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The power of the normal bundle associated to an immersion of RP^n, its complexification and extendibility

Dedicated to Professor Takao Matumoto on his 60th birthday

Yutaka Hemmi, Teiichi Kobayashi and Min Lwin Oo

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Abstract. The purpose of this paper is to establish the formulas on the power of the normal bundle associated to an immersion of the real projective space RP^n in the Euclidean space R^{n+k}, and apply them to the problem of extendibility and stable extendibility. Furthermore, we give an example of a 2-dimensional R-vector bundle over RP^2 that is stably extendible to RP^3 but is not extendible to RP^3.

1. Introduction

Let F stand for either the real number field R or the complex number field C, and let X be a space and A its subspace. A t-dimensional F-vector bundle ζ over A is said to be extendible (respectively stably extendible) to X, if and only if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ, that is, if and only if ζ is equivalent (respectively stably equivalent) to the induced bundle $i^*\alpha$ of a t-dimensional F-vector bundle α over X under the inclusion map $i : A \to X$ (cf. [11, p. 20], [12, p. 191 and p. 209] and [3, p. 273–p. 274]). For simplicity, we use the same letter for a vector bundle and its equivalence class.

We study the question:

Determine the dimension n for which an F-vector bundle over RP^n is extendible (or stably extendible) to RP^m for every $m \geq n$.

The answers are obtained for the tangent bundle $\tau = \tau(RP^n)$ of the real projective n-space RP^n in [8, Theorem 4.2], its complexification $c\tau$ in [7, Theorem 1], the square $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ and its complexification $c\tau^2$ in [5, Theorems 4 and 5], and the power $\tau^r = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ (r-fold) and its complexification $c\tau^r$ in [10, Theorems A and B], where $\otimes$ denotes the tensor product and $r > 0$.

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The purpose of this paper is to study the question for the power $\nu^r = \nu \otimes \cdots \otimes \nu$ (r-fold) of the normal bundle ν of an immersion of RP^n in R^{n+k} and its complexification $c\nu^r$, where $r > 0$ and $k > 0$. The partial answer to the question above are obtained in [9, Theorems B and D] and [6, Theorems 2 and 4].

The following results on ν and ν^2 were obtained.

THEOREM 1.1 (cf. [9, Theorem B]). *Let ν be the normal bundle associated to an immersion of RP^n in R^{n+k}, and let $n+1 \le k \le n+12$. Then the following three conditions are equivalent:*
(i) *ν is extendible to RP^m for every $m \ge n$.*
(ii) *ν is stably extendible to RP^m for every $m \ge n$.*
(iii) $1 \le n \le 8$.

THEOREM 1.2 (cf. [6, Theorem 2]). *Let ν^2 be the square of the normal bundle ν associated to an immersion of RP^n in R^{2n+1}. Then the following three conditions are equivalent:*
(i) *ν^2 is extendible to RP^m for every $m \ge n$.*
(ii) *ν^2 is stably extendible to RP^m for every $m \ge n$.*
(iii) $1 \le n \le 17$ *or* $n = 20, 21$.

The first purpose of this paper is to obtain the answer for the r-fold power ν^r. Denote by $\phi(n)$ the number of integers s such that $0 < s \le n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$. Then we have

THEOREM A. *Let k, n and r be positive integers with $n < k^r$, and let $\nu^r = \nu \otimes \cdots \otimes \nu$ be the r-fold power of the normal bundle ν associated to an immersion of RP^n in R^{n+k}. Then the following three conditions are equivalent:*
(i) *ν^r is extendible to RP^m for every $m \ge n$.*
(ii) *ν^r is stably extendible to RP^m for every $m \ge n$.*
(iii) *There is an integer a satisfying*

$$(2n+k+2)^r - k^r \le a2^{\phi(n)+1} \le (2n+k+2)^r + k^r.$$

If $r = 1$ and $n+1 \le k \le n+12$, condition (iii) is equivalent to the condition: $1 \le n \le 8$, and if $r = 2$ and $k = n+1$, it is equivalent to the condition: $1 \le n \le 17$ or $n = 20, 21$. (Note that $2^{\phi(n)} > 2n+13$ for $n \ge 9$ and that there does not exist any integer a such that $4(n+1)^2 \le a2^{\phi(n)} \le 5(n+1)^2$ for $n = 18, 19$ and $n \ge 22$.) Hence Theorem A is a generalization of Theorems 1.1 and 1.2.

The following results on $c\nu$ and $c\nu^2$ were obtained.

THEOREM 1.3 (cf. [9, Theorem D]). *Let $c\nu$ be the complexification of the normal bundle ν associated to an immersion of RP^n in R^{n+k}, and let $n+1 \le k \le n+8$. Then the following three conditions are equivalent:*

(i) *cv is extendible to RP^m for every $m \geq n$.*
(ii) *cv is stably extendible to RP^m for every $m \geq n$.*
(iii) $1 \leq n \leq 9$.

Theorem 1.4 (cf. [6, Theorem 4]). *Let cv^2 be the complexification of the square v^2 of the normal bundle v associated to an immersion of RP^n in R^{2n+1}. Then the following three conditions are equivalent:*
(i) *cv^2 is extendible to RP^m for every $m \geq n$.*
(ii) *cv^2 is stably extendible to RP^m for every $m \geq n$.*
(iii) *$1 \leq n \leq 18$ or $n = 20, 21$.*

The second purpose of this paper is to obtain the answer for the complexification cv^r of the r-fold power v^r. For a real number x, let $[x]$ denote the largest integer n with $n \leq x$ and let $\langle x \rangle$ denote the smallest integer n with $x \leq n$. Then we have

Theorem B. *Let k, n and r be positive integers with $\langle n/2 \rangle \leq k^r$. Then, for the complexification $cv^r = c(v \otimes \cdots \otimes v)$ of the r-fold power v^r of the normal bundle v associated to an immersion of RP^n in R^{n+k}, the following three conditions are equivalent:*
(i) *cv^r is extendible to RP^m for every $m \geq n$.*
(ii) *cv^r is stably extendible to RP^m for every $m \geq n$.*
(iii) *There is an integer b satisfying*

$$(2n + k + 2)^r - k^r \leq b2^{[n/2]+1} \leq (2n + k + 2)^r + k^r.$$

If $r = 1$ and $n + 1 \leq k \leq n + 8$, condition (iii) is equivalent to the condition: $1 \leq n \leq 9$, and if $r = 2$ and $k = n + 1$, it is equivalent to the condition: $1 \leq n \leq 18$ or $n = 20, 21$. (Note that $2^{[n/2]} > 2n + 9$ for $n \geq 10$ and that there does not exist any integer b such that $4(n+1)^2 \leq b2^{[n/2]} \leq 5(n+1)^2$ for $n = 19$ and $n \geq 22$.) Hence Theorem B is a generalization of Theorems 1.3 and 1.4.

In the proof of Theorem 2.2 of [9], it is shown that the tangent bundle $\tau(S^n)$ of the standard n-sphere S^n is stably extendible to S^{n+1} but is not extendible to S^{n+1} if $n \neq 1, 3, 7$. In this paper we have

Theorem C. *There is a 2-dimensional R-vector bundle over RP^2 that is stably extendible to RP^3 but is not extendible to RP^3.*

This paper is arranged as follows. In Section 2 we establish the formulas on the r-fold power $v^r = v \otimes \cdots \otimes v$ of the normal bundle v associated to an immersion of the real projective n-space RP^n in the Euclidean $(n + k)$-space R^{n+k}. In Section 3 we apply the results in Section 2 to the problem of extendibility and stable extendibility of the r-fold power v^r and give a proof of Theorem A. In Section 4 we establish the formulas on the complexification

$cv^r = c(v \otimes \cdots \otimes v)$ of v^r. In Section 5 we study the problem of extendibility and stable extendibility of cv^r and prove Theorem B. In Section 6 we recall some known results and prove Theorem C by using them.

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2. The r-fold power of the normal bundle associated to an immersion of RP^n in R^{n+k}

Let ξ_n denote the canonical line bundle over RP^n. Then we have

THEOREM 2.1. *Let $v^r = v \otimes \cdots \otimes v$ be the r-fold power of the normal bundle v associated to an immersion of RP^n in R^{n+k}, where $k > 0$. Then the following holds in the Grothendieck group $KO(RP^n)$.*

$$v^r = -2^{-1}\{(2n+k+2)^r - k^r\}\xi_n + 2^{-1}\{(2n+k+2)^r + k^r\}.$$

PROOF. Let $\tau = \tau(RP^n)$ be the tangent bundle of RP^n. Then $\tau \oplus 1 = (n+1)\xi_n$ and $\tau \oplus v = n+k$, where $\oplus$ denotes the Whitney sum. Hence $v = n+k+1-(n+1)\xi_n$ in $KO(RP^n)$. So the equality holds for $r = 1$.

Assume that the formula holds for $r \geq 1$. Then

$$\begin{aligned} v^{r+1} &= v \otimes v^r \\ &= \{-(n+1)\xi_n + (n+k+1)\} \\ &\quad \times [-2^{-1}\{(2n+k+2)^r - k^r\}\xi_n + 2^{-1}\{(2n+k+2)^r + k^r\}] \\ &= -2^{-1}\{(2n+k+2)^{r+1} - k^{r+1}\}\xi_n + 2^{-1}\{(2n+k+2)^{r+1} + k^{r+1}\} \end{aligned}$$

since $\xi_n \otimes \xi_n = 1$. Hence the equality holds for any positive integer r by induction on r. □

Using Theorem 2.1, we obtain

THEOREM 2.2. *For any positive integer r and any integer a, the following holds in $KO(RP^n)$.*

$$v^r = 2^{-1}\{a2^{\phi(n)+1} - (2n+k+2)^r + k^r\}\xi_n + 2^{-1}\{(2n+k+2)^r + k^r - a2^{\phi(n)+1}\}.$$

PROOF. Adding $a2^{\phi(n)}(\xi_n - 1) = 0$ (cf. [1, Theorem 7.4]) to the equality in Theorem 2.1, we have the desired equality. □

THEOREM 2.3. *Assume that there is an integer a satisfying*

$$(2n+k+2)^r - k^r \leq a2^{\phi(n)+1} \leq (2n+k+2)^r + k^r.$$

Then

$$\nu^r = 2^{-1}\{a2^{\phi(n)+1} - (2n+k+2)^r + k^r\}\xi_n \oplus 2^{-1}\{(2n+k+2)^r + k^r - a2^{\phi(n)+1}\}$$

holds as R-vector bundles if $n < k^r$.

Proof. Set $X = 2^{-1}\{a2^{\phi(n)+1} - (2n+k+2)^r + k^r\}$ and $Y = 2^{-1}\{(2n+k+2)^r + k^r - a2^{\phi(n)+1}\}$. Then, by the assumption, $X \geq 0$ and $Y \geq 0$, and, by Theorem 2.2, $\nu^r = X\xi_n + Y$ in $KO(RP^n)$. If $n(= \dim RP^n) < k^r(= \dim \nu^r = \dim(X\xi_n \oplus Y))$, we have $\nu^r = X\xi_n \oplus Y$ as R-vector bundles (cf. [2, Theorem 1.5, p. 100]). □

3. Stable extendibility of the r-fold power of the normal bundle associated to an immersion of RP^n in R^{n+k}

We recall the following result.

Theorem 3.1 (cf. [8, Theorem 4.1]). *Let ζ be a t-dimensional R-vector bundle over RP^n. Assume that there is a positive integer l such that ζ is stably equivalent to $(t+l)\xi_n$ and $t+l < 2^{\phi(n)}$. Then $n < t+l$ and ζ is not stably extendible to RP^{t+l}.*

We prove

Theorem 3.2. *Let ν^r be the r-fold power of the normal bundle ν associated to an immersion of RP^n in R^{n+k}, where $r > 0$ and $k > 0$. Then ν^r is stably extendible to RP^m for every $m \geq n$ if and only if there is an integer a satisfying*

$$(2n+k+2)^r - k^r \leq a2^{\phi(n)+1} \leq (2n+k+2)^r + k^r.$$

Proof. First, we prove the "if" part. Let X and Y be defined as in the proof of Theorem 2.3. Then $X \geq 0$ and $Y \geq 0$ by the assumption, and $\nu^r = X\xi_n + Y$ by Theorem 2.2. Since $X\xi_n \oplus Y$ is extendible to RP^m for every $m \geq n$ and since $\dim \nu^r = k^r = \dim(X\xi_n \oplus Y)$, ν^r is stably extendible to RP^m for every $m \geq n$ (cf. [7, Lemma 2.1]).

To prove the "only if" part, we prove the contraposition. Assume that every integer a satisfies

$$a2^{\phi(n)+1} < (2n+k+2)^r - k^r \qquad \text{or} \qquad (2n+k+2)^r + k^r < a2^{\phi(n)+1}.$$

Clearly there is an integer a with $a2^{\phi(n)+1} < (2n+k+2)^r - k^r$. Define A as the maximum integer such that $A2^{\phi(n)+1} < (2n+k+2)^r - k^r$. If A satisfies $A2^{\phi(n)+1} \leq (2n+k+2)^r + k^r - 2^{\phi(n)+1}$, we have $(2n+k+2)^r - k^r \leq (A+1)2^{\phi(n)+1} \leq (2n+k+2)^r + k^r$ and these are inconsistent with our assumption. Hence A satisfies

$$A2^{\phi(n)+1} > (2n+k+2)^r + k^r - 2^{\phi(n)+1}.$$

Putting $\zeta = \nu^r$, $t = k^r$ and $l = 2^{-1}\{A2^{\phi(n)+1} - (2n+k+2)^r - k^r + 2^{\phi(n)+1}\}$ in Theorem 3.1, we see that ν^r is not stably extendible to RP^{t+l}, since $t + l < 2^{\phi(n)}$ and $l > 0$.

Clearly there is an integer a with $(2n+k+2)^r + k^r < a2^{\phi(n)+1}$. Define B as the minimum integer such that $(2n+k+2)^r + k^r < B2^{\phi(n)+1}$. If B satisfies $B2^{\phi(n)+1} \geq (2n+k+2)^r - k^r + 2^{\phi(n)+1}$, we have $(2n+k+2)^r - k^r \leq (B-1)2^{\phi(n)+1} \leq (2n+k+2)^r + k^r$ and these are inconsistent with our assumption. Hence B satisfies

$$B2^{\phi(n)+1} < (2n+k+2)^r - k^r + 2^{\phi(n)+1}.$$

Putting $\zeta = \nu^r$, $t = k^r$ and $l = 2^{-1}\{B2^{\phi(n)+1} - (2n+k+2)^r - k^r\}$ in Theorem 3.1, we see that ν^r is not stably extendible to RP^{t+l}, since $t + l < 2^{\phi(n)}$ and $l > 0$. □

Proof of Theorem A. According to Theorem 2.2 in [9], for $m \geq n$, ν^r is extendible to RP^m if and only if ν^r is stably extendible to RP^m, provided $n < k^r$. Hence the result follows from Theorem 3.2. □

4. The complexification of the r-fold power of the normal bundle associated to an immersion of RP^n in R^{n+k}

Complexifying the equality in Theorem 2.1, we have

Theorem 4.1. *Let $c\nu^r = c(\nu \otimes \cdots \otimes \nu)$ be the complexification of the r-fold power ν^r of the normal bundle ν associated to an immersion of RP^n in R^{n+k}, where $r > 0$ and $k > 0$. Then the following holds in the Grothendieck group $K(RP^n)$.*

$$c\nu^r = -2^{-1}\{(2n+k+2)^r - k^r\}c\xi_n + 2^{-1}\{(2n+k+2)^r + k^r\},$$

where $c\xi_n$ denotes the complexification of ξ_n.

Using Theorem 4.1, we obtain

Theorem 4.2. *In $K(RP^n)$*

$$c\nu^r = 2^{-1}\{b2^{[n/2]+1} - (2n+k+2)^r + k^r\}c\xi_n + 2^{-1}\{(2n+k+2)^r + k^r - b2^{[n/2]+1}\}$$

holds for any integer b.

Proof. Adding $b2^{[n/2]}(c\xi_n - 1) = 0$ (cf. [1, Theorem 7.3]) to the equality in Theorem 4.1, we have the desired equality. □

Using Theorem 4.2, we obtain

Theorem 4.3. *Assume that there is an integer b satisfying*

$$(2n+k+2)^r - k^r \le b2^{[n/2]+1} \le (2n+k+2)^r + k^r.$$

Then

$$cv^r = 2^{-1}\{b2^{[n/2]+1} - (2n+k+2)^r + k^r\}c\xi_n \oplus 2^{-1}\{(2n+k+2)^r + k^r - b2^{[n/2]+1}\}$$

holds as C*-vector bundles, if* $\langle n/2 \rangle \le k^r$.

Proof. Set $V = 2^{-1}\{b2^{[n/2]+1} - (2n+k+2)^r + k^r\}$ and $W = 2^{-1}\{(2n+k+2)^r + k^r - b2^{[n/2]+1}\}$. Then, by the assumption, $V \ge 0$ and $W \ge 0$, and by Theorem 4.2, $cv^r = Vc\xi_n + W$ in $K(RP^n)$. If $\langle n/2 \rangle (= \langle (\dim RP^n)/2 \rangle) \le k^r (= \dim cv^r = \dim(Vc\xi_n \oplus W))$, we have $cv^r = Vc\xi_n \oplus W$ as C-vector bundles (cf. [2, Theorem 1.5, p. 100]). □

5. Stable extendibility of the complexification cv^r

We recall the following result.

Theorem 5.1 (cf. [8, Theorem 2.1]). *Let* ζ *be a* t*-dimensional* C*-vector bundle over* RP^n. *Assume that there is a positive integer* l *such that* ζ *is stably equivalent to* $(t+l)c\xi_n$ *and* $t+l < 2^{[n/2]}$. *Then* $[n/2] < t+l$ *and* ζ *is not stably extendible to* RP^{2t+2l}.

Theorem 5.2. *Let* v^r *be the* r*-fold power of the normal bundle* v *associated to an immersion of* RP^n *in* R^{n+k}, *where* $r > 0$ *and* $k > 0$, *and let* cv^r *be its complexification. Then* cv^r *is stably extendible to* RP^m *for every* $m \ge n$ *if and only if there is an integer* b *satisfying*

$$(2n+k+2)^r - k^r \le b2^{[n/2]+1} \le (2n+k+2)^r + k^r.$$

Proof. First, we prove the "if" part. Let V and W be defined as in the proof of Theorem 4.3. Then $V \ge 0$ and $W \ge 0$ by the assumption, and $cv^r = Vc\xi_n + W$ by Theorem 4.2. Since $Vc\xi_n \oplus W$ is extendible to RP^m for every $m \ge n$ and since $\dim cv^r = k^r = \dim(Vc\xi_n \oplus W)$, cv^r is stably extendible to RP^m for every $m \ge n$ (cf. [7, Lemma 2.1]).

To prove the "only if" part, we prove the contraposition. Assume that every integer b satisfies

$$b2^{[n/2]+1} < (2n+k+2)^r - k^r \quad \text{or} \quad (2n+k+2)^r + k^r < b2^{[n/2]+1}.$$

Clearly there is an integer b with $b2^{[n/2]+1} < (2n+k+2)^r - k^r$. Define C as the maximum integer such that $C2^{[n/2]+1} < (2n+k+2)^r - k^r$. If C satisfies $C2^{[n/2]+1} \le (2n+k+2)^r + k^r - 2^{[n/2]+1}$, we have $(2n+k+2)^r - k^r \le (C+1)2^{[n/2]+1} \le (2n+k+2)^r + k^r$ and these are inconsistent with our assumption. Hence C satisfies

$$C2^{[n/2]+1} > (2n+k+2)^r + k^r - 2^{[n/2]+1}.$$

Putting $\zeta = cv^r$, $t = k^r$ and $l = 2^{-1}\{C2^{[n/2]+1} - (2n+k+2)^r - k^r + 2^{[n/2]+1}\}$ in Theorem 5.1, we see that cv^r is not stably extendible to RP^{2t+2l}, since $t + l < 2^{[n/2]}$ and $l > 0$.

Clearly there is an integer b with $(2n+k+2)^r + k^r < b2^{[n/2]+1}$. Define D as the minimum integer such that $(2n+k+2)^r + k^r < D2^{[n/2]+1}$. If D satisfies $D2^{[n/2]+1} \geq (2n+k+2)^r - k^r + 2^{[n/2]+1}$, we have $(2n+k+2)^r - k^r \leq (D-1)2^{[n/2]+1} \leq (2n+k+2)^r + k^r$ and these are inconsistent with our assumption. Hence D satisfies

$$D2^{[n/2]+1} < (2n+k+2)^r - k^r + 2^{[n/2]+1}.$$

Putting $\zeta = cv^r$, $t = k^r$ and $l = 2^{-1}\{D2^{[n/2]+1} - (2n+k+2)^r - k^r\}$ in Theorem 5.1, we see that cv^r is not stably extendible to RP^{2t+2l}, since $t + l < 2^{[n/2]}$ and $l > 0$. □

PROOF OF THEOREM B. According to Theorem 2.3 in [9], for $m \geq n$, cv^r is extendible to RP^m if and only if cv^r is stably extendible to RP^m, provided $\langle n/2 \rangle \leq k^r$. Hence the result follows from Theorem 5.2. □

6. Proof of Theorem C

We recall some known results.

THEOREM 6.1 (cf. [11, Theorem 1]). *Let ζ be a 2-dimensional R-vector bundle over RP^n, where $n \geq 3$. Then ζ is equivalent to a sum of two line bundles, that is, ζ is equivalent to one of the bundles: 2, $2\xi_n$ or $\xi_n \oplus 1$.*

THEOREM 6.2 (cf. [4, p. 490 and p. 501]). *Let ζ be a non-orientable n-dimensional R-vector bundle over RP^n, where n is even. Then there is an infinite number of equivalent classes of n-dimensional R-vector bundles over RP^n which are stably equivalent to ζ.*

PROOF OF THEOREM C. Consider the bundle $\xi_2 \oplus 1$ over RP^2. By Theorem 6.2, there are infinitely many 2-dimensional R-vector bundles over RP^2 which are not equivalent to each other but are stably equivalent to $\xi_2 \oplus 1$, since $\xi_2 \oplus 1$ is non-orientable. Let α be a 2-dimensional R-vector bundle over RP^2 which is stably equivalent to $\xi_2 \oplus 1$ but is not equivalent to $\xi_2 \oplus 1$. Then α is stably extendible to RP^3, since $\xi_2 \oplus 1$ is extendible to RP^3. But α is not extendible to RP^3. In fact, if there is a 2-dimensional R-vector bundle β over RP^3 such that the restriction of β to RP^2 is equivalent to α, then β is equivalent to one of the bundles: 2, $2\xi_3$ or $\xi_3 \oplus 1$, by Theorem 6.1. So α is equivalent to one of the bundles: 2, $2\xi_2$ or $\xi_2 \oplus 1$. This is impossible. □

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Yutaka Hemmi
Department of Mathematics
Faculty of Science
Kochi University
Kochi 780-8520, *Japan*
E-mail: hemmi@math.kochi-u.ac.jp

Teiichi Kobayashi
Asakura-ki 292-21
Kochi 780-8066, *Japan*
E-mail: kteiichi@lime.ocn.ne.jp

Min Lwin Oo
Department of Mathematics
Graduate School of Science
Kochi University
Kochi 780-8520, *Japan*
E-mail: minl@cc.kochi-u.ac.jp