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Singular limit of a degenerate chemotaxis-Fisher equation

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Abstract. We study the singular limit of a degenerate nonlinear diffusion equation which appears in a chemotaxis-growth model. We prove the convergence to the solution of a free boundary problem where the motion equation of the interface involve the gradient of the chemotactic concentration and the critical velocity of a degenerate Fisher equation.

1 Introduction

The equation that we consider in this paper arises in biomathematics. It is a simplified version of a model of pattern formation during bacterial growth that has attracted a lot of attention in the recent years. It actually describes aggregation phenomena in bacteria colonies in the presence of an attractant. The first model for this so-called chemotaxis phenomenon has been introduced by Keller and Segel in [16] and [17] and reads as follows

$$\begin{cases} u_t = \nabla.(D(u)\nabla u) - \nabla.(u\nabla\chi(v)) \\ \tau v_t = \Delta v + u - \gamma v \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \end{cases}$$

where u is the density of biological population (eg. slime mold), v is the concentration of the chemotaxis substance, ν is the outward normal vector on the boundary of the domain and $\tau \geq 0$ is a chemotaxis time. Here, $D(u) \geq 0$ is the mobility coefficient of the slime mold and $\gamma > 0$ represents the degradation rate of v. The second term in the equation is accounting for the chemotaxis effect, $\chi(v)$ being the sensitivity function of the chemotactic aggregation [23].

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The function χ satisfies the assumption that $\chi' > 0$ to take into account the attraction of the slime mold towards the regions of higher concentration of v.

For the Keller-Segel system, in the case that $D(u)$ is constant, blow-up of the u component of a solution may occur, which corresponds to an aggregation phenomena; we refer in particular to [12], [7], [15], [22], [8], [11], [28]. A recent survey of blow-up results can be found in [14].

Chemotaxis models including a growth effect have been considered (see [19], [5] and [29]). Such a model is typically a reaction-diffusion system of the form

$$\begin{cases} u_t = \nabla.(D(u)\nabla u) - \nabla.(u\nabla\chi(v)) + f(u) \\ \tau v_t = \Delta v + u - \gamma v, \end{cases}$$

where $f(u)$ is the growth term. Two types of hypotheses are classically considered for the function f, (i) f is logistic (or of Fisher-type) or (ii) f is bistable. Mimura and Tsujikawa [19] consider a chemotactic model with cubic-like nonlinear growth (case (ii)) and studied the problem of existence and stability of standing pulse solutions of

$$\begin{cases} u_t = \varepsilon D_u \Delta u - k\nabla.(u\nabla\chi(v)) + \dfrac{1}{\varepsilon} f(u) \\ \varepsilon\tau v_t = \Delta v + u - \gamma v, \end{cases}$$

in $\mathbf{R}^2$ and for $0 < \varepsilon \ll 1$ sufficiently small. Bonami, Hilhorst, Logak and Mimura [3] studied the homogeneous Neumann boundary value problem for this system of equations on a bounded domain in $\mathbf{R}^N$ in the bistable case and for $\tau = 0$ and $D_u = \frac{1}{\varepsilon}$. They established that the limiting behavior of the solution of the chemotaxis problem as $\varepsilon \to 0$ is given by a free boundary problem involving mean curvature and the normal derivative of the chemotaxis sensitivity.

In this paper, we consider the case (i) assuming that the growth term is logistic. More precisely, we choose

$$f(u) = u(1 - u^p), \tag{1.1}$$

with $p > 0$. Besides we consider the case of a degenerate diffusion, taking into account population density pressure (see Morishita [20], Shigesada [24]). This corresponds to the mobility coefficient $D(u) = D_0 u^{m-1}$, for $D_0 > 0$ and $m > 1$. Precisely, we have the following system

$$\begin{cases} u_t = \varepsilon\nabla(D_0 u^{m-1}\nabla u) - k\nabla.(u\nabla\chi(v)) + \dfrac{1}{\varepsilon} u(1 - u^p) \\ \varepsilon\tau v_t = \Delta v + u - \gamma v. \end{cases}$$

Due to the non uniqueness of the travelling wave of this problem and the degeneracy of the diffusivity, the analysis of this model becomes extremely complex. Therefore, to well understand the phenomena we have to consider the above model in the simpler case of only one equation, assuming that v is given. More precisely, we consider in this paper the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \Delta(u^m) - \nabla.(u\nabla v) + \dfrac{1}{\varepsilon} u(1-u^p) & \text{in } \Omega \times (0,T] \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = g^\varepsilon(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $v = v(x,t)$ represents here the sensitivity of the chemotaxis and is supposed to be a given smooth function. Here $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain and ν denotes the outward normal vector on $\partial\Omega$. We assume that $m > 1$, which makes the equation degenerate parabolic. Therefore we define below the notion of a weak solution to this equation. We study the limiting behavior of the solution u^ε to Problem (1.2) as ε goes to zero. For a suitable choice of initial data g^ε compactly supported in Ω, we prove that as ε goes to zero, the function u^ε converges to u^0 on a time interval $[0,T]$ for some $T > 0$. The limiting function u^0 is the characteristic function of a smooth moving domain $\Omega_t \subset\subset \Omega$, whose motion law is related to v by

$$V_n = \frac{\partial v}{\partial n} + c_m \qquad \text{on } \Gamma_t,\ t \in [0,T], \tag{1.3}$$

where $\Gamma_t = \partial\Omega_t$, n is the outward normal vector on Γ_t, V_n is the normal velocity at a point on Γ_t and c_m is a constant defined as the critical velocity of the following degenerate travelling wave Fisher problem

$$\begin{cases} (U^m)'' + cU' + U(1-U^p) = 0, \\ U(-\infty) = 1, \quad U(+\infty) = 0. \end{cases} \tag{1.4}$$

More precisely, c_m is defined in the following theorem, established in [10] together with other properties of the travelling fronts of reaction-diffusion-advection equations (see also [1]).

Theorem 1.1 [10]. *Assume that $m > 1$. Then there exists $c_m > 0$ such that*

(i) *For $0 < c < c_m$, there is no weak solution to* (1.4).

(ii) *For any $c \geq c_m$, there is a weak solution U_c to* (1.4), *which is unique up to translation. For $c > c_m$, U_c is strictly positive and strictly decreasing on* $\mathbf{R}$. *For $c = c_m$, U_{c_m} is compactly supported from the right. We can uniquely define U_{c_m} by imposing the condition that*

$$U_{c_m} > 0 \quad on \ (-\infty, 0) \qquad and \qquad U_{c_m} = 0 \quad on \ [0, +\infty). \tag{1.5}$$

Moreover we have the following properties, recalled in [13].

Proposition 1.2 [13]. *Let U be the unique solution of*

$$\begin{cases} (U^m)'' + c_m U' + U(1 - U^p) = 0 \\ U(-\infty) = 1, \quad U > 0 \ on \ (-\infty, 0), \quad U = 0 \ on \ [0, +\infty). \end{cases} \tag{1.6}$$

Then there exists $k_1 > 0$ and $\alpha > 0$ such that

$$\begin{aligned} |(U^m)'(z)| &\le k_1 U(z) && for \ all \ z \in \mathbf{R}, \\ |1 - U(z)| &\le k_1 e^{-\alpha z} && for \ all \ z \in \mathbf{R}_-, \\ |zU'(z)| &\le k_1 U(z) && for \ all \ z \le -1. \end{aligned} \tag{1.7}$$

In the case where $v = 0$, the problem (1.2) has been considered in [13]. The authors establish in this case the local-in-time convergence of the solution u^ε to a step function taking values 1 and 0 on both sides of an interface moving with constant normal velocity c_m. Thus our results here extend the results of [13] when a chemotaxis term arises in the equation for the density of bacteria u. Our methods are similar to those used in [13]. But the determination of the sub- and super-solution is more complicated.

The organization of the paper is as follows. In section 2, we give a formal derivation of the motion law (1.3). In section 3, we show that if a smooth initial interface Γ_0 is given, the corresponding initial-value problem for the front admits a unique smooth local-in-time solution $\Gamma = (\Gamma_t)_{t \in [0, T]}$, for some $T > 0$. In section 4, we define a notion of weak solution to Problem (1.2) and define the modified distance function to the limit interface. In section 5 we prove the convergence result using sub- and super-solutions. Finally in section 6 we study an example of the radially symmetric case which has non trivial solution in contrast to the model discused in [13].

2 Formal derivation of the interface motion equation

We show in this section how the front propagation law (1.3) can be derived formally from Problem (1.2). Using the results of Theorem 1.1 in [10], we consider the unique solution $U = U_{c_m}$ to the problem (1.4) for $c = c_m$ satisfying (1.5). Let us consider a smooth moving boundary $\Gamma_t = \partial \Omega_t$ and let d be the signed distance function to Γ_t defined in the neighborhood of Γ_t by

$$d(x, t) = \begin{cases} -dist(x, \Gamma_t) & \text{for } x \in \Omega_t \\ dist(x, \Gamma_t) & \text{for } x \in \Omega \backslash \overline{\Omega_t} \end{cases}$$

and smoothly extended in Ω in order to satisfy

$$d(x,t) < 0 \quad \text{for all } x \in \Omega_t, \qquad d(x,t) > 0 \quad \text{for all } x \in \Omega \backslash \overline{\Omega_t}.$$

In particular, $d = 0$ on Γ_t and $|\nabla d(x,t)| = 1$ in a neighborhood of Γ_t.

As in the formal derivation proposed in [4], we make the assumption that for ε small enough, the solution u^ε to Problem (1.2) can be approximated by the function

$$\tilde{u}(x,t) = U\left(\frac{d(x,t)}{\varepsilon}\right).$$

Note that by the definitions of U_{c_m} and of d, the function $\tilde{u}(.,t)$ approximates for $\varepsilon > 0$ small enough the characteristic function χ_{Ω_t}. Our purpose here is to find the evolution law for $d(x,t)$, assuming that the profile of u^ε is given by the travelling wave U_{c_m}.

For a given smooth function v, we define the operator L by

$$Lu := u_t - \varepsilon \Delta u^m + \nabla u.\nabla v - \frac{1}{\varepsilon} u(1 - u^p) + u\Delta v. \tag{2.1}$$

An easy computation gives that

$$L\tilde{u} = \frac{(U^m)''}{\varepsilon}(1 - |\nabla d|^2) + \frac{U'}{\varepsilon}(d_t + \nabla d \nabla v + c_m) + (U^m)'\Delta d + U\Delta v.$$

Note that for x close to Γ_t, $|\nabla d(x,t)| = 1$ so that

$$L\tilde{u} = \frac{U'}{\varepsilon}(d_t + \nabla d \nabla v + c_m) + O(1).$$

Since u^ε satisfies $Lu^\varepsilon = 0$, we impose that for the approximation $\tilde{u}$, the $\frac{1}{\varepsilon}$ term drops which implies that

$$-d_t = \nabla d \nabla v + c_m$$

on Γ_t. Since ∇d is the outward normal vector on Γ_t, this can be rewritten as

$$V_n = \frac{\partial v}{\partial n} + c_m \qquad \text{on } \Gamma_t,$$

which is equation (1.3).

3 The limit interface motion: Well-posedness

We consider the following problem

$$\begin{cases} V_n = \dfrac{\partial v}{\partial n} + c_m & \text{on } \Gamma_t,\ t \in (0,T] \\ \Gamma_{t|t=0} = \Gamma_0 \end{cases} \tag{3.1}$$

and we establish the following result.

Theorem 3.1. *Let v be a given smooth function, $\Omega_0 \subset\subset \Omega$ is a smooth domain and $\Gamma_0 = \partial\Omega_0$ is the zero-level of a smooth function f_0. Then there exists a time $T > 0$ such that Problem* (3.1) *has a unique smooth solution $\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]}$.*

Proof. Let f_0 be a smooth function, assume that Γ_0 is given by

$$\Gamma_0 = \{x \in \Omega, f_0(x) = 0\} \tag{3.2}$$

and that Ω_0 is the connected component of $\Omega\backslash\Gamma_0$ which contains

$$\{x \in \Omega, f_0(x) < 0\}.$$

Let $T > 0$ be a fixed constant that will be chosen later. For $0 \le t \le T$, we parameterize the interface $\Gamma = (\Gamma_t)_{t\in[0,T]}$ as follows.

$$\Gamma_t = \{x \in \Omega, f(x,t) = 0\}, \tag{3.3}$$

where $f : \Omega \times [0,T] \to \mathbf{R}$ is the unknown function. We consider the restriction of f on the interface Γ so that we have

$$f(x,t) = 0 \qquad \text{on } \Gamma_t$$

after derivation with respect to t we obtain that

$$-f_t = \nabla f . \frac{\partial x}{\partial t}.$$

Since

$$V_n = -\frac{\partial x}{\partial t} . \frac{\nabla f}{|\nabla f|}$$

the equation (3.1) implies that

$$f_t = \nabla v . \nabla f + c_m |\nabla f|.$$

Finally we can rewrite Problem (3.1) as a first order nonlinear evolution equation for $f(x,t)$ of the form

$$(P_f) \begin{cases} f_t = a(x,t).\nabla f + c_m|\nabla f|, & x \in \Omega, t \in [0,T] \\ f(x,0) = f_0(x), & x \in \Omega \end{cases}$$

where a is a smooth function. Problem (P_f) is therefore a first-order Hamilton-Jacobi type problem. The local-in-time well-posedness of Problem (P_f) can be found in [2] and [18].

We define the front Γ_t at time t by (3.3) where f is the unique solution of (P_f) then it is shown in [6] and [9] that the propagation of Γ_t depends only on the sets Γ_0 and Ω_0 but not on the choice of f_0 satisfying (3.2). This ends the proof of Theorem 3.1.

4 Weak solutions and modified distance function

Since the equation for u in Problem (1.2) is parabolic degenerate, we need to define a weak notion of solution to this problem.

Definition of a weak solution. A function $u : \Omega \times \mathbf{R}_+ \to \mathbf{R}$ is a weak solution of Problem (1.2) if for all $T > 0$, (i) $u \in C(\overline{\Omega} \times [0, T])$ and $\nabla(u^m) \in L^2(\Omega \times [0, T])$, (ii) for any function $\phi \in C^1(\overline{\Omega} \times [0, T])$ with $\phi \geq 0$ in $\overline{\Omega} \times [0, T]$, u satisfies the integral identity

$$\int_\Omega u(T)\phi(T) = \int_\Omega u(0)\phi(0) + \int_0^T \int_\Omega \left(u\phi_t - \varepsilon \nabla(u^m)\nabla\phi + u\nabla v \nabla\phi + \frac{1}{\varepsilon} f(u)\phi \right).$$

By a sub- (resp. super-) solution of Problem (1.2), we mean a function u_- (resp. u_+) which satisfies (i) and (ii) with the equality replaced by $\leq$ (resp. $\geq$).

It follows from these definitions that u is a weak solution of Problem (1.2) if for all $T > 0$ we have

$$\text{(a)} \quad u \in C(\overline{\Omega} \times [0, T]) \qquad \text{and} \qquad \nabla(u^m) \in C(\overline{\Omega} \times [0, T]),$$

and

$$\text{(b)} \quad Lu = 0 \qquad a.e. \qquad \text{in } \overline{\Omega} \times [0, T].$$

Similarly, u_- (resp. u_+) is a weak sub- (resp. super-) solution of Problem (1.2) if it satisfies a) and $Lu_- \leq 0$ (resp. $Lu_+ \geq 0$), $a.e.$

Modified distance function. Let $\Gamma = (\Gamma_t)_{t \in [0, T]}$ be the solution to the front propagation law given in Theorem 3.1. We define for x close to Γ_t, the distance function $\bar{d}(x, t)$ by

$$\bar{d}(x, t) = \begin{cases} -dist(x, \Gamma_t) & \text{for } x \in \Omega_t \\ dist(x, \Gamma_t) & \text{for } x \in \Omega \backslash \overline{\Omega_t} \end{cases}$$

We choose $d_0 > 0$ such that $x \mapsto \bar{d}(x, 0)$ is smooth in the tubular neighborhood of Γ_0

$$\Gamma(2d_0) = \{x \in \overline{\Omega}, |\bar{d}(x, 0)| < 2d_0\}.$$

We define now d a smooth modification of $\bar{d}$ such that

$$d = \bar{d} \qquad \text{in } \Gamma(d_0) = \{(x, t) \in \overline{\Omega} \times [0, t], |\bar{d}(x, t)| < d_0\},$$

$$d_0 < |d| \leq 2d_0 \qquad \text{and} \qquad d\bar{d} > 0 \qquad \text{in } \Gamma(2d_0) \backslash \Gamma(d_0),$$

$$|d| = 2d_0 \qquad \text{and} \qquad d\bar{d} > 0 \qquad \text{in } \overline{\Omega} \times [0, t] \backslash \Gamma(2d_0).$$

In particular, $d = 0$ on Γ_t and $|\nabla d(x,t)| = 1$ in $\Gamma(d_0)$, and we have

$$d_t + c_m + \nabla d \nabla v = 0 \qquad \text{on } \Gamma_t = \{x \in \Omega, d(x,t) = 0\}$$

which implies that there exists $k_2 > 0$ such that for all $(x,t) \in \overline{\Omega} \times [0,T]$ we have

$$|(d_t + c_m |\nabla d|^2 + \nabla d \nabla v)(x,t)| \le k_2 |d(x,t)| \tag{4.1}$$

and

$$|\nabla d(x,t)| + |\Delta d(x,t)| \le k_2. \tag{4.2}$$

We prove here the following result.

Theorem 4.1. *Let Ω_0 be a smooth bounded domain with $\Omega_0 \subset\subset \Omega$ and $\Gamma = \bigcup_{t \in [0,T]} (\Gamma_t \times \{t\})$ defined by* (3.1) *in Theorem* 3.1 *for $T > 0$ small enough. Then there exist initial data g^ε satisfying*

$$\lim_{\varepsilon \to 0} g^\varepsilon(x) = \chi_{\Omega_0}(x)$$

for all $x \in \Omega$ such that the corresponding solution u^ε of (1.2) *satisfies*

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = \chi_{\Omega_t}(x)$$

for all $x \in \Omega$ and $t \in [0,T]$.

5 Sub-solution and super-solution

We define the functions $u_\pm$ as

$$u_\pm(x,t) = a_\pm(\varepsilon) U\left(\frac{d(x,t) \mp \varepsilon m_1 e^{m_2 t}}{\varepsilon}\right)$$

with $a_\pm(\varepsilon) = 1 \pm C\varepsilon$ for some constants $m_1 > 1$ and $C, m_2 > 0$, and we shall show the following

Proposition 5.1. *Let $\varepsilon > 0$ small enough and $U = U_{c_m}$ be defined in* (1.4)–(1.5). *Then there exist $m_1 > 1$ and $C, m_2 > 0$ such that u_- and u_+ defined above are respectively sub- and super-solution of Problem* (1.2).

Proof. To prove this proposition it is sufficient to prove that

$$Lu_+ \ge 0 \qquad \text{and} \qquad Lu_- \le 0 \qquad a.e. \qquad \text{on } \overline{\Omega} \times [0,T].$$

We first establish that

$$Lu_+ \ge 0 \qquad a.e. \qquad \text{on } \overline{\Omega} \times [0,T].$$

Since $u_+ = 0$ if $d(x,t) > \varepsilon m_1 e^{m_2 t}$ then

$$Lu_+ = 0 \qquad \text{if } d(x,t) > \varepsilon m_1 e^{m_2 t}.$$

Hence we establish $Lu_+ \geq 0$ on the subset where $d(x,t) \leq \varepsilon m_1 e^{m_2 t}$.

An easy computation gives that

$$\begin{aligned}
\varepsilon Lu_+ &= aU'[d_t - \varepsilon m_1 m_2 e^{m_2 t} + \nabla d \nabla v + c_m a^{m-1} |\nabla d|^2] \\
&\quad + U[-a + a^m |\nabla d|^2 + U^p (a^{p+1} - a^m |\nabla d|^2) + \varepsilon a \Delta v] - \varepsilon (U^m)' a^m \Delta d \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

We first analyze $T_1 = aU' T_1'$, where

$$\begin{aligned}
T_1' &= d_t - \varepsilon m_1 m_2 e^{m_2 t} + \nabla d \nabla v + c_m a^{m-1} |\nabla d|^2 \\
&= d_t + c_m |\nabla d|^2 + \nabla d \nabla v - \varepsilon m_1 m_2 e^{m_2 t} + c_m (a^{m-1} - 1) |\nabla d|^2 \\
&\leq k_2 |d - \varepsilon m_1 e^{m_2 t}| + \varepsilon (k_2 - m_2) m_1 e^{m_2 t} + c_m |a^{m-1} - 1| k_2^2
\end{aligned}$$

where k_2 is defined in (4.1) and (4.2).

*First case $0 \leq d \leq \varepsilon m_1 e^{m_2 t}$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have

$$\varepsilon m_1 e^{m_2 T} \leq d_0$$

which implies that $0 \leq d < d_0$ so that

$$|d - \varepsilon m_1 e^{m_2 t}| \leq \varepsilon m_1 e^{m_2 t}.$$

Therefore we obtain that

$$T_1' \leq k_2 |d - \varepsilon m_1 e^{m_2 t}| + \varepsilon (k_2 - m_2) m_1 e^{m_2 t} + c_m k_2^2 |a^{m-1} - 1|.$$

We choose $a = a_+ = 1 + C\varepsilon$, where C is determined later.

Hence

$$T_1' \leq \varepsilon [(2k_2 - m_2) m_1 e^{m_2 t} + 2 c_m C k_2^2 (m-1)].$$

We choose m_2 large enough to obtain $T_1' \leq 0$ in $[0, T]$ conclude that $T_1 \geq 0$.

*Second case $d \leq 0$. We have that

$$T_1' \leq k_2 |d - \varepsilon m_1 e^{m_2 t}| + \varepsilon (k_2 - m_2) m_1 e^{m_2 t} + c_m |a^{m-1} - 1| k_2^2.$$

With our choice of m_2 in the first case we obtain

$$T_1' \leq k_2 |d - \varepsilon m_1 e^{m_2 t}|.$$

Since $d \leq 0$ then

$$\frac{d - \varepsilon m_1 e^{m_2 t}}{\varepsilon} \le -m_1 \le -1.$$

Using (1.7) and the fact that $U' \le 0$ we obtain

$$T_1 = aU'T_1' \ge -\varepsilon k_1 k_2 a U.$$

We analyze now $T_2 + T_3$. We have

$$|T_3| = |(U^m)' \Delta d a^m| \le \varepsilon a^m k_1 k_2 U$$

which implies that

$$\begin{aligned} T_2 + T_3 &\ge U[-a + a^m|\nabla d|^2 + U^p(a^{p+1} - a^m|\nabla d|^2) + \varepsilon a \Delta v - \varepsilon k_1 k_2 a^m] \\ &\ge U[-a + a^m + (|\nabla d|^2 - 1)a^m(1 - U^p) \\ &\quad + U^p(a^{p+1} - a^m) - \varepsilon a K - \varepsilon k_1 k_2 a^m]. \end{aligned}$$

*If $-d_0 \le d \le \varepsilon m_1 e^{m_2 t}$ then $|\nabla d| = 1$. Using that

$$0 \le (1 - U^p)\left(\frac{d - \varepsilon m_1 e^{m_2 t}}{\varepsilon}\right) \le (1 - U^p)\left(-\frac{d_0}{\varepsilon}\right) \le k_1 e^{-\alpha(d_0/\varepsilon)}$$

then we have

$$|(|\nabla d|^2 - 1)a^m(1 - U^p)| \le (1 + k_2^2)a^m k_1 e^{-\alpha(d_0/\varepsilon)} \le \frac{\varepsilon k_1 a^m(1 + k_2^2)}{\alpha d_0}.$$

Hence

$$T_2 + T_3 \ge U\left[-a + a^m + U^p(a^{p+1} - a^m) - \varepsilon a K - \varepsilon k_1 k_2 a^m - \frac{\varepsilon k_1 a^m(1 + k_2^2)}{\alpha d_0}\right]$$

therefore

$$\begin{aligned} \varepsilon L u_+ \ge U\Bigg[&-a + a^m + U^p(a^{p+1} - a^m) - \varepsilon a K \\ &- \varepsilon k_1 k_2 a^m - \varepsilon k_1 k_2 a - \frac{\varepsilon k_1 a^m(1 + k_2^2)}{\alpha d_0}\Bigg]. \end{aligned}$$

-First case $1 < m \le p + 1$. We choose $a = a_+ = 1 + C\varepsilon$ such that

$$(1 + \varepsilon K + \varepsilon k_1 k_2)a - \left(1 - \varepsilon k_1 k_2 - \varepsilon k_1 \frac{1 + k_2^2}{\alpha d_0}\right)a^m \le 0$$

which is satisfied if we choose

$$C > C_0 = \frac{K + 2k_1 k_2 + k_1 \frac{1 + k_2^2}{\alpha d_0}}{m - 1}.$$

-Second case $m \geq p+1$. We choose $a = a_+ = 1 + C\varepsilon$ such that

$$(1 + \varepsilon K + \varepsilon k_1 k_2)a - a^{p+1} + \varepsilon\left(k_1 k_2 + \varepsilon k_1 \frac{1 + k_2^2}{\alpha d_0}\right)a^m \leq 0$$

which is satisfied if we choose

$$C > C_0 = \frac{1}{p}\left(K + 2k_1 k_2 + k_1 \frac{1 + k_2^2}{\alpha d_0}\right).$$

Therefore for this choice of a_+ we obtain $Lu_+ \geq 0$ *a.e.* on $\overline{\Omega} \times [0, T]$.

We now prove that $Lu_- \leq 0$ *a.e.* on $\overline{\Omega} \times [0, T]$. We have

$$\begin{aligned}
\varepsilon Lu_- &= aU'[d_t + \varepsilon m_1 m_2 e^{m_2 t} + \nabla d \nabla v + c_m a^{m-1}|\nabla d|^2] \\
&\quad + U[-a + a^m|\nabla d|^2 + U^p(a^{p+1} - a^m|\nabla d|^2) + \varepsilon a \Delta v] - \varepsilon(U^m)' a^m \Delta d \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

We first analyze $T_1 = aU'T_1'$, where

$$\begin{aligned}
T_1' &= d_t + \varepsilon m_1 m_2 e^{m_2 t} + \nabla d \nabla v + c_m a^{m-1}|\nabla d|^2 \\
&= d_t + c_m|\nabla d|^2 + \nabla d \nabla v + \varepsilon m_1 m_2 e^{m_2 t} + c_m(a^{m-1} - 1)|\nabla d|^2 \\
&\geq -k_2|d + \varepsilon m_1 e^{m_2 t}| + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - c_m|a^{m-1} - 1|\,|\nabla d|^2.
\end{aligned}$$

There exists $\varepsilon_0 > 0$ such that $d_0 > 2\varepsilon m_1 e^{m_2 t}$ for all $\varepsilon < \varepsilon_0$.

*$-d_0 < -2\varepsilon m_1 e^{m_2 t} < d < -\varepsilon m_1 e^{m_2 t}$. We have $|\nabla d| = 1$ then

$$T_1' \geq -k_2|d + \varepsilon m_1 e^{m_2 t}| + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - c_m|a^{m-1} - 1|.$$

We choose $a = a_- = 1 - C\varepsilon$ where C is determined later. Hence

$$\begin{aligned}
T_1' &\geq -k_2|d + \varepsilon m_1 e^{m_2 t}| + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - \varepsilon(m-1)Cc_m \\
&\geq k_2(d + \varepsilon m_1 e^{m_2 t}) + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - \varepsilon(m-1)Cc_m \\
&\geq \varepsilon(m_1 m_2 e^{m_2 t} - Cc_m(m-1)).
\end{aligned}$$

We choose m_1 and m_2 large enough to obtain

$$m_1 m_2 e^{m_2 t} - Cc_m(m-1) \geq 0.$$

Hence $T_1' \geq 0$ and then

$$T_1 \leq 0.$$

*$-d_0 - \varepsilon m_1 e^{m_2 t} \leq d \leq -2\varepsilon m_1 e^{m_2 t}$. We have $|\nabla d| = 1$ and

$$T_1' \geq -k_2|d + \varepsilon m_1 e^{m_2 t}| + \varepsilon m_1(k_2 + m_2)e^{m_2 t} + c_m(a^{m-1} - 1)$$
$$\geq -k_2|d + \varepsilon m_1 e^{m_2 t}|$$

which implies that

$$T_1 \leq \varepsilon a k_1 k_2 U.$$

*$d \leq -d_0 - \varepsilon m_1 e^{m_2 t}$. We have

$$T_1' \geq -k_2|d + \varepsilon m_1 e^{m_2 t}| + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - \varepsilon k_3 k_2^2 c_m.$$

Hence

$$T_1 \leq aU'[k_2(d + \varepsilon m_1 e^{m_2 t}) + \varepsilon(k_2 + m_2)m_1 e^{m_2 t} - \varepsilon k_3 k_2^2 c_m]$$
$$\leq \frac{\varepsilon a k_1 U}{d_0}[k_2 d_0 - \varepsilon(k_2 + m_2)m_1 e^{m_2 t} + \varepsilon k_3 k_2^2 c_m]$$
$$\leq \varepsilon a k_4 U.$$

Therefore

$$\varepsilon L u_- \leq U[-a + a^m(|\nabla d|^2 - 1)(1 - U^p) + a^m + U^p(a^{p+1} - a^m)$$
$$+ \varepsilon a K + \varepsilon k_1 k_2 a^m] + T_1$$
$$\leq U[-a + a^m(|\nabla d|^2 - 1)(1 - U^p) + a^m(1 + \varepsilon k_1 k_2)$$
$$+ U^p(a^{p+1} - a^m) + \varepsilon a(K + k_4)]$$
$$\leq U[-a + a^m(1 + 2\varepsilon k_1 k_2) + U^p(a^{p+1} - a^m) + \varepsilon a(K + k_4)]$$
$$\leq U[-a(1 - \varepsilon k_5) + a^m(1 + 2\varepsilon k_1 k_2) + U^p(a^{p+1} - a^m)].$$

-First case $1 < m \leq p + 1$. We choose $a = a_- = 1 - C\varepsilon$ such that

$$(1 - \varepsilon k_5)a - (1 + 2\varepsilon k_1 k_2)a^m \geq 0$$

which is satisfied if we choose

$$C > C_0 = \frac{k_5 + 2k_1 k_2}{m - 1}.$$

-Second case $m \geq p + 1$. We choose $a = a_- = 1 - C\varepsilon$ such that

$$(1 - \varepsilon k_5)a - a^{p+1} - 2\varepsilon k_1 k_2 a^m \geq 0$$

which is satisfied if we choose

$$C > C_0 = \frac{1}{p}(k_5 + 2k_1 k_2).$$

Therefore for these choices of a_+ and a_- we obtain

$$Lu_- \le 0 \le Lu_+ \qquad a.e. \qquad \text{on } \bar{\Omega} \times [0,T].$$

This ends the proof of Proposition 5.1.

Finally the result of Theorem 4.1 is obtained for any initial data g^ε satisfying

$$(1 - C\varepsilon)U\left(\frac{d(x,0) + \varepsilon m_1}{\varepsilon}\right) \le g^\varepsilon(x) \le (1 + C\varepsilon)U\left(\frac{d(x,0) - \varepsilon m_1}{\varepsilon}\right). \quad (5.1)$$

6 The interface motion in the radially symmetric case

In this section we consider the particular case of radial symmetric where the sensitivity function v is of the form $v(t,x) = h(|x|) + k(t)$ with h a smooth even function.

We prove here the following result.

Theorem 6.1. *Let* $\Omega = \mathscr{B}_{(0,R)}$, $\Omega_0 = \mathscr{B}_{(0,r_0)}$ *with* $0 < r_0 < R$ *two ball in* $\mathbf{R}^n$ *and let* u^ε *be the solution of Problem* (1.2) *with a radially symmetric initial data* g^ε *satisfying* (5.1). *Then there are two cases*

1. *either* $h'(r) > -c_m$ *on* $[r_0, \infty)$ *and* $\int_{r_0}^{\infty} \frac{dr}{c_m + h'(r)} = T < +\infty$ *then*

$$\lim_{t \to T} \lim_{\varepsilon \to 0} u^\varepsilon(t,x) = 1 \qquad \text{for all } x \in \Omega$$

2. *or there exists* $r_\infty > 0$ *depending only on* h *and* r_0 *such that*

$$\lim_{t \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon(t,x) = \begin{cases} 1 & \text{in } \mathscr{B}_{(0,r_\infty)} \\ 0 & \text{in } \mathscr{B}_{(0,R)} \backslash \mathscr{B}_{(0,r_\infty)} \end{cases}$$

Proof. In this case, the solution of Problem (3.1) is $\Gamma_t = \partial \mathscr{B}_{(0,r(t))}$ where $r(t)$ is a solution to the following

$$\begin{cases} r' = h'(r) + c_m & t > 0 \\ r(0) = r_0 \end{cases} \quad (6.1)$$

which is a first-order ODE. We know that the solution of this problem is either a constant r_0 or is a strictly monotonous function. We distinguish here the following three cases.

*First case $h'(r_0) = -c_m$. This implies that r is a constant function and then $r_\infty = r_0$.

*Second case $h'(r_0) < -c_m$. Since h is an even function then $h'(0) = 0$ which implies that there exists $0 < r < r_0$ such that $h'(r) = -c_m$. We obtain then that $r_\infty = \sup\{r \in (0, r_0), h'(r) = -c_m\}$.

*Third case $h'(r_0) > -c_m$. In this case there are three possibilities.

1) There exists $r > r_0$ such that $h'(r) = -c_m$. We obtain then that $r_\infty = \inf\{R, \inf\{r > r_0, h'(r) = -c_m\}\}$.

2) $h'(r) > -c_m$ on $[r_0, \infty)$ and $\int_{r_0}^{\infty} \frac{dr}{c_m + h'(r)} = T < +\infty$, the solution $r(t)$ of 6.1 blows up at this time T.

3) $h'(r) > -c_m$ on $[r_0, \infty)$ and $\int_{r_0}^{\infty} \frac{dr}{c_m + h'(r)} = +\infty$ then $r_\infty = \inf\{R, \infty\} = R$.

According now to the result obtained in Theorem 4.1 this ends the proof of Theorem 6.1.

Remark 6.2. *This result give the equilibrium points. The equilibrium r_∞ is stable if $h''(r_\infty) < 0$.*

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