (I) such that $(y_0, z_0) \notin \Gamma_1$. Then we get a solution $z(y)$ of (R) from the solution of (D) passing (y_0, z_0), and a solution $x(t)$ of (E) and (I) from $z(y)$ through (T). Conversely from $x(t)$ and (T) we define the same solution (y, z) of (D) whose orbit is $z = z(y)$. Here recall that (ω_-, ω_+) denotes the domain of $x(t)$.

Consider the case $t \to \omega_+$. Then there exist the following possibilities:

(4.1) $\omega_+ < \infty, \qquad \lim_{t \to \omega_+} x(t) = 0$

(4.2) $\omega_+ < \infty, \qquad \lim_{t \to \omega_+} x(t) = \infty$

(4.3) $\omega_+ = \infty, \qquad \lim_{t \to \omega_+} x(t) = 0$

(4.4) $\omega_+ = \infty, \qquad 0 < \lim_{t \to \omega_+} x(t) < \infty$

(4.5) $\omega_+ = \infty, \qquad \lim_{t \to \omega_+} x(t) = \infty.$

In the cases (4.1), (4.3) and (4.4), we get $\lim_{t \to \omega_+} y = \infty$ from (T). Here we show the fact that z does not vanish for $y > 1$.

Suppose the contrary. Then z vanishes for some $y (>1)$. Therefore (y, z) is in the region P of Figure 2. However if $dy/ds \neq 0$, then from (T) and (D) we get

$$\frac{ds}{dt} = \frac{dy}{dt} \bigg/ \frac{dy}{ds} = \frac{1}{\alpha y} < 0.$$

Furthermore $dy/ds \neq 0$ holds for almost every s, since $dy/ds = 0$ now occurs only if $z = 0$ and implies $dz/ds \neq 0$ in (D). Thus s decreases as t increases. Therefore from Lemma 3.2 (y, z) tends to $(1, 0)$ as $t \to \omega_+$ since (y, z) also satisfies (3.7). This contradicts $\lim_{t \to \omega_+} y = \infty$. Hence we conclude the above fact.

Consequently if $\alpha \neq -2$, then Lemma 2.3 implies $\omega_+ < \infty$. Namely (4.3) and (4.4) do not occur. So we suppose $\alpha \neq -2$ and (4.1).

Notice here that we have $-\infty < x'(\omega_+) \leq 0$ from $x''(t) > 0$ and $x(\omega_+) = 0$. Then if $\alpha < -2$, we get from (T) and (1.1)

$$\lim_{t \to \omega_+} y^{-3/2} z = \lim_{t \to \omega_+} \frac{\alpha x'(t)}{y^{1/2} x(t)} = -\alpha \lambda e^{-\alpha \lambda \omega_+/2} \lim_{t \to \omega_+} \frac{x'(t)}{x(t)^{(\alpha+2)/2}} = 0.$$

Namely using ξ and w defined in (2.1), from the above fact we have a solution w of (2.2) with $\lim_{\xi \to 0} w = \infty$. Therefore from Corollary 2.4 there does not exist $x(t)$ satisfying (4.1).

Suppose $-2 < \alpha < -1$ now. Then if $x'(\omega_+) = 0$, l'Hospital's theorem implies

$$\left(\lim_{t \to \omega_+} y^{-3/2} z \right)^2 = \alpha^2 \lambda^2 e^{-\alpha \lambda \omega_+} \lim_{t \to \omega_+} \frac{x'(t)^2}{x(t)^{\alpha+2}} = \frac{2\alpha^2 \lambda^2}{\alpha + 2}.$$

Now we show $\lim_{t \to \omega_+} y^{-3/2} z \ge 0$. If $(y, z) \in P$ for some t, then from the reasoning used for showing the above fact, (y, z) tends to $(1, 0)$, which contradicts $y \to \infty$ as $t \to \omega_+$. Moreover if $(y, z) \notin P$ and $z < 0$ for some t (cf. the point p in Figure 4), then y decreases as t increases from $y' = z < 0$. Hence since $y \to \infty$ as $t \to \omega_+$, (y, z) enters a region $z > 0$ in the end. In this case (y, z) exists below Γ_1. Therefore (y, z) gets into P and we have a contradiction $(y, z) \to (1, 0)$ again. Thus we obtain $z > 0$ namely $\lim_{t \to \omega_+} y^{-3/2} z \ge 0$, and

(4.6) $$\lim_{t \to \omega_+} y^{-3/2} z = \rho^{-1} \qquad (\rho = (1/\alpha\lambda)\sqrt{(\alpha+2)/2})$$

(cf. the point q in Figure 4).

Here, recall the proof of Lemma 2.3. Then the orbit of (D) satisfying (4.6) (namely $\gamma = \rho$) is represented only as (2.5) in the neighborhood of $y = \infty$. Hence this exists uniquely. So let $z = z_2(y)$ be the orbit with (4.6). Then Γ_2 appearing in Section 1 is the orbit $z = z_2(y)$. As shown below, Γ_2
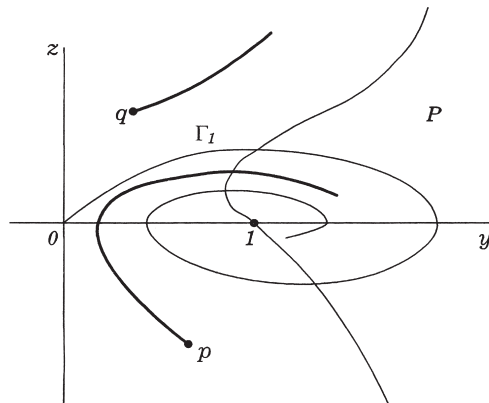

Figure 4

tends to $(0,0)$ as $s \to \infty$. Now if we take (t_0, a, b) satisfying $(y_0, z_0) \in \Gamma_2$, we get (1.6) from Corollary 2.4.

On the other hand, if $x'(\omega_+) \neq 0$ namely $-\infty < x'(\omega_+) < 0$ we have

$$\lim_{t \to \omega_+} y^{-3/2}z = \infty.$$

Therefore if $z = z(y)$ is an orbit of (y, z) satisfying this, then we obtain $z(y) > z_2(y)$ and (t_0, a, b) satisfies $(y_0, z_0) \in R_1$ (cf. Figure 1). Moreover we get (1.9) from Corollary 2.4.

Next if $\alpha = -2$, then Lemma 2.5 implies that (4.1), (4.3) and (4.4) do not occur. Indeed it is already shown that z does not vanish.

Now, suppose (4.2). Then we get $\lim_{t \to \omega_+} y = 0$. Since $y > 0$, $z(= y')$ attains a negative value and hence from the same reasoning as in the case (4.1) and $-2 < \alpha < -1$ we conclude a contradiction $(y, z) \to (1, 0)$ as $t \to \omega_+$ (cf. the point p of Figure 4).

Suppose (4.5). Then if $|x'(\omega_+)| < \infty$, we have

$$\lim_{t \to \omega_+} \frac{x(t)}{e^{-\lambda t}} = \lim_{t \to \omega_+} \frac{x'(t)}{-\lambda e^{-\lambda t}} = 0$$

and

$$\lim_{t \to \omega_+} y = \lim_{t \to \omega_+} \lambda^{-2} \left(\frac{x(t)}{e^{-\lambda t}}\right)^\alpha = \infty.$$

Hence from Lemmas 2.3, 2.5 and 3.2 we get a contradiction as in the cases (4.1), (4.3) and (4.4).

If $x'(t) \to \infty$ as $t \to \omega_+$, then we have

$$\lim_{t \to \omega_+} \frac{x(t)}{e^{-\lambda t}} = \lim_{t \to \omega_+} \frac{x''(t)}{\lambda^2 e^{-\lambda t}} = \lim_{t \to \omega_+} \frac{1}{\lambda^2} \left(\frac{x(t)}{e^{-\lambda t}}\right)^{1+\alpha}.$$

However the sign of $z = y'$ is definite as the point q of Figure 4, or a solution (y, z) of (D) gets into P and is on the curve tending to $(1, 0)$ spirally as the point p of Figure 4. Therefore $\lim_{t \to \omega_+} y$ exists. Hence $\lim_{t \to \omega_+} x(t)/e^{-\lambda t}$ also exists. If c denotes this, then we obtain $c = (1/\lambda^2)c^{1+\alpha}$. Therefore we get $c = 0, \lambda^{2/\alpha}, \infty$ and $\lim_{t \to \omega_+} y = \infty, 1, 0$ respectively. However $\lim_{t \to \omega_+} y = 0$ is impossible as in the case (4.2), and $\lim_{t \to \omega_+} y = \infty$ deduces a contradiction from Lemmas 2.3, 2.5 and 3.2. Hence we conclude $\lim_{t \to \omega_+} y = 1$. In this case the solution (y, z) of (D) tends to $(1, 0)$, turning. Moreover if $-2 < \alpha < -1$ and $z = z(y)$ is the orbit of (y, z), then we get $z(y) < z_2(y)$ and hence take (t_0, a, b) such that (y_0, z_0) belongs to a region below Γ_2. Furthermore from the same reasoning as in Section 3 we have (1.4) and $\omega_+ = \infty$.

Next we consider the case $t \to \omega_-$. Then there exist the following possibilities:

$$(4.7) \qquad \omega_- > -\infty, \qquad \lim_{t \to \omega_-} x(t) = 0$$

$$(4.8) \qquad \omega_- > -\infty, \qquad \lim_{t \to \omega_-} x(t) = \infty$$

$$(4.9) \qquad \omega_- = -\infty, \qquad \lim_{t \to \omega_-} x(t) = 0$$

$$(4.10) \qquad \omega_- = -\infty, \qquad 0 < \lim_{t \to \omega_-} x(t) < \infty$$

$$(4.11) \qquad \omega_- = -\infty, \qquad \lim_{t \to \omega_-} x(t) = \infty.$$

Define (y, z) from $x(t)$ and (T) again. Then as stated in the proof of Lemma 3.2, z is not bounded as $y \to \infty$. Therefore if $(y, z) \in P$ and $z < 0$ for some t, then it follows from $z = y'$ and Lemma 2.1 that as t decreases, y increases, (y, z) gets into the region $z > 0$, and eventually y decreases.

Therefore if (4.7) is valid, then it suffices to follow the discussion done in the cases (4.1), (4.3) and (4.4). Hence we conclude that if $\alpha < -2$, then there does not exist $x(t)$ satisfying (4.7) and if $-2 < \alpha < -1$ and $x'(\omega_-) = 0$, then z satisfies (2.6) in the neighborhood of $y = \infty$. Here let Γ_3 be the orbit of (D) which is represented uniquely as (2.6) in the neighborhood of $y = \infty$. Then since Γ_3 lies in the region $z < 0$ and $(y, z) \notin \bar{P}$ as p of Figure 4, Γ_3 tends to $(1, 0)$ as $s \to -\infty$. Furthermore (t_0, a, b) satisfies $(y_0, z_0) \in \Gamma_3$ and we get (1.8) in the neighborhood of $t = \omega_-$ from Corollary 2.4. Moreover we conclude that if $-2 < \alpha < -1$ and $x'(\omega_-) \neq 0$, then (t_0, a, b) satisfies $(y_0, z_0) \in R_3$ and we get (1.10) in the neighborhood of $t = \omega_-$. If $\alpha = -2$, Lemmas 2.5 and 3.2 imply that (4.7) does not occur.

In the cases (4.8), (4.10) and (4.11), we get from (T)

$$(4.12) \qquad \lim_{t \to \omega_-} y = 0.$$

Now we suppose (4.9). If $x'(\omega_-) \neq 0$, then we have (4.12) immediately. On the other hand, if $x'(\omega_-) = 0$, then from the same discussion as in the case (4.5) we obtain $\lim_{t \to \omega_-} y = \infty, 1, 0$. If $\lim_{t \to \omega_-} y = 1$, then the orbit (y, z) of (D) tends to $(1, 0)$. However since Γ_1 surrounds $(1, 0)$, (y, z) cannot tend to $(1, 0)$, going around this anticlockwise. Therefore from Lemma 2.1 and $z = y'$, $\lim_{t \to \omega_-} y = 1$ is impossible. Moreover from Lemmas 2.3, 2.5 and 3.2, $\lim_{t \to \omega_-} y = \infty$ deduces a contradiction. Hence we get (4.12).

Recall that (y, z) is a solution of (D) passing (y_0, z_0). Then if (y, z) is in the region $0 < z < z_1(y)$ and tends to $(0, 0)$ as $t \to \omega_-$ (as the point p of Figure 5), then z is a solution of (R) with (3.2). However a solution of (R)
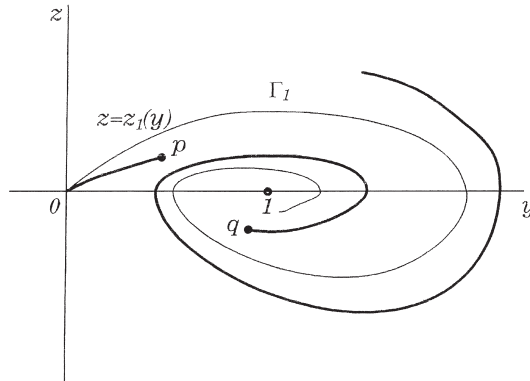
Figure 5

satisfying (3.2) and (3.4) is only $z_1(y)$. Therefore (3.2) holds and (3.4) does not. This is a condition (9) of [12] and hence from discussion after this we obtain $z > z_1(y)$. This is a contradiction and (y, z) does not tend to $(0, 0)$ as $t \to \omega_-$, remaining in $0 < z < z_1(y)$.

So, suppose that (y, z) is in the region $z < 0$ (as the point q of Figure 5). Then y increases as t decreases since $y' = z < 0$. Therefore due to (4.12) and the above, (y, z) enters the region $z > 0$, going around $(1, 0)$. In this case $(y_0, z_0) \notin \Gamma_3 \cup R_3$ must hold. If (y, z) is in the region $z > 0$, then as t decreases y decreases from $y' = z > 0$ and if t is sufficiently close to ω_-, then (y, z) is outside P. On the other hand, s increases as y decreases in terms of $dy/ds = \alpha yz < 0$. Therefore as t decreases, z decreases from Lemma 2.1 and Poincaré-Bendixon's theorem implies that (y, z) converges to $(0, 0)$ as $t \to \omega_-$, since (y, z) satisfies the 2-dimensional dynamical system (3.7). Hence z is a solution of (R) satisfying (3.2). Here notice that from the discussion just done, Γ_2 also tends to $(0, 0)$ as $t \to \omega_-$ (namely $s \to \infty$).

Moreover from the uniqueness of $z_1(y)$, z satisfies a condition (9) of [12] again and from discussions after this we obtain

(4.13) $$x(t) \sim ct \qquad \text{as } t \to -\infty$$

where c is a negative constant. Hence integrating (E) from $-\infty$ to t twice we get (1.7) (cf. [6]).

Here for proving (v) of Theorem I, suppose that (t_0, a, b) satisfies $(y_0, z_0) \in \Gamma_2$. Then in the above discussions of the case $t \to \omega_+$, we get only the case when (4.1), $-2 < \alpha < -1$ and $x'(\omega_+) = 0$ hold and do not get the other cases. Moreover in those of the case $t \to \omega_-$, we do not have the case (4.7) and have the other cases. Since all cases to occur have been examined, we therefore conclude (v) of Theorem I. Similarly we get (vi) through (xi) of Theorems I, II and III. Now the proof is completed.

At the end of this paper, let us notice about the solution $x_\pm(t)$ of $x'' = \pm e^{\alpha\lambda t}x^{1+\alpha}$ satisfying (4.13) which appeared also in [12]. From the same proof as of (1.7), $x_\pm(t)$ satisfies

$$x_\pm(t) = ct + d \pm \frac{(ct)^{1+\alpha}}{\alpha^2\lambda^2}e^{\alpha\lambda t}(1 + o(1))$$

as $t \to -\infty$. Here the double signs correspond in the same order.

References

[1] R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.

[2] H. Dulac, Solutions d'un système d'équations différentielles dans le voisinage de valeurs singulières, Bull. Soc. Math. Fr. **40** (1912), 324–363.

[3] E. Hille, Ordinary Differential Equations in the Complex Domain, John-Wiley, 1976.

[4] M. Hukuhara, Sur les points singuliers d'une équation différentielle ordinaire du premier ordre, I, Mem. Fac. Eng. Kyushu Imp. Univ. **8** (1937), 203–247.

[5] M. Hukuhara, T. Kimura, and T. Matuda, Équations Différentielles Ordinaires du Premier Ordre dans le Champ Complexe, Publ. Math. Soc. Japan (1961).

[6] K. Kamo, Classification of proper solutions of some Emden-Fowler equations, Hiroshima Math. J. **29** (1999), 459–477.

[7] M. Naito, Nonoscillatory solutions of second order differential equations with integrable coefficients, Proc. Amer. Math. Soc. **109** (1990), 769–774.

[8] T. Saito, On bounded solutions of $x'' = t^\beta x^{1+\alpha}$, Tokyo J. Math. **1** (1978), 57–75.

[9] T. Saito, Solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ with movable singularity, Tokyo J. Math. **2** (1979), 262–283.

[10] I. Tsukamoto, T. Mishina, and M. Ono, On solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$, Keio Sci. Tech. Rep. **35** (1982), 1–36.

[11] I. Tsukamoto, On solutions of $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$, Tokyo J. Math. **12** (1989), 181–203.

[12] I. Tsukamoto, Remarks on the papers "On solutions of $x'' = \pm e^{\alpha\lambda t}x^{1+\alpha}$", Funkcial. Ekvac. **37** (1994), 363–371.

[13] I. Tsukamoto, On the generalized Thomas-Fermi differential equations and applicability of Saito's transformation, Tokyo J. Math. **20** (1997), 107–121.

[14] I. Tsukamoto, Asymptotic behavior of solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ where $-1 < \alpha < 0$, Osaka J. Math. **40** (2003), 595–620.

[15] I. Tsukamoto, On solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ starting at some positive t, Hokkaido Math. J. **32** (2003), 523–538.

[16] I. Tsukamoto, On solutions of $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$ where $\alpha < 0$, Japan J. Math. **31** (2005), 1–24.

Ichiro Tsukamoto
3-10-38 Higashi-kamagaya
Kamagaya-shi, Chiba
273-0104, Japan
E-mail: itkmt@k9.dion.ne.jp