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Oscillation criteria for first order nonlinear delay differential equations

Jianchu JIANG

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ABSTRACT. In this paper, some new oscillation criteria are obtained for the first order nonlinear delay differential equation

$$x'(t) + p(t)f(x(t-\tau_1), \dots, x(t-\tau_m)) = 0$$

and the corresponding advanced differential equation. Our results improve the known results in the literature. And an example is given to demonstrate the advantage of our results.

1. Introduction

The oscillatory behavior of differential equations with deviating arguments has been studied by many authors. For some contributions in this area see the papers [1–11].

Consider the first order delay differential equation

$$x'(t) + p(t)f(x(t-\tau_1), \dots, x(t-\tau_m)) = 0, \qquad t \geq t_0 \tag{1}$$

and the advanced differential equation

$$x'(t) - p(t)f(x(t+\tau_1), \dots, x(t+\tau_m)) = 0, \qquad t \geq t_0, \tag{2}$$

where $p(t) \geq 0$ is a continuous function, $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m$, and the function f satisfies the following conditions:

(H_1). f is continuous on $\mathbf{R}^m$ and such that

$$y_i > 0 \qquad \text{for } i = 1, \dots, m \Rightarrow f(y_1, \dots, y_m) > 0$$

and

$$y_i < 0 \qquad \text{for } i = 1, \dots, m \Rightarrow f(y_1, \dots, y_m) < 0;$$

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(H_2). there exist $\varepsilon > 0$, $M \geq 0$ and $r > 0$ such that

$$\left| f(u_1, \ldots, u_m) - \prod_{j=1}^{m} u_j^{\alpha_j} \right| \leq M \Bigl(\max_{1 \leq j \leq m} \{|u_j|\} \Bigr)^r \prod_{j=1}^{m} |u_j|^{\alpha_j} \qquad \text{for } u_i \in (-\varepsilon, \varepsilon),$$

where $\alpha_j > 0$, $j = 1, \ldots, m$ are rational numbers with denominator of positive odd integers, and $\sum_{j=1}^{m} \alpha_j = 1$.

The sepcial forms of equation (1), (2) are the equations

$$x'(t) + p(t)x(t - \tau) = 0, \qquad t \geq t_0, \tag{3}$$

$$x'(t) - p(t)x(t + \tau) = 0, \qquad t \geq t_0, \tag{4}$$

$$x'(t) + p(t) \prod_{j=1}^{m} [x(t - \tau_j)]^{\alpha_j} = 0, \qquad t \geq t_0, \tag{5}$$

and

$$x'(t) - p(t) \prod_{j=1}^{m} [x(t + \tau_j)]^{\alpha_j} = 0, \qquad t \geq t_0. \tag{6}$$

Equation (3) is a basic delay differential equation, which plays a crucial role in many investigations and therefore is always in the center of interest. So far, there have been many oscillatory results for equation (3), we refer to the monographies [5–7] and the reference cited therein. One basic oscillation criteria is [2]

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}. \tag{7}$$

In condition (7), the constant $1/e$ is the best possible [5].

Yu [8] extended the above results (7) for (3) to the nonlinear differential equation (5) and (6) and proved the following theorems.

THEOREM A. *Assume that*

$$\liminf_{t \to \infty} \sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} p(s) > \frac{1}{e}. \tag{8}$$

Then every solution of (5) *oscillates.*

THEOREM B. *Assume that*

$$\liminf_{t \to \infty} \sum_{j=1}^{m} \alpha_j \int_{t}^{t+\tau_j} p(s)ds > \frac{1}{e}. \tag{9}$$

Then every solution of (6) *oscillates.*

Very recently, Tang and Yu [9] obtained some new oscillation criteria for (5) and (6), which improve Theorems A and B.

We remark that Theorems A and B are easily extended to the more general nonlinear equations (1) and (2) respectively. In this paper, we will establish some new oscillation criteria for (1) and (2) which contain and improve condition (8) and (9) and other results. More precisely, we obtain the following theorems.

Theorem 1. *Assume that* (H_1), (H_2) *hold, and that*

$$\liminf_{t\to\infty}\sum_{j=1}^{m}\alpha_j\int_t^{t+\tau_j}p(s)ds>0. \tag{10}$$

Suppose also that there exists $T_0\geq t_0>0$ *such that*

$$\int_{T_0}^{\infty}p(t)\ln\left(e\sum_{j=1}^{m}\alpha_j\int_t^{t+\tau_j}p(s)ds\right)dt=\infty. \tag{11}$$

Then every solution of (1) *oscillates.*

Theorem 2. *Assume that* (H_1), (H_2) *hold, and that*

$$\liminf_{t\to\infty}\sum_{j=1}^{m}\alpha_j\int_{t-\tau_j}^{t}p(s)ds>0. \tag{12}$$

Suppose that there exists $T_0\geq t_0>0$ *such that*

$$\int_{T_0}^{\infty}p(t)\ln\left(e\sum_{j=1}^{m}\alpha_j\int_{t-\tau_j}^{t}p(s)ds\right)dt=\infty. \tag{13}$$

Then every solution of (2) *oscillates.*

2. Some lemmas

Lemma 1. *Assume that* (H_1), (H_2) *hold, and*

$$\int_{t_0}^{\infty}p(t)dt=\infty. \tag{14}$$

Then every nonoscillatory solution of (1) *converges to zero monotonically as* $t\to\infty$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. Then there exists a $t_1\geq t_0$ such that $x(t-\tau_m)>0$, $x'(t)\leq 0$ for

$t \geq t_1$. Hence the limit $\lim_{t\to\infty} x(t) = \beta$ exists and $\beta \geq 0$. If $\beta > 0$, then there exists $T > t_1$ such that $f(x(t-\tau_1), \dots, x(t-\tau_m)) > \frac{1}{2} f(\beta, \dots, \beta) > 0$ for $t \geq T$. Integrating (1) from T to $t \geq T$, we have

$$x(t) - x(T) \leq -\frac{1}{2} f(\beta, \dots, \beta) \int_T^t p(s)ds,$$

which, together with (14), implies that

$$\lim_{t\to\infty} x(t) = -\infty.$$

This contradicts the fact that $x(t)$ is eventually positive. The proof of Lemma 1 is complete.

LEMMA 2. *Assume that* (H_1), (H_2) *and* (14) *hold. If* (1) *has a nonoscillatory solution, then*

$$\int_t^{t+\tau_m} p(s)ds \leq \frac{2}{\alpha_m} \tag{15}$$

eventually.

PROOF. Suppose that $x(t)$ is a nonoscillatory solution of (1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 1 and (1), there exists $t_1 > 0$ such that

$$\varepsilon > x(t-\tau_m) \geq x(t-\tau_{m-1}) \geq \cdots \geq x(t-\tau_1) \geq x(t) > 0, \qquad \text{for } t \geq t_1,$$

and $\lim_{t\to\infty} x(t) = 0$, where ε is given by condition (H_2). From this and (H_2), there exists $t_2 \geq t_1$ such that

$$f(x(t-\tau_1), \dots, x(t-\tau_m)) \geq \frac{1}{2} \prod_{j=1}^m [x(t-\tau_j)]^{\alpha_j}, \qquad t \geq t_2. \tag{16}$$

It follows from (1) that

$$x'(t) + \frac{1}{2} p(t) \prod_{j=1}^m [x(t-\tau_j)]^{\alpha_j} \leq 0, \qquad t \geq t_2. \tag{17}$$

Consequently, we have

$$x'(t) + \frac{1}{2} p(t) [x(t)]^{1-\alpha_m} [x(t-\tau_m)]^{\alpha_m} \leq 0, \qquad t \geq t_2. \tag{18}$$

Set $y(t) = [x(t)]^{\alpha_m}$ for $t \geq t_2 + \tau_m$. Then

$$y'(t) + \frac{1}{2} \alpha_m p(t) y(t-\tau_m) \leq 0, \qquad t \geq t_2. \tag{19}$$

From (19), it is easy to show that (15) holds eventually [5, 11]. The proof of Lemma 2 is complete.

Lemma 3. *Assume that* (H_1), (H_2) *and* (10) *hold. If* $x(t)$ *is a nonoscillatory solution of* (1), *then* $x(t-\tau_m)/x(t)$, *which is well defined for large* t, *is bounded.*

Proof. We shall assume $x(t)$ to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. Set $y(t)$ as in the proof of Lemma 2. Then (19) holds. From (19), it is easy to show (see [5, 11]) that Lemma 3 is true. The proof of Lemma 3 is complete.

Lemma 4. *Assume that* (H_1), (H_2) *and* (14) *hold. If* $x(t)$ *is a nonoscillatory solution of* (1), *then there exist* $A > 0$ *and* $T > 0$ *such that*

$$|x(t)| \le A \exp\left(-\frac{1}{2}\int_T^t p(s)ds\right), \qquad t \ge T. \tag{20}$$

Proof. We shall assume $x(t)$ to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 1 there exists $t_1 > 0$ such that

$$0 < x(t) \le x(t-\tau_1) \le \cdots \le x(t-\tau_m) < \varepsilon \qquad \text{for } t \ge t_1,$$

where ε is given by condition (H_2). Similar to the proof of Lemma 2, it is easy to show that there exists $t_2 > t_1$ such that (17) holds. From (17), we have

$$x'(t) + \frac{1}{2}p(t)x(t) \le 0, \qquad t \ge t_2. \tag{21}$$

This yields (20), where $A = x(T)$ and $T = t_2$. The proof of Lemma 4 is complete.

3. Proof of theorems

Proof of Theorem 1. Assume that (1) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 1, there exists $t_1 \ge T_0$ such that

$$0 < x(t) \le x(t-\tau_1) \le \cdots \le x(t-\tau_m) < \varepsilon, \qquad t \ge t_1. \tag{22}$$

where ε is given by condition (H_2). From (22) and (H_2) we have

$$f(x(t-\tau_1),\dots,x(t-\tau_m)) \ge \prod_{j=1}^m [x(t-\tau_j)]^{\alpha_j} - M[x(t-\tau_m)]^{1+r}, \qquad t \ge t_1. \tag{23}$$

Substituing (23) into (1), we obtain

$$\frac{x'(t)}{x(t)} + p(t)\prod_{j=1}^m \left(\frac{x(t-\tau_j)}{x(t)}\right)^{\alpha_j} - Mp(t)\frac{[x(t-\tau_m)]^{1+r}}{x(t)} \le 0, \qquad t \ge t_1. \tag{24}$$

Set $\lambda(t) = -x'(t)/x(t)$ for $t \geq t_1$. Then $\lambda(t) \geq 0$ for $t \geq t_1$, and from (24), we have

$$\lambda(t) \geq p(t) \exp\left(\sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} \lambda(s)ds\right) - Mp(t)\frac{x(t-\tau_m)}{x(t)}[x(t-\tau_m)]^r, \qquad t \geq t_1 + \tau_m. \tag{25}$$

By Lemmas 2–4, there exists $T > t_1 + \tau_m$, $A > 0$ and $M_1 > 0$ such that

$$x(t-\tau_m) \leq A \exp\left(-\frac{1}{2}\int_T^{t-\tau_m} p(s)ds\right), \qquad t \geq T, \tag{26}$$

$$\int_{t-\tau_m}^{t} p(s)ds \leq \frac{2}{\alpha_m}, \qquad t \geq T, \tag{27}$$

$$\frac{x(t-\tau_m)}{x(t)} \leq M_1, \qquad t \geq T. \tag{28}$$

From these and (25), we have

$$\begin{aligned}\lambda(t) &\geq p(t) \exp\left(\sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} \lambda(s)ds\right) \\ &\quad - MM_1 p(t)\left(A \exp\left(-\frac{1}{2}\int_T^{t-\tau_m} p(s)ds\right)\right)^r \\ &\geq p(t) \exp\left(\sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} \lambda(s)ds\right) \\ &\quad - MM_1 A_1 p(t) \exp\left(-\frac{r}{2}\int_T^{t} p(s)ds\right), \qquad t \geq T,\end{aligned}$$

where $A_1 = e^{r/\alpha_m}A^r$. It follows that

$$\begin{aligned}&\lambda(t)\sum_{j=1}^{m} \alpha_j \int_{t}^{t+\tau_j} p(s)ds \\ &\quad \geq p(t)\left(\sum_{j=1}^{m} \alpha_j \int_{t}^{t+\tau_j} p(s)ds\right) \exp\left(\sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} p(s)ds\right) \\ &\quad\quad - MM_1 A_1 p(t)\left(\sum_{j=1}^{m} \alpha_j \int_{t}^{t+\tau_j} p(s)ds\right) \exp\left(-\frac{r}{2}\int_T^{t} p(s)ds\right), \quad t \geq T.\end{aligned} \tag{29}$$

One can easily show that $\gamma e^x \geq x + \ln e\gamma$ for $\gamma > 0$. Hence,

$$\lambda(t) \sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds$$

$$\geq p(t) \sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} \lambda(s)ds + p(t) \ln\left(e \sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \right)$$

$$- MM_1 A_1 p(t) \left(\sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \right) \exp\left(-\frac{r}{2} \int_T^t p(s)ds \right), \quad t \geq T. \tag{30}$$

Set

$$D(t) = MM_1 A_1 p(t) \left(\sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \right) \exp\left(-\frac{r}{2} \int_T^t p(s)ds \right).$$

Then (30) can be written as

$$\lambda(t) \sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \geq p(t) \sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} \lambda(s)ds$$

$$+ p(t) \ln\left(e \sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \right) - D(t), \qquad t \geq T. \tag{31}$$

Integrating (31) from T to $N > T + 2\tau_m$, we get

$$\sum_{j=1}^{m} \alpha_j \int_T^N \lambda(t) \int_t^{t+\tau_j} p(s)dsdt$$

$$\geq \sum_{j=1}^{m} \alpha_j \int_T^N p(t) \int_{t-\tau_j}^{t} \lambda(s)dsdt$$

$$+ \int_T^N p(t) \ln\left(e \sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \right) dt - \int_T^N D(t)dt. \tag{32}$$

By (27), we have

$$\sum_{j=1}^{m} \alpha_j \int_t^{t+\tau_j} p(s)ds \leq \frac{2}{\alpha_m}, \qquad t \geq T. \tag{33}$$

Therefore,

$$\begin{aligned}\int_T^\infty D(t)dt &\le \frac{2MM_1A_1}{\alpha_m}\int_T^\infty p(t)\exp\left(-\frac{r}{2}\int_T^t p(s)ds\right)dt\\ &= \frac{4MM_1A_1}{\alpha_m}\int_0^\infty e^{-ru}\,du\\ &= \frac{4MM_1A_1}{r\alpha_m} < \infty, \qquad (34)\end{aligned}$$

Interchanging the order of integration, we find

$$\int_T^N p(t)\int_{t-\tau_j}^t \lambda(s)dsdt \ge \int_T^{N-\tau_j}\lambda(s)\int_s^{s+\tau_j} p(t)dtds.$$

Substituting this into (32) we have

$$\begin{aligned}\sum_{j=1}^m \alpha_j\int_{N-\tau_j}^N \lambda(t)\int_t^{t+\tau_j} p(s)dsdt \ge \int_T^N p(t)\ln\left(e\sum\alpha_j\int_t^{t+\tau_j} p(s)ds\right)dt\\ -\int_T^N D(t)dt. \qquad (35)\end{aligned}$$

From (11), (34) and (35), we have

$$\lim_{N\to\infty}\sum_{j=1}^m \alpha_j\int_{N-\tau_j}^N \lambda(t)\int_t^{t+\tau_j} p(s)dsdt = \infty. \qquad (36)$$

On the other hand, by (33) and Lemma 3, we have

$$\begin{aligned}&\lim_{N\to\infty}\sum_{j=1}^m \alpha_j\int_{N-\tau_j}^N \lambda(t)\int_t^{t+\tau_j} p(s)dsdt\\ &\le \limsup_{N\to\infty}\sum_{j=1}^m \frac{2\alpha_j}{\alpha_m}\int_{N-\tau_j}^N \lambda(t)dt\\ &= \frac{2}{\alpha_m}\limsup_{N\to\infty}\left[\frac{x(N-\tau_j)}{x(N)}\right]^{\alpha_j}\\ &\le \frac{2}{\alpha_m}\limsup_{N\to\infty}\left[\frac{x(N-\tau_m)}{x(N)}\right]^{\alpha_j} < \infty,\end{aligned}$$

which contradicts (36). The proof of Theorem 1 is complete.

Theorem 2 could be proved by the method similarly to those of Theorem 1, and so we omit it here.

4. An example

EXAMPLE. Consider the following delay differential equation

$$x'(t) + \frac{3}{4\pi e}(1+\cos t)[\exp(x(t-\pi))^{2/3}(x(t-3\pi))^{1/3} - 1] = 0, \qquad t \geq 0. \quad (37)$$

where

$$p(t) = \frac{3}{4\pi e}(1+\cos t), \qquad f(u_1,u_2) = \exp(u_1^{2/3}u_2^{1/3}) - 1.$$

Obviously, $f(u_1,u_2)$ satisfies condition (H_1) and (H_2). We observe that $\tau_1 = \pi$, $\tau_2 = 3\pi$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, and

$$\alpha_1 \int_t^{t+\tau_1} p(s)ds + \alpha_2 \int_t^{t+\tau_2} p(s)ds = \frac{1}{4\pi e}(5\pi - 6\sin t),$$

$$\alpha_1 \int_{t-\tau_1}^{t} p(s)ds + \alpha_2 \int_{t-\tau_2}^{t} p(s)ds = \frac{1}{4\pi e}(5\pi + 6\sin t).$$

It follows that

$$\liminf_{t\to\infty} \sum_{j=1}^{2} \alpha_j \int_t^{t+\tau_j} p(s)ds = \liminf_{t\to\infty} \sum_{j=1}^{2} \alpha_j \int_{t-\tau_j}^{t} p(s)ds = \frac{5\pi - 6}{4\pi e} < \frac{1}{e}.$$

This shows neither (8) nor (9) are true. But

$$\begin{aligned}
&\int_0^{2\pi} p(t)\ln\left[e\sum_{j=1}^{2}\alpha_j\int_t^{t+\tau_j} p(s)ds\right]dt \\
&= \frac{3}{4\pi e}\int_0^{2\pi}(1+\cos t)\ln\left[\frac{1}{4\pi}(5\pi - 6\sin t)\right]dt \\
&= \frac{3}{2e}\ln\frac{5}{4} - \frac{3}{4\pi e}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{6}{5\pi}\right)^{2n}\int_0^{2\pi}\sin^{2n} t\,dt \\
&> \frac{3}{2e}\ln\frac{5}{4} - \frac{3}{8e}\sum_{n=1}^{\infty}\left(\frac{6}{5\pi}\right)^{2n} \approx 0.27/e > 0.
\end{aligned}$$

Hence

$$\int_0^{\infty} p(t)\ln\left(e\sum_{j=1}^{2}\alpha_j\int_t^{t+\tau_j} p(s)ds\right)dt = \infty.$$

In view of Theorem 1, every solution of equation (37) is oscilatory.

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Department of Mathematics
Loudi Teachers' College
Loudi, Hunan 417000
P. R. China
e-mail: ldjjc@mail.ld.hn.cn