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Linearly normal curves with degenerate general hyperplane section

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Abstract. We study linearly normal projective curves with degenerate general hyperplane section, in terms of the "amount of degeneracy" of it, giving a characterization and/or a description of such curves.

1. Introduction

We work over an algebraically closed field K of arbitrary characteristic.

By "curve" we always mean a locally Cohen-Macaulay purely one-dimensional projective scheme.

Recall that a non-degenerate curve $Y \subseteq \mathbf{P}^n$ is called *linearly normal* if the natural map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(Y, \mathcal{O}_Y(1))$ is bijective, or equivalently if $H^1(\mathbf{P}^n, \mathcal{I}_Y(1)) = 0$.

Linearly normal curves occur in a very natural setting. Indeed if we start with an abstract projective scheme Y and a line bundle $\mathcal{L} \in \mathbf{Pic}(Y)$, with $\mathcal{L}$ very ample, it is very natural to consider the complete embedding of Y in the projective space $\mathbf{P}(H^0(Y, \mathcal{L}))$ induced by $H^0(Y, \mathcal{L})$.

The aim of the paper is to give a description and, whenever possible, to characterize linearly normal curves in terms of the "amount of degeneracy" of the general hyperplane section.

We fix some notation. Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve and let $C := Y_{red}$. Set $d := \deg(Y)$, $\delta := \deg(C)$ and $s = s(Y) := \dim(\langle Y \cap H \rangle)$, where H is a general hyperplane.

Our first result is the following characterization of linearly normal curves with maximally degenerate general hyperplane section (i.e. $s = 1$).

Theorem A. *Let $Y \subseteq \mathbf{P}^n$ $(n \geq 3)$ be a non-degenerate curve. Then the following are equivalent:*
(i) *$s = 1$ and Y is linearly normal;*
(ii) *$\deg(Y) = 2$ and $p_a(Y) = -n + 2$.*

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A complete classification and a description of such curves is given in Example 5.1.

Our next theorem deals with curves in $\mathbf{P}^5$.

Recall that if $Y \subseteq \mathbf{P}^5$ is a non-degenerate curve, with degenerate general hyperplane section having C irreducible, not a line, then $\langle C \rangle$ is a plane and $s = 3$ (see [1], Th. 2.1 and [2], Th. 2.5).

Theorem B. *Let $Y \subseteq \mathbf{P}^5$ be a linearly normal curve, with degenerate general hyperplane section and assume C irreducible of degree $\delta > 1$.*

Let Y'' be the maximal locally Cohen-Macaulay subcurve of Y contained in the plane $\pi := \langle C \rangle$ and set $d'' := \deg(Y'')$. Then:

$$d'' + \delta \leq d \leq 2d''$$

A description of many such curves is given in Example 5.2 and Lemma 5.11; in particular we show that for any given C and Y'' all the degrees d's compatible with Theorem B are really attained (see Remark 5.12).

The content of the paper is organized as follows: after Section 2, where we introduce some preliminaries, Section 3 is devoted to prove Theorem A and Theorem B.

Next, in Section 4, we discuss the case $s = 2$ and we give a complete classification, in characteristic zero, of the linearly normal multiple lines Y with $s(Y) = 2$ and $\deg(Y) \geq 5$.

Finally in Section 5 we show examples and make further remarks.

2. Preliminaries

Let $Y \subseteq \mathbf{P}^n$ be a linearly normal curve, and let $\mathbf{B} \subseteq \mathbf{P}^n$ be the scheme-theoretic base locus of all quadric hypersurfaces containing Y.

Remark 2.1. If $H \subseteq \mathbf{P}^n$ is a general hyperplane, then H contains no irreducible components of Y_{red}. Thus its equation is not a zero-divisor of $\mathcal{O}_Y$ and we have the exact sequence:

$$0 \to \mathcal{I}_Y(1) \to \mathcal{I}_Y(2) \to \mathcal{I}_{Y \cap H, H}(2) \to 0.$$

Since $h^1(\mathcal{I}_Y(1)) = 0$, every quadric hypersurface of H containing $Y \cap H$ is the intersection with H of a quadric hypersurface of $\mathbf{P}^n$ containing Y.

Thus $\mathbf{B} \cap H$ is the scheme-theoretic base locus of all quadric hypersurfaces of H containing $Y \cap H$. In particular $\mathbf{B} \cap H \subseteq \langle Y \cap H \rangle$.

Remark 2.2. If $s \leq n - 2$ we have:

(i) $\mathbf{B} \cap H \neq \langle Y \cap H \rangle$, whence $\dim(\langle \mathbf{B} \cap H \rangle) < s$;

(ii) $\mathbf{B} \cap H$ is cut out by the quadrics of $\langle Y \cap H \rangle$ containing $Y \cap H$.

Indeed, we already observed that $\mathbf{B} \cap H \subseteq \langle Y \cap H \rangle$ (see Remark 2.1). If equality holds, then by Remark 2.1 we have that $\mathbf{B}$ is the scheme-theoretical union of a linear space L of dimension $s+1 \leq n-1$ and, possibly, a zero-dimensional scheme.

Since Y has no zero-dimensional components, it follows that $Y \subseteq L$, a contradiction. This proves (i), and (ii) immediately follows.

Remark 2.3. Assume that Y is linearly normal with degenerate general hyperplane section. Then $\mathbf{B} \cap H$ cannot be arithmetically Cohen-Macaulay of dimension $r > 0$. Indeed, if this is the case, then $\mathbf{B}$ is the scheme-theoretic union of a zero-dimensional scheme and of a scheme $\mathbf{B}'$ of pure dimension $r+1$ such that $\mathbf{B}' \cap H = \mathbf{B} \cap H$. Then $\mathbf{B}'$ is arithmetically Cohen-Macaulay and, in particular, the restriction map $H^0(\mathscr{I}_{\mathbf{B}'}(1)) \to H^0(\mathscr{I}_{\mathbf{B}' \cap H, H}(1))$ is surjective, whence $\mathbf{B}'$ is degenerate since $\mathbf{B}' \cap H$ is. Since $Y \subseteq \mathbf{B}'$, this is a contradiction.

Remark 2.4. Let $Y \subseteq \mathbf{P}^n$ be a linearly normal curve and $H \subseteq \mathbf{P}^n$ be a general hyperplane. Then from the exact sequences $0 \to \mathscr{I}_Y \to \mathscr{I}_Y(1) \to \mathscr{I}_{Y \cap H, H}(1) \to 0$ and $0 \to \mathscr{I}_Y \to \mathscr{O}_{\mathbf{P}^n} \to \mathscr{O}_Y \to 0$ it is easy to see that $s(Y) = n - 1 - h^1(\mathscr{I}_Y) = n - h^0(\mathscr{O}_Y)$.

Remark 2.5. If Y is a projective abstract curve and we fix a very ample line bundle $\mathscr{L}$, we obtain the embedding of Y in the projective space $\mathbf{P}(H^0(Y, \mathscr{L}))$, say of dimension n; the same Y, by choosing a different very ample line bundle $\mathscr{L}'$, can be embedded by the linear system $H^0(Y, \mathscr{L}')$ in the projective space $\mathbf{P}(H^0(Y, \mathscr{L}'))$, say of dimension n'. Hence by Remark 2.4 we have that the classification of linearly normal curves isomorphic to Y with large s is equivalent to the classification very ample line bundles, which seems very difficult.

3. Proofs of Theorems A and B

Proof of Theorem A

Proof. Assume (i) holds. If $\deg(Y) \geq 3$ the scheme $\mathbf{B} \cap H$ is a line for a general hyperplane H. But this is impossible by Remark 2.3. Thus $\deg(Y) = 2$.

Hence $h^1(\mathscr{I}_{Y \cap H, H}(j)) = 0$ for all $j \geq 1$. Then from the exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{I}_Y(1) \to \mathscr{I}_{Y \cap H, H}(1) \to 0,$$

and Serre' vanishing it follows that $h^1(\mathscr{O}_Y) = h^2(\mathscr{I}_Y) = 0$.

Since $h^0(\mathscr{O}_Y) = n - 1$, by Remark 2.4 it follows $p_a(Y) = -n + 2$.

This shows that (i) $\Rightarrow$ (ii). Reversing the argument it is easy to prove the opposite implication. □

Remark 3.1. (i) Notice that, when $n = 3$, with the assumption "linearly normal" we exclude from the classification given in [8] all positive characteristic multiple lines which are not double lines and almost all double lines.

(ii) Notice also that, if $n = 3$, Y is either the union of two skew lines or a double line (as a Cartier divisor) on a smooth quadric surface. If $n \geq 4$, Y is a unique double line (up to projective equivalence) described in Example 5.1.

Proof of Theorem B

Proof. First of all we show that $d'' + \delta \leq d$. Since C is irreducible and $Y \neq C$, there are integers $e \geq 2$, $e'' \geq 1$ such that $d = \delta e$ and $d'' = \delta e''$. Since $Y \neq Y''$ we have also $e > e''$, whence our claim.

To prove the other inequality we proceed in several steps.

Step 1. Let's recall first the definition of generic spanning increasing given in [2].

If $H \subseteq \mathbf{P}^n$ is a general hyperplane, let $P \in H \cap Y_{red}$ and let $Z(P)$ be the largest subscheme of $Y \cap H$ supported by P. It turns out that the integer $\dim\langle Y_{red} \cup Z(P)\rangle - \dim\langle Y_{red}\rangle$ does not depend on the choice of P and H; it is called generic spanning increasing of Y, and denoted by z_Y or simply by z whenever no explicit reference to Y is needed.

By [2], Theorem 2.5, we have $z + 2\dim(\langle Y_{red}\rangle) \leq s + 2$, whence $s = 3$ by our assumption; we also have that $z > 0$ (loc. cit.), and this implies $z = 1$.

Step 2. If $d'' = \delta$ (i.e. $Y'' = C$), then $d = 2\delta = 2d''$ (i.e. Y is a double structure on C).

Proof of Step 2. By contradiction, assume that $d = e\delta$, with $e \geq 3$. Set $\ell := H \cap \pi$ and let P be a point of $H \cap C = \ell \cap C$. By definition of z we have:

$$\dim\langle Z(P)\rangle - \dim\langle Z(P) \cap \pi\rangle = z = 1$$

and since $Z(P) \cap \pi = P$ (with reduced structure), we have $\dim\langle Z(P)\rangle = 1$.

Let $\{P_1, \dots, P_\delta\} = C \cap H$, and for each $i = 1, \dots, \delta$ denote by ℓ_i the line $\langle Z(P_i)\rangle$.

Let $Q_1, \dots, Q_r$ be a basis of the linear system of quadrics in $\langle Y \cap H\rangle$ containing $Y \cap H$ (note that $r \geq 1$ by Remark 2.2).

By our assumption it follows that $\deg(Z(P_i) \cap \ell_i) \geq 3$, whence $Q_i \supseteq \ell_1 \cup \cdots \cup \ell_\delta$ for all $i = 1, \dots, r$.

Observe that $\langle H \cap Y\rangle \supseteq \ell_1 \cup \cdots \cup \ell_\delta \supseteq H \cap Y$, whence $\langle \ell_1 \cup \cdots \cup \ell_\delta\rangle = \langle H \cap Y\rangle$, which means that at least two of the lines are skew.

If $\delta = 2$ this readily implies that $\mathbf{B} \cap H = \ell_1 \cup \ell_2$, whence $\mathbf{B}$ is the union of two distinct planes and a zero-dimensional scheme.

Since C is irreducible, this implies that Y is contained in one of the two planes, absurd.

If $\delta \geq 3$, then $Q_i \supseteq \ell \cup \ell_1 \cup \cdots \cup \ell_\delta$ for all $i = 1, \dots, r$.

If $r = 1$, then $\mathbf{B} \cap H = Q_1$, a contradiction by Remark 2.4.

If $r > 1$ we show that $\mathbf{B} \cap H$ is a plane (whence a contradiction by the above argument). This is clear by Bézout if $\delta \geq 4$. If $\delta = 3$, let $D := \ell \cup \ell_1 \cup \ell_2 \cup \ell_3$. Then either $p_a(D) = 0$ (and D cannot be a complete intersection of two quadrics), or three of the four lines are coplanar. In either case we have the conclusion.

Step 3. If $d'' > \delta$, then $d \leq 2d''$.

Proof of Step 3. Since $d'' > \delta$, we have $\dim\langle Z(P_i) \cap \pi \rangle = 1$ and $\deg(Z(P_i) \cap \ell) \geq 2$ $(i = 1, \dots, \delta)$ whence, in particular, $\deg(Z \cap H \cap \ell) \geq 4$ and $\ell \subseteq Q_i$ for all $i = 1, \dots, r$.

Moreover $\pi_i := \langle Z(P_i) \rangle$ is a plane (use the same argument as in Step 1) which contains ℓ, because $Y'' \neq C$.

If $\mathbf{B} \cap H = \pi_i$ for some i, then we get a contradiction as above.

If not, then $Q_i \cap \pi_j$ is a conic supported by ℓ, hence it is a double line in π_j.

This means that the zero-dimensional scheme $Z(P_i)$ is contained in a double line, and an easy calculation shows that

$$\deg(Z(P_i)) \leq 2 \deg(Z(P_i) \cap \ell).$$

But $\deg(Z(P_i)) = e$ and $\deg(Z(P_i) \cap \ell) = d''$, whence our conclusion. $\square$

4. The case $s = 2$

We recall the following result (see [5], Step 3 in the proof of Prop. 1.4):

Proposition 4.1. *Assume* $\mathrm{char}(K) = 0$. *Let* $Y \in \mathbf{P}^n$ *be a non-degenerate curve with* $s \leq n - 2$. *Then a general hyperplane section* $H \cap Y$ *is contained in at least* s *independent quadrics of the linear space* $\langle H \cap Y \rangle$.

Proposition 4.2. *Assume* $\mathrm{char}(K) = 0$. *Let* $Y \in \mathbf{P}^n$ *be a non-degenerate curve of degree* $d \geq 5$, *with* $s = 2$ *and let* H *be a general hyperplane. Then:*
(i) $H \cap Y$ *contains a collinear subscheme of degree* $d - 1$;
(ii) Y *contains a planar subcurve of degree* $d - 1$.

Proof. By Proposition 4.1 $H \cap Y$ is contained in two independent conics of the plane $\langle H \cap Y \rangle$. Since $d \geq 5$ this easily implies (i). We can get (ii) from (i) and [6], Cor. 4.4. $\square$

Remark 4.3. If Y_{red} is irreducible and $s(Y) = 2$, then Y_{red} is a line. Indeed from [2], Th. 2.5 we have $2 \dim\langle Y_{red} \rangle \leq s(Y) + 1$. If the characteristic is zero these multiple lines of degree ≥ 5 are characterized as the ones containing a planar multiple line of degree $d - 1$.

Now we want to show how the above multiple lines can be constructed and we want to characterize the linearly normal ones among them. This can be done in arbitrary characteristic.

Remark 4.4. (i) Let $Y \subseteq \mathbf{P}^n$ be a multiple line of degree d, containing a planar subcurve Z of degree $d-1$ and let $C := Y_{red}$. Let $\mathscr{L}$ be the kernel of the natural map $\mathcal{O}_Y \to \mathcal{O}_Z$.

Then it is easy to see that $\mathscr{N}\mathscr{L} = 0$, where $\mathscr{N}$ is the nilradical of $\mathcal{O}_Y$; therefore $\mathscr{L}$ is in a natural way an $\mathcal{O}_C$-module. Moreover one can easily show that $\mathscr{L}$ is torsion free, hence invertible.

Then we have exact sequences

$$0 \to \mathscr{I}_Y \to \mathscr{I}_Z \to \mathcal{O}_C(a) \to 0$$

and

$$0 \to \mathscr{L} \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

for some integer a.

(ii) From the above cohomology sequences we get that the following conditions are equivalent:

(a) Y is linearly normal;

(b) the natural map $H^0(\mathscr{I}_Z(1)) \to H^0(\mathcal{O}_C(a))$ is an isomorphism;

(c) $a = n-4$ and Y is non-degenerate;

(d) Y is non-degenerate and $p_a(Y) = \frac{1}{2}(d-2)(d-3) + 2 - n$.

(iii) One can construct all the curves described above by using a well-known contruction (for details see e.g. [5], proof of Prop. 1.11; here the case $n = 4$ is treated, but the extension to arbitrary n is straightforward).

Let $C \subseteq \mathbf{P}^n$ be a line and let Z be a planar curve of degree $d - 1 \geq 2$ with $Z_{red} = C$. Then for every line bundle $\mathscr{L} := \mathcal{O}_C(a)$ with $a \geq n-4$, one can construct many non-degenerate curves Y of degree d whose ideal sheaf fits into the exact sequences seen in (i). If $a = n-4$, then Y is linearly normal by (ii).

5. Examples and further remarks

5.1. Ribbons. Here we consider linearly normal structures corresponding to double lines and double conics, using the notion of ribbon introduced in [3].

Let X be a locally Cohen-Macaulay projective scheme such that $T := X_{red} \cong \mathbf{P}^1$ and the nilpotent sheaf $\mathscr{I}$ of $\mathcal{O}_X$ has $\mathscr{I}^2 = 0$. Thus $\mathscr{I}$ may be seen as a coherent $\mathcal{O}_T$-module.

Here we assume $\mathscr{I} \in \mathbf{Pic}(T)$ and call X a rational ribbon or, just, a ribbon.

Set $x := \deg(\mathscr{I})$ as a line bundle on T.

We have an exact sequence:

$$0 \to \mathscr{I} \to \mathcal{O}_X \to \mathcal{O}_T \to 0.$$

Thus $\chi(\mathcal{O}_X) = 2 + x$ and $p_a(X) = -x - 1$.

The ribbon is said to be split if the quotient map $\mathcal{O}_X \to \mathcal{O}_T$ has a retraction.

For any fixed integer x there is a unique split ribbon with $\deg(\mathcal{I}) = x$ (see [3], pp. 724–725). If $x \geq -3$ every ribbon with invariant x is split ([3], Cor. 1.4), while if $x \leq -4$ the set of all isomorphic classes of non-split ribbons with invariant x is parametrized by a projective space of dimension $-x-4$.

Here we will need only split ribbons.

We have the exact sequence of sheaves of groups

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_T^* \to 0$$

and, since $H^0(T, \mathcal{O}_T^*) \cong \mathbf{K}^*$, the natural map $H^0(X, \mathcal{O}_X^*) \to H^0(T, \mathcal{O}_T^*)$ is surjective.

Hence we obtain $\ker(\rho) \cong H^1(T, \mathcal{I})$, where ρ is the natural surjective map $\mathbf{Pic}(X) \to \mathbf{Pic}(T) \cong \mathbf{Z}$.

Example 5.1. We want to show that for every $n \geq 3$ there is exactly one double line $Y \subseteq \mathbf{P}^n$ of genus $-n+2$, up to projective equivalence.

First we prove the uniqueness of Y as a complete embedding of a ribbon X by some very ample line bundle $\mathcal{L} \in \mathbf{Pic}(X)$ such that $\mathcal{L}_{|T}$ has degree 1.

By the previous exact sequence we also have $x = n-3 \geq -3$, whence X is unique, because it is a split ribbon.

Since $H^1(T, \mathcal{I}) = 0$, for any $n \geq 3$ there is at most one such pair $(X, \mathcal{L})$.

It is easy to see the existence of such pair $(X, \mathcal{L})$ in the following way.

Let $S_{n-3} \subseteq \mathbf{P}^n$ be the minimal degree smooth rational normal surface scroll with S_{n-3} isomorphic to the Hirzebruch surface F_{n-3}.

We have $\mathbf{Pic}(S_{n-3}) \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathbf{Pic}(S_{n-3})$ a fiber, f, of the ruling of S_{n-3} and a minimal degree section, h, of the ruling of S_{n-3}.

Hence $h^2 = 3-n$, $h \cdot f = 1$, $f^2 = 0$ and S_{n-3} is embedded by the very ample complete linear system $|h + (n-2)f|$; in particular S_{n-3} is unique up to a projective transformation and h is embedded as a line.

We take as Y the Cartier divisor $2h$ of S_{n-3} and as $\mathcal{L}$ the restriction of $\mathcal{O}_{S_{s-3}}(h + (n-2)f)$ to Y.

Since $\omega_{S_{n-3}} \sim -2h - (n-1)f$ by the adjunction formula we get $p_a(Y) = -n+2$.

Example 5.2. We want to show that for every $n \geq 5$ there is a unique linearly normal double conic $Y \subseteq \mathbf{P}^n$ with $s(Y) = 3$ (up to projective equivalence).

We proceed as in Example 5.1.

By Remark 2.4 we must have $h^0(\mathcal{O}_X) = n-3$, whence $x = n-5 \geq 0$. Whence, as before, there is at most one pair $(X, \mathcal{L})$ with $\deg(\mathcal{L}_{|T}) = 2$. This shows uniqueness.

To show the existence, let $S \subseteq \mathbf{P}^n$ be the embedding of the Hirzebruch surface F_{n-5} by the very ample complete linear system $|h + (n-3)f|$; in particular S is unique up to a projective transformation and h is embedded as a conic.

We take as Y the Cartier divisor $2h$ of S and as $\mathscr{L}$ the restriction of $\mathcal{O}_S(h + (n-3)f)$ to Y.

Denote by D a general hyperplane section of S.

From the exact sequence

$$0 \to \mathscr{I}_S \to \mathscr{I}_Y \to \mathcal{O}_S(-Y) \to 0$$

we get $h^0(\mathscr{I}_Y(1)) = h^0(\mathcal{O}_S(D - Y)) = 0$ and $h^1(\mathscr{I}_Y(1)) = h^1(\mathcal{O}_S(D - Y))$.

By duality it is easy to see that $h^2(\mathcal{O}_S(D - Y)) = 0$, hence by Riemann-Roch we get

$$-h^1(\mathcal{O}_S(D - Y)) = \frac{1}{2}(D - Y) \cdot (D - Y - \omega_S) + 1 = 0$$

In conclusion Y is linearly normal.

Let now H be a general hyperplane. Then $H \cap Y$ is the union of two disjoint 0-dimensional subschemes, each spanning a fiber of S. Hence $s(Y) = 3$.

5.2. Doublings. Here we produce examples of linearly normal locally Gorenstein curves $Y \subseteq \mathbf{P}^5$ with $s = 3$ and $\dim\langle Y_{red}\rangle = 2$.

We recall first some facts from [4].

Definition 5.3. Let $X \subseteq Z \subseteq \mathbf{P}^n$ be closed subschemes. Then Z is said to be a *doubling* of X if $\mathscr{I}_Z \supseteq \mathscr{I}_X^2$ and the following holds: if T' is an irreducible component of $X_{red} = Z_{red}$ and X' (resp. Z') is the largest subscheme of X (resp. of Z) supported by T', then $\deg(Z') = 2\deg(X')$.

Remark 5.4. (i) Let Z be a doubling of X. We have an exact sequence

$$0 \to \mathscr{L} \to \mathcal{O}_Z \to \mathcal{O}_X \to 0$$

where $\mathscr{L}$ is an $\mathcal{O}_X$-module, canonically isomorphic to $\mathscr{I}_X/\mathscr{I}_Z$.

(ii) If $\mathscr{L}$ is an invertible $\mathcal{O}_X$-module and $\phi : \mathscr{I}_X/\mathscr{I}_X^2 \to \mathscr{L}$ is a surjective map, then the scheme Z such that $\mathscr{I}_Z/\mathscr{I}_X^2 = \ker\phi$ is a doubling of X.

(iii) If Z is a doubling of X corresponding to $\mathscr{L} := \mathscr{I}_X/\mathscr{I}_Z$, then $\mathscr{L}$ is invertible in the following two cases:

(a) X and Z are locally Gorenstein;

(b) X is smooth and Z is S_2.

Now we consider the following situation: $\pi \subseteq \mathbf{P}^5$ is a plane and $D \subseteq \pi$ is a curve of degree $\delta > 1$. We have:

Lemma 5.5. *Let D be as above, Y a locally Gorenstein doubling of D and put $\mathscr{L} := \mathscr{I}_D/\mathscr{I}_Y$. Then the following are equivalent:*
(i) *Y is linearly normal and $s(Y) = 3$;*
(ii) *$h^0(\mathscr{L}) = 1$ and the map $H^0(\mathscr{I}_D(1)) \to H^0(\mathscr{L}(1))$ is an isomorphism (in particular $h^0(\mathscr{L}(1)) = 3$).*

Proof. (i) $\Rightarrow$ (ii) We have the exact sequence

$$0 \to \mathscr{L} \to \mathscr{O}_Y \to \mathscr{O}_D \to 0$$

and $H^0(\mathscr{O}_Y) \to H^0(\mathscr{O}_D)$ is obviously surjective.

Since $h^0(\mathscr{O}_D) = 1$ and $h^0(\mathscr{O}_Y) = 2$ (see Remark 2.4) we have $h^0(\mathscr{L}) = 1$.

The conclusion follows from the exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{I}_D \to \mathscr{L} \to 0,$$

since $H^0(\mathscr{I}_Y(1)) = H^1(\mathscr{I}_Y(1)) = 0$.

For the converse, reverse the above argument. □

Now we discuss the existence of doublings as above.

Lemma 5.6. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_D$-module such that $h^0(\mathscr{L}) = 1$, $h^0(\mathscr{L}(1)) = 3$ and $\mathscr{L}$ is generically generated.*

Then there is a linearly normal curve Y, which is a doubling of D associated to $\mathscr{L}$ and with $s(Y) = 3$.

Proof. By Remark 5.4(ii) we have to find a surjective map of $\mathscr{O}_D$-modules $\phi : \mathscr{I}_D/\mathscr{I}_D^2 \to \mathscr{L}$.

Since $\mathscr{I}_D/\mathscr{I}_D^2 = 3\mathscr{O}_D(-1) \oplus \mathscr{O}_D(-\delta)$, ϕ is determined by three sections $f_1, f_2, f_3 \in H^0(\mathscr{L}(1))$ and one section $f_4 \in H^0(\mathscr{L}(\delta))$.

Choose f_1, f_2, f_3 linearly independent.

By assumption there is an exact sequence

$$0 \to \mathscr{O}_D(\delta - 1) \to \mathscr{L}(\delta - 1) \to \mathscr{F} \to 0,$$

where $\mathscr{F}$ has 0-dimensional support.

Then $H^1(\mathscr{L}(\delta - 1)) = H^1(\mathscr{O}_D(\delta - 1)) = 0$, whence $\mathscr{L}(\delta)$ is generated by global sections ([9], Th. 2, p. 41).

Then it is easy to see that there is f_4 such that the map ϕ defined by $f_1, \ldots, f_4$ is surjective.

The doubling Y of D determined in this way has the required property by Lemma 5.5(i). □

Finally we give a characterization of the line bundles $\mathscr{L}$ corresponding to linearly normal doublings of Y with $s(Y) = 3$.

Proposition 5.7. *Let $D \subseteq \mathbf{P}^2$ be a curve, E an effective Cartier divisor on D and put $\mathscr{L} := \mathscr{O}_D(E)$. Then the following are equivalent:*
(i) $h^0(\mathscr{L}) = 1$ *and* $h^0(\mathscr{L}(1)) = 3$;
(ii) $h^1(\mathscr{L}(1)) = \frac{1}{2}(\delta - 2)(\delta - 3) - \deg(E)$;
(iii) *If* $\delta \le 3$ *then* $E = 0$; *if* $\delta \ge 4$ E *(considered as a subscheme of* $\mathbf{P}^2$*) imposes independent conditions to the curves of degree* $\delta - 4$ *(i.e.* $H^1(\mathscr{I}_E(\delta - 4)) = 0$*).*

Proof. The equivalence of (i) and (ii) follows easily from the exact sequence

$$0 \to \mathscr{O}_D \to \mathscr{L} \to \mathscr{O}_E \to 0$$

and the equivalence of (ii) and (iii) follows from Serre's duality and the exact sequence

$$0 \to \mathscr{I}_D \to \mathscr{I}_E \to \mathscr{O}_D(-E) \to 0. \qquad \square$$

Remark 5.8. It is clear that if (iii) of the above Proposition is satisfied, then

$$0 \le \deg(E) \le \frac{1}{2}(\delta - 2)(\delta - 3)$$

Conversely, if $D = C$ is an integral curve and $0 \le t \le \frac{1}{2}(\delta - 2)(\delta - 3)$, then there are reduced divisors E of degree t satisfying $h^1(\mathscr{I}_E(\delta - 4)) = 0$ (easy induction on t).

Proposition 5.9. *Let $C \subseteq \mathbf{P}^5$ be an integral plane curve of degree $\delta > 1$ and let Y be a locally Gorenstein doubling of C. Then we have:*
(i) *if Y is linearly normal and $s(Y) = 3$, then:*

$$2g - 1 - \frac{1}{2}(\delta - 2)(\delta - 3) \le p_a(Y) \le 2g - 1$$

where $g = p_a(C) = \frac{1}{2}(\delta - 1)(\delta - 2)$;
(ii) *every values for $p_a(Y)$ as in* (i) *can be attained.*

Proof. Let $\mathscr{L}$ be the invertible sheaf associated with Y.
Then $p_a(Y)$ can be computed from the exact sequence

$$0 \to \mathscr{L} \to \mathscr{O}_Y \to \mathscr{O}_C \to 0.$$

Moreover $\mathscr{L} = \mathscr{O}_C(E)$ and $0 \le \deg(E) \le \frac{1}{2}(\delta - 2)(\delta - 3)$.
An easy calculation shows that (i) holds. Finally (ii) follows from Remark 5.8. $\square$

5.3. All integers prescribed by Theorem B can occur.

Remark 5.10. Let $\pi \subseteq \mathbf{P}^5$ be a plane. We recall (see [2], Ex. 13) that there is a non-degenerate double structure X on π such that $s(X) = 3$, defined by the exact sequences:

$$0 \to \mathscr{I}_X \to \mathscr{I}_\pi \to \mathscr{O}_\pi \to 0$$

$$0 \to \mathscr{O}_\pi \to \mathscr{O}_X \to \mathscr{O}_\pi \to 0$$

From the cohomology sequences of the above sequences it is easy to see that $h^1(\mathscr{I}_X(1)) = 0$ and $h^1(\mathscr{O}_X(j)) = 0$ for all negative j's.

Now we can prove the following:

Lemma 5.11. *Let X be as above and let F be a hypersurface of degree $r > 1$ not containing π. Let $W := F \cap X$ and let $Y' \subseteq \pi$ be a curve such that $\langle W \cap Y' \rangle = \pi$.*
Then $Y := W \cup Y'$ is linearly normal and $s(Y) = 3$.

Proof. From the exact sequence $0 \to \mathscr{I}_W \to \mathscr{I}_F \oplus \mathscr{I}_X$ we get $h^0(\mathscr{I}_W(1)) = 0$, whence $h^0(\mathscr{I}_Y(1)) = 0$.

Moreover, from the exact sequence $0 \to \mathscr{I}_W \to \mathscr{I}_X \to \mathscr{O}_X(-r) \to 0$ and Remark 5.10, we get $h^1(\mathscr{I}_W(1)) = 0$.

The Mayer-Vietoris sequence gives:

$$0 \to H^0(\mathscr{I}_W(1)) \oplus H^0(\mathscr{I}_{Y'}(1)) \to H^0(\mathscr{I}_{W \cap Y'}(1))$$

$$\to H^1(\mathscr{I}_Y(1)) \to H^1(\mathscr{I}_W(1)) \oplus H^1(\mathscr{I}_{Y'}(1)) \to 0.$$

By assumption we have $h^0(\mathscr{I}_{W \cap Y'}(1)) = 3 = h^0(\mathscr{I}_{Y'}(1))$ and obviously $h^1(\mathscr{I}_{Y'}(1)) = 0$.

Since $h^1(\mathscr{I}_W(1)) = 0 = h^0(\mathscr{I}_W)(1))$ we get $h^1(\mathscr{I}_Y(1)) = 0$.

This shows that Y is linearly normal.

Finally we have $3 = s(X) \geq s(Y) \geq s(W) \geq 3$ and the conclusion follows. $\square$

Now we are ready to show that all integers prescribed by Theorem B can occur.

Remark 5.12. Let $\pi \subseteq X \subseteq \mathbf{P}^5$ be as before.

Let $C \subseteq \pi$ be an integral curve of degree $\delta > 1$ and choose a homogeneous polynomial G such that $C = V(G) \cap \pi$.

Fix integers $e, f > 0$ and put $d'' := e\delta$, $d = f\delta$ and assume $d'' + \delta \leq d \leq 2d''$ (i.e. $e + 1 \leq f \leq 2e$) (see Theorem B).

Put $Y' := V(G^e) \cap \pi$, $W := V(G^{f-e}) \cap X$ and $Y := W \cup Y'$.

Since $1 \leq f - e \leq e$ it is clear that $Y' \cap W$ is a 1-dimensional scheme of degree $\delta(f - e) \geq 2$, whence $\langle Y' \cap W \rangle = \pi$. Then we can apply Lemma 5.11 to show that Y is linearly normal and $s(Y) = 3$.

Since $f - e \leq e$ it is clear that Y' is the largest subcurve of Y contained in π (hence $Y' = Y''$, with the notation of Theorem B).

Finally, by restricting to a general hyperplane we see that

$$\deg(Y) = \deg(Y') + \deg(W) - \deg(Y' \cap W) = d.$$

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